Relevant Implication and Ordered Geometry

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Abstract

This paper shows that model structures for $\mathbf{R}^+$, the system of positive relevant implication, can be constructed from ordered geometries. This extends earlier results building such model structures from projective spaces. A final section shows how such models can be extended to models for the full system $\mathbf{R}$.

1 Introduction

This paper is a sequel to an earlier paper by the author [6] showing how to construct models for the logic **KR** from projective spaces. In that article, the logical relation $Rabc$ is an extension of the relation $Cabc$, where a, b, c are points in a projective space and $Cabc$ is read as "a, b, c are distinct and collinear."

In the case of the projective construction, the relation $Rabc$ is totally symmetric. This is of course not the case in general in $\mathbf{R}$ models, where the first two points a and b can be permuted, but not the third. For the third point, we have only the implication $Rabc \Rightarrow Rac^*b^*$.

On the other hand, if we read $Rabc$ as "c is between a and b", where we are operating in an ordered geometry, then we can permute a and b, but not b and c, just as in the case of $\mathbf{R}$. Hence, these models are more natural than the ones constructed from projective spaces, from the point of view of relevance logic. In the remainder of the paper, we carry out the plan outlined here, by constructing models for $\mathbf{R}^+$ from ordered geometries. This confirms the earlier intuitions of J.M. Dunn, who dubbed one of the crucial postulates of $\mathbf{R}$ model structures the "Pasch Law."

2 Ordered Geometry

The system of ordered geometry, originally due to Moritz Pasch [3] and simplified by Veblen [9], is formulated as a system involving a non-empty universe of *points* and a ternary relation $Babc$ between the points, to be read as "b is between a and c." Here we follow the exposition of Coxeter [2, Chapter 12].

Before stating the axioms, we introduce some useful definitions. For a, b distinct points, the *segment* (ab) is the set of points p for which $Bapb$. The *interval* $[ab]$ is the set $(ab) \cup \{a\} \cup \{b\}$. The *ray* a/b is the set of points p for which $Bpab$. The *line* L_{ab} is $a/b \cup [ab] \cup b/a$. A point c is *on* the line L_{ab} if

$c \in L_{ab}$. Points lying on the same line are said to be *collinear*. Three non-collinear points determine a *triangle abc*, consisting of these three points, called *vertices*, together with the three segments (ab), (ac) and (bc), called *sides*. If $a_1, \ldots, a_k$ are points, then we write $\#a_1 \ldots a_k$ to denote that all of the points are pairwise distinct.

There are seven axioms, adapted from §12.2 of Coxeter's text [2]. We omit Coxeter's remaining three axioms, two dealing with the third dimension, the third stating a continuity postulate.

Axiom 1: $\exists ab(a \neq b)$;

Axiom 2: $a \neq b \Rightarrow \exists c(Babc)$;

Axiom 3: $Babc \Rightarrow a \neq c$;

Axiom 4: $Babc \Rightarrow (Bcba \wedge \neg Bbca)$;

Axiom 5: $(a \neq b \wedge c \neq d \wedge c, d \in L_{ab}) \Rightarrow a \in L_{cd}$;

Axiom 6: $a \neq b \Rightarrow \exists c(c \notin L_{ab})$;

Axiom 7: $(acd \text{ is a triangle } \wedge Bacb \wedge Bced) \Rightarrow \exists f(f \in L_{be} \wedge Bafd)$.

We now list some theorems of this system; for proofs, the reader can consult §12.2 of Coxeter's monograph [2], or the chapter [9] by Oswald Veblen.

Theorem 1: $Babc \Rightarrow \neg Bcab$;

Theorem 2: $Babc \Rightarrow \#abc$;

Theorem 3: $a, b \notin (ab)$;

Theorem 4: $(c \neq d \wedge c, d \in L_{ab}) \Rightarrow L_{ab} = L_{cd}$;

Theorem 5: $(\#abc \wedge \exists de(a, b, c \in L_{de}) \Rightarrow (Babc \vee Bbca \vee Bcab)$;

Theorem 6: $a \neq b \Rightarrow \exists c(Bacb)$;

Theorem 7: $(acd \text{ is a triangle } \wedge Bacb \wedge Bced) \Rightarrow \exists f(Bafd \wedge Bbef)$;

Theorem 8: $(Babc \wedge Bbcd) \Rightarrow Babd$;

Theorem 9: $(Babc \wedge Babd \wedge c \neq d) \Rightarrow [(Bbcd \vee Bbdc) \wedge (Bacd \vee Badc)]$;

Theorem 10: $(Babd \wedge Bacd \wedge b \neq c) \Rightarrow (Babc \vee Bacb)$;

Theorem 11: $(Babc \wedge Bacd) \Rightarrow (Bbcd \wedge Babd)$.

3 R^+-frames and geometry

3.1 Model theory of R^+

The logic $\mathbf{R}^+$ is the positive fragment of the relevant logic $\mathbf{R}$, that is to say, it is the family of all theorems of $\mathbf{R}$ that do not involve negation. In this subsection, we give the basic model theory for this system.

An $\mathbf{R}^+$-frame (or $\mathbf{R}^+$ model structure) is a triple $\langle 0, K, R\rangle$, where K is a set, $0 \in K$, and R is a ternary relation on K, satisfying the postulates:

P 1: $R0aa$,

P 2: $Raaa$,

P 3: $(Rabc \wedge Rcde) \Rightarrow \exists f(Radf \wedge Rfbe)$,

P 4: $(R0da \wedge Rabc) \Rightarrow Rdbc$,

for $a, b, c, d \in K$. J.M. Dunn dubbed **P 3** *Pasch's Law* because of its similarity in form to the famous postulate introduced by Pasch into geometry (reading $Rabc$, 'c is between a and b'). Axiom 7 and Theorem 7 above are an expression of Pasch's Law. In the following subsection, we shall show that the similarity observed by Dunn can be strengthened to an identity in models constructed from ordered geometries.

If we define $a \leq b$ as $R0ab$, then it is not hard to show that the relation $\leq$ is reflexive and transitive – in fact, we can assume in addition that it is a partial ordering, though this is not necessary for soundness. A subset S of K is *increasing* if it satisfies the condition: $(a \in S \wedge a \leq b) \Rightarrow b \in S$.

A *valuation* in an $\mathbf{R}^+$-frame assigns an increasing subset $\Phi(P) \subseteq K$ to each propositional variable P. Given a valuation in an $\mathbf{R}^+$-frame, the forcing relation $\models$ for elements of K and formulas of $\mathbf{R}^+$ is defined by:

1. $a \models P \Leftrightarrow a \in \Phi(P)$,
2. $a \models A \wedge B \Leftrightarrow a \models A$ and $a \models B$,
3. $a \models A \vee B \Leftrightarrow a \models A$ or $a \models B$,
4. $a \models A \rightarrow B \Leftrightarrow \forall bc((b \models A \wedge Rabc) \Rightarrow c \models B)$.

A formula A is *valid* in an $\mathbf{R}^+$-frame if $0 \models A$ for all valuations in the frame. Routley and Meyer [4, §10] gave a completeness proof for $\mathbf{R}^+$ relative to this semantics, proved by a canonical model construction.

Theorem 3.1 *A formula is a theorem of* $\mathbf{R}^+$ *if and only if it is valid in all* $\mathbf{R}^+$*-frames.*

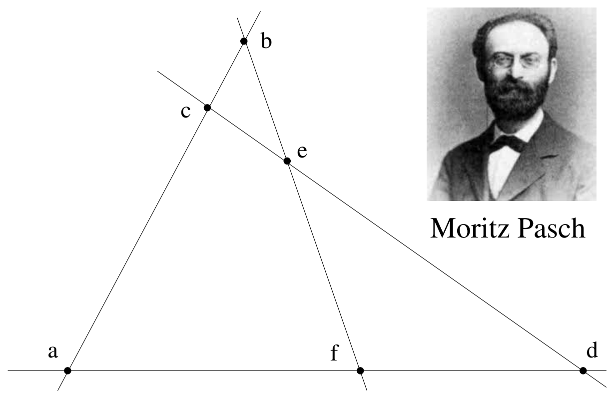

Figure 1: The Pasch axiom

3.2 Constructing R^+-frames from ordered geometries

This subsection contains the main construction, showing how, starting from an ordered geometry, we can define an $\mathbf{R}^+$-frame.

Definition 3.2 *Let $\mathcal{G} = \langle S, B \rangle$ be an ordered geometry, where S is the set of points, and B the betweenness relation on S. The structure $\mathcal{F}(\mathcal{G}) = \langle S, R \rangle$ is defined by $R = \{(a, b, c) : Bacb\} \cup \{(a, a, a) : a \in S\}$.*

Lemma 3.3 *If $\mathcal{G}$ is an ordered geometry, then $\mathcal{F}(\mathcal{G})$ satisfies postulates* **P 2** *and* **P 3** *in the definition of an $\mathbf{R}^+$-frame.*

Proof. The postulate **P 2** holds by the definition of $\mathcal{F}(\mathcal{G})$. It remains to prove **P 3**, or *Pasch's Law*. Accordingly, let us assume that $Rabc$ and $Rcde$; we aim to show that there is an $f \in S$ so that $Radf$ and $Rfbe$. Various cases arise. In the proofs below, uses of Axiom 4 are mostly tacit.

Case 1.1: The points a, c, d form a triangle, $Bacb$ and $Bced$. By Theorem 7, there is a point f so that $Bafd$ and $Bbef$, so that $Radf$ and $Rfbe$. Figure 1 illustrates this case.

Case 1.2: The points a, c, d are collinear, $Bacb$ and $Bced$. By Theorem 2, $\#abc$ and $\#cde$; hence, by Theorem 4, the points a, b, c, d, e are all collinear. For this case, we shall assume that $\#abcde$; we deal with the case of coincidences below. Various cases now arise, depending on the positions of d and e relative to a, b and c. Since $d \in L_{ab}$, it follows by Theorem 5 that either $Bbad$, $Badb$ or $Babd$.

Case 1.2.1: $Bbad$. Since $Bbca$ and $Bbad$, by Theorem 11, we have $Bcad$. Because $a \neq e$ and $Bced$, it follows by Theorem 10 that either $Bcae$ or $Baec$.

In the first case, by Theorem 6, there is a point f so that $Bdfe$; from $Bcae$ and $Bced$, we infer by Theorem 11 that $Baed$. Hence, from $Bdfe$ and $Bdea$, we infer $Bdfa$, that is, $Radf$. By Theorem 11 again, from $Bdfe$ and $Bdea$, we have $Bfea$. Since $Bdea$ and $Bdac$, by Theorem 11, we have $Beac$; since $Bacb$, by Theorem 8, we have $Beab$. From $Bfea$ and $Beab$, by Theorem 8 again, we deduce $Bfeb$, that is to say, $Rfbe$, concluding the first case.

In the second case, we have $Baec$. By Theorem 6, there is an f so that $Bafd$, so that $Radf$. Since $Baec$ and $Bacb$, by Axiom 11, we have $Baeb$. Again by Axiom 11, from $Bdfa$ and $Bdab$ we infer $Bfab$. By Axiom 4, we have $Bbea$ and $Bbaf$, so by Axiom 11, $Bbef$, that is, $Rfbe$, completing the proof of the second case.

Case 1.2.2: $Badb$. Since $Bacb$, by Theorem 10, either $Bacd$ or $Badc$.

In the first case, by Theorem 6, there is a point f so that $Bcfe$. Since $Bced$, by Theorem 11, $Bcfd$; we have $Bacd$, so by Theorem 11 again, $Bafd$, that is to say, $Radf$. We have $Bafd$ and $Badb$, so by Theorem 11, $Bfdb$. From $Bcfe$ and $Bced$, we deduce $Bfed$ by Theorem 11; from $Bfdb$ and $Bfed$, we obtain by Theorem 11 again $Bfeb$, that is, $Rfbe$.

In the second case, by Theorem 6, there is a point f so that $Bafd$, so that $Radf$. Repeatedly applying Axiom 4 and Theorem 11, from $Bafd$ and $Badc$, we deduce $Bfdc$; from $Bced$ and $Bcdf$ we deduce $Bfec$. From $Bafd$ and $Badc$ we have $Bafc$; hence using $Bacb$ we infer $Bfcb$. Finally, from $Bfec$ and $Bfcb$, we obtain $Bfeb$, that is, $Rfbe$.

Case 1.2.3: $Babd$. Applying Theorem 11, from $Bacb$ and $Babd$ we infer $Bcbd$. Since $Bced$, by Theorem 10, we have $Bcbe$ or $Bceb$.

In the first case, we have $Bcbe$. By Theorem 6, there is a point f so that $Befd$. As in the cases above, we apply Axiom 4 and Theorem 11. From $Bcbe$ and $Bced$, we deduce $Bcbd$; then from $Bacb$ and $Bcbd$ by Theorem 8, we infer $Bdca$. We deduce $Bdfc$ from $Bdfe$ and $Bdec$; combining this with $Bdca$, we conclude $Bdfa$, or $Radf$. From $Bcbe$ and $Bced$, we get $Bdeb$, hence, using $Bdfe$, we obtain $Bfeb$, that is, $Rfbe$.

In the second case, we have $Bceb$. By Theorem 6, there is a point f so that $Bcfe$. From $Bcfe$ and $Bceb$, we infer $Bfeb$, that is, $Rfbe$. From $Bcfe$ and $Bced$, we deduce $Bcfd$, and from $Bcfe$ and $Bceb$, we deduce $Bcfb$. $Bbfc$ and $Bbca$ imply $Bfca$; from $Bcfd$ and $Bfca$, we conclude $Bdfa$ by Theorem 8, that is to say, $Radf$. This concludes Case 1.2.

Case 1.3: In this case, we assume $Bacb$ and $Bced$, but do not assume $\#abcde$, so that coincidences can occur.

We start by analysing which coincidences are possible. In view of the conditions $\#abc$ and $\#cde$, the possible identifications are: $a = d$, $a = e$, $b = d$ and $b = e$. The joint coincidences $a = d \wedge a = e$, $a = d \wedge b = d$, $a = e \wedge b = e$, and $b = d \wedge b = e$ are all ruled out either by $\#abc$ or $\#cde$. This leaves two possible joint coincidences.

First, if $a = d$ and $b = e$, then $Bacb$ is equivalent to $Bdce$; by Axiom 4, this contradicts $Bced$. Second, if $a = e$ and $b = d$, then $Bacb$ implies $Becd$, contradicting $Bced$, again by Axiom 4. Thus we have ruled out the remaining possible joint coincidences, showing that only single coincidences can occur.

Case 1.3.1: $a = d$ and $\#abce$. Set $f = a = d$, so that $Radf$. We have $Bfec$ and $Bfcb$, so by Theorem 11, $Bfeb$, that is, $Rfbe$.

Case 1.3.2: $a = e$ and $\#abcd$. By Theorem 6, there is a point f so that $Bdfe$, so that $Radf$. From $Bdfe$ and $Bdec$, we deduce $Bfec$. Then $Bfec$ and $Becb$ imply that $Bfeb$, that is to say, $Rfbe$.

Case 1.3.3: $b = d$ and $\#abce$. Set $f = c$. Since $Bdec$, we have $Bbef$, or $Rfbe$; since $Bacb$, we have $Bafd$, or $Radf$.

Case 1.3.4: $b = e$ and $\#abcd$. Set $f = b = e$. Then $Rfbe$ by definition. Since $Bacb$ and $Bcbd$, we have $Bacd$. From $Bdfc$ and $Bdca$, we infer $Bdfa$, that is to say, $Radf$, concluding Case 1.3.

Case 1.4: In this last case, we have $a = b = c$ or $c = d = e$.

In the first case, $Raaa$ and $Baed$. By Theorem 6, there is a point f so that $Bdfe$. From $Bdfe$ and $Bdea$, we infer $Bdfa$, or $Radf$. Since $Bdfe$ and $Bdea$, we have $Bfea$, or $Rfbe$.

In the second case, $Bacb$ and $Rccc$. By Theorem 6, there is a point so that $Bafc$, so that $Radf$. From $Bafc$ and $Bacb$ we infer $Bfcb$, or $Rfbe$, concluding Case 1.4 and the proof of Pasch's Law. □

The preceding lemma supplies us with the basic material for our construction; to complete it, we need to show how to add a a zero point to the frame.

Definition 3.4 *Let $\mathcal{F} = \langle S, R\rangle$ be a structure consisting of a non-empty set S with a ternary relation on S. Let 0 be an element not in S. Then the structure $\mathcal{F} + 0 = \langle K, R^0\rangle$ is defined as follows:*

1. $K = S \cup \{0\}$;
2. $R^0 = R \cup \{(0, a, a), (a, 0, a) : a \in K\}$.

The relation R^0 can be decomposed into four mutually exclusive parts, namely:

1. $\{(a, b, c) : a, b, c \in S\}$;
2. $\{(0, a, a) : a \in S\}$;
3. $\{(a, 0, a) : a \in S\}$;
4. $\{(0, 0, 0)\}$.

This decomposition is useful in the case distinctions of the following theorem.

Theorem 3.5
1. *If $\mathcal{F} = \langle S, R\rangle$ satisfies postulates* **P 2** *and* **P 3** *in the definition of an $\mathbf{R}^+$-frame, then $\mathcal{F} + 0$ is an $\mathbf{R}^+$-frame.*
2. *If $\mathcal{G} = \langle S, B\rangle$ is an ordered geometry, then $\mathcal{F}(\mathcal{G}) + 0$ is an $\mathbf{R}^+$-frame.*

Proof. Assume that $\mathcal{F} = \langle S, R\rangle$ satisfies postulates **P 2** and **P 3** in the definition of an $\mathbf{R}^+$-frame. By construction, $\mathcal{F}(\mathcal{G}) + 0$ satisfies the postulates **P 1** and **P 2**. If $R^0 0ad$, then $a = d$, so **P 4** holds as well. It remains to show that $\mathcal{F}(\mathcal{G}) + 0$ satisfies **P 3** as well.

Assume that R^0abc and R^0cde, where $a, b, c \in K$. We aim to show that there is an $f \in K$ so that R^0adf and R^0fbe.

(1) Assume that $c = 0$. In that case, $a = b = c = 0$, and R^0cde has the form $R^0 0dd$, for $d \in K$. Set $f := d$, so that $R^0 0dd$ and $R^0 d0d$, that is, R^0adf and R^0fbe.

(2) Assume that $a, b, c \in S$. If $c, d, e \in S$, then $\exists f(R^0adf \wedge R^0fbe)$ holds because $\mathcal{F}$ satisfies **P 3**. On the other hand, if $0 \in \{c, d, e\}$, then R^0cde must have the form R^0c0c, so $d = 0$ and $e = c$. Set $f := a$, so that R^0a0a and R^0abc, that is, $R^0adf \wedge R^0fbe$.

(3) Assume that $c, d, e \in S$, but $0 \in \{a, b, c\}$. If $a = 0$, then $b = c$. Set $f := d$, so that $R^0 0dd \wedge R^0dce$, that is to say, $R^0adf \wedge R^0fbe$. If $b = 0$, then $a = c$. Set $f := e$, so that $R^0cde \wedge R^0e0e$, that is, $R^0adf \wedge R^0fbe$.

(4) Assume that $c \in S$, but $0 \in \{a, b, c\}$ and $0 \in \{c, d, e\}$. If $a = 0$ then $b = c$, $d = 0$ and $e = c$. Set $f := 0$, so that $R^0 000$ and $R^0 0cc$, so that $R^0adf \wedge R^0fbe$. If $b = 0$, then $a = c = e$ and $d = 0$. Set $f := c$, so that $R^0c0c \wedge R^0c0c$, that is to say, $R^0adf \wedge R^0fbe$, completing the proof that $\mathcal{F}(\mathcal{G}) + 0$ satisfies **P 3**. $\square$

The $\mathbf{R}^+$-frame $\mathcal{F}(\mathcal{G}) + 0$ constructed in Theorem 3.5 has a rich geometrical structure, since the geometry $\mathcal{G}$ can be recovered from it by a restriction to the non-zero points. It is the geometrical structure of certain $\mathbf{R}^+$-frames that accounts for the undecidability of the principal relevant logics [7].

4 R-frames and geometry

4.1 Model theory of R

An **R**-frame (or **R** model structure) is a quadruple $\langle 0, K, R, {}^* \rangle$, where K is a set, $0 \in K$, R is a ternary relation on K, and * is a function defined on K satisfying the postulates:

P 1: $R0aa$,

P 2: $Raaa$,

P 3: $(Rabc \wedge Rcde) \Rightarrow \exists f(Radf \wedge Rfbe)$,

P 4: $(R0da \wedge Rabc) \Rightarrow Rdbc$,

P 5: $Rabc \Rightarrow Rac^*b^*$,

P 6: $a^{**} = a$.

for $a, b, c, d \in K$.

As in §3.1, we define $a \leq b$ as $R0ab$. A subset S of K is *increasing* if it satisfies the condition: $(a \in S \wedge a \leq b) \Rightarrow b \in S$. A *valuation* in an **R**-frame assigns an increasing subset $\Phi(P) \subseteq K$ to each propositional variable P. Given a valuation in an **R**-frame, the forcing relation $\models$ for elements of K and formulas of **R** is defined by:

1. $a \models P \Leftrightarrow a \in \Phi(P)$,

2. $a \models A \wedge B \Leftrightarrow a \models A$ and $a \models B$,

3. $a \models A \vee B \Leftrightarrow a \models A$ or $a \models B$,

4. $a \models A \rightarrow B \Leftrightarrow \forall bc((b \models A \wedge Rabc) \Rightarrow c \models B)$,

5. $a \models \neg A \Leftrightarrow a^* \not\models A$.

A formula A is *valid* in an **R**-frame if $0 \models A$ for all valuations in the frame. Routley and Meyer [4, §7] provide a completeness proof for **R** relative to this semantics, proved by a canonical model construction.

Theorem 4.1 *A formula is a theorem of* **R** *if and only if it is valid in all* **R**-*frames.*

4.2 Model theory of KR

The logic **KR** results by adding to **R** the axiom *ex falso quodlibet*, that is to say, $(A \wedge \neg A) \rightarrow B$. The model theory for **KR** is elegantly simple. The usual ternary relational semantics for **R** includes an operation $*$ designed to deal with the truth condition for negation

$$x \models \neg A \Leftrightarrow x^* \not\models A.$$

The effect of adding *ex falso quodlibet* to **R** is to identify x and x^*; this in turn has a notable effect on the ternary accessibility relation. The postulates for an **R** model structure include the following implication:

$$Rxyz \Rightarrow (Ryxz \,\&\, Rxz^*y^*).$$

The result of the identification of x and x^* is that the ternary relation in a **KR** model structure (KRms) is *totally symmetric*. A **KR**-frame $\mathcal{K} = \langle S, R, 0\rangle$ is a 3-place relation R on a set containing a distinguished element 0, and satisfying the postulates:

1. $R0ab \Leftrightarrow a = b$;

2. $Raaa$;

3. $Rabc \Rightarrow (Rbac \,\&\, Racb)$ (total symmetry);

4. $(Rabc \,\&\, Rcde) \Rightarrow \exists f(Radf \,\&\, Rfbe)$ (Pasch's Law).

Although **KR** contains the irrelevant axiom $(A \wedge \neg A) \rightarrow B$, it is more closely related to the relevant logic **R** than might be thought initially. Define an **R**-frame $\mathcal{F}$ to be *order-trivial* if it satisfies the first postulate for a **KR**-frame, that is, $\forall ab(R0ab \Leftrightarrow a = b)$.

We define a *homomorphism* to be a morphism in the category of ternary relational structures with 0, that is to say, a function φ from $\mathcal{M}_1 = \langle S_1, R_1, 0\rangle$ to $\mathcal{M}_2 = \langle S_2, R_2, 0\rangle$ satisfying the conditions

$$\varphi(0) = 0, \text{ and } \forall x, y, z \in S_1[R_1xyz \Rightarrow R_2\varphi(x)\varphi(y)\varphi(z)].$$

The notion of homomorphism given here is weaker than that of "frame morphism" (corresponding to the concept of "p-morphism" in modal logic) as defined (for example) in [8].

Definition 4.2 *Let $\mathcal{F} = \langle 0, K, R, ^* \rangle$ be an order-trivial* **R**-*frame. The relational structure $[\mathcal{F}] = \langle [K], [R], \{0\} \rangle$ is defined as follows:*

1. $[K] = \{\{a, a^*\} : a \in K\}$;
2. *For $\alpha, \beta, \gamma \in [K]$, $[R]\alpha\beta\gamma$ holds if and only if for some $a \in \alpha, b \in \beta, c \in \gamma$, $Rabc$.*

Theorem 4.3 *Let $\mathcal{F} = \langle 0, K, R, ^* \rangle$ be an order-trivial* **R**-*frame. Then $[\mathcal{F}]$ is a* **KR**-*frame that is a homomorphic image of $\mathcal{F} = \langle 0, K, R \rangle$ under the mapping $r(x) = \{x, x^*\}$.*

Proof. The first, second and fourth postulates for a **KR**-frame follows immediately from the corresponding postulates for the **R**-frame $\mathcal{F}$ and its order-triviality. Hence, we only need to check the third.

Assume that $[R]\alpha\beta\gamma$ holds in $[\mathcal{F}]$. Thus there are $a, b, c \in K$ so that $a \in \alpha$, $b \in \beta$, $c \in \gamma$ and $Rabc$. Then $[R]\beta\alpha\gamma$ holds since $Rbac$ holds in $\mathcal{F}$. Since $Rabc$, we have Rac^*b^*. Since $a \in \alpha$, $c^* \in \gamma$ and $b^* \in \beta$, we have $[R]\alpha\gamma\beta$, showing that the third postulate holds in $[\mathcal{F}]$.

The fact that $[\mathcal{F}]$ is a homomorphic image of $\mathcal{F} = \langle 0, K, R \rangle$ under the mapping $r(x) = \{x, x^*\}$ follows by the definition of $[\mathcal{F}]$. □

4.3 Constructing R-frames from KR-frames

Theorem 4.3 shows that there is a canonical way to associate a **KR**-frame with any order-trivial **R**-frame. The linear subspaces of a **KR**-frame form a modular lattice, so this construction also associates a modular lattice with any order-trivial **R**-frame. Conversely [5], any modular lattice gives rise to a **KR**-frame. Thus, we can classify order-trivial **R**-frames by their associated modular lattices – which we can consider as generalized geometries. These order-trivial **R**-frames are of course not uniquely determined by the associated modular lattices; we discuss this problem in more detail in what follows.

If $\mathcal{F}$ is an order-trivial **R**-frame, then the **KR**-frame $[\mathcal{F}]$ is a homomorphic image of $\mathcal{F}$ under the map $\varphi(x) = \{x, x^*\}$ – taking the image of the $*$ function in $\mathcal{F}$ to be the identity on $[\mathcal{F}]$.

Following up the thoughts on classification above, we might ask : "What can we say about the **R**-frames that are inverse images of a given **KR**-frame?" This seems to be a rather difficult problem in general, but we can say a few things about it.

Starting from a **KR**-frame $\mathcal{F} = \langle S, R, 0 \rangle$, let $S^\sharp$ be a set of points disjoint from S; we assume a bijection $x \mapsto x^\sharp$ between S and $S^\sharp$. Let P be a set of points in S where $0 \notin P$; these are the points in S that we choose to *split*. That is to say, if a point x is in P, then we replace x by $r(x) = \{x, x^\sharp\}$, while if

$x \notin P$, then we replace x by $r(x) = \{x\}$. Let $r(S)$ be the set resulting from these replacements, and define $S(P)$ to be $\bigcup r(S)$.

Definition 4.4 *The structure $\mathcal{F}^r$ is defined as $\langle r(S), R^r, \{0\}\rangle$, where the ternary relation R^r on $r(S)$ is defined as follows:*

$$R^r\alpha\beta\gamma \Leftrightarrow \exists abc \in S\ [r(a) = \alpha \wedge r(b) = \beta \wedge r(c) = \gamma \wedge Rabc].$$

Lemma 4.5 *If $\mathcal{F}$ is a* **KR**-*frame, then the map r is an isomorphism between $\mathcal{F}$ and $\mathcal{F}^r$, so that $\mathcal{F}^r$ is also a* **KR**-*frame.*

Proof. The map r is a bijection between S and $r(S)$, and $r(0) = \{0\}$. If $Rabc$, then by definition, $R^r r(a)r(b)r(c)$. Conversely, if $R^r\alpha\beta\gamma$ then by definition there are $a, b, c \in S$ so that $Rabc$, where $r(a) = \alpha$, $r(b) = \beta$, $r(c) = \gamma$, so that $Rr^{-1}(\alpha)r^{-1}(\beta)r^{-1}(\gamma)$, completing the proof that r is an isomorphism. $\Box$

On the set $S(P)$, define a $*$-structure by setting $x^* = x$ if $x \notin P$, and for $x \in P$, $x^* = x^\sharp$ and $(x^\sharp)^* = x$. The result $\langle S(P), ^*\rangle$ of this construction we call *the $*$-structure determined by S and P.*

The $*$-structure determined by S and P is sufficient to interpret the "extensional" connectives $\wedge, \vee, \neg$, but to deal with implication, we need to add a ternary relation to the $*$-structure $\langle S(P), ^*\rangle$. Can we always make this extension so that the result is an **R**-frame? The answer is "yes!" as follows from the simple construction given below.

Definition 4.6 *Let $\mathcal{F} = \langle S, R, 0\rangle$ be a* **KR**-*frame, $P \subseteq S \setminus \{0\}$ and $\langle S(P), ^*\rangle$ the $*$-structure determined by S and P. Let δ be a choice function defined on $r(S)$ so that for $\alpha \in r(S)$, $\delta(\alpha) \in \alpha$. Define a ternary relation R^δ_P on $S(P)$ as follows:*

$$R^\delta_P = \{(\delta(\alpha), \delta(\beta), \delta(\gamma)) : R^r\alpha\beta\gamma \wedge \alpha, \beta, \gamma \in r(S)\},$$

and define:

$$R_P = \bigcup\{R^\delta_P : \delta \textit{ a choice function } r(S) \mapsto S(P)\}$$

The structure $\mathcal{F}(P)$ is defined as $\langle 0, S(P), R_P, ^*\rangle$.

Lemma 4.7 *Let $\mathcal{F} = \langle S, R, 0\rangle$ be a* **KR**-*frame, $P \subseteq S \setminus \{0\}$ and $\mathcal{F}(P)$ the structure $\langle 0, S(P), R_P, ^*\rangle$ defined by $\mathcal{F}$ and P. For $\alpha, \beta, \gamma \in r(S)$ and $a, b, c \in S(P)$, if $R^r\alpha\beta\gamma$, $a \in \alpha$, $b \in \beta$ and $c \in \gamma$, then R_Pabc.*

Proof. Let $\alpha, \beta, \gamma \in r(S)$ and $a, b, c \in S(P)$, $R^r\alpha\beta\gamma$, $a \in \alpha$, $b \in \beta$ and $c \in \gamma$. Let δ be a choice function on $S(P)$ so that $\delta(\alpha) = a$, $\delta(\beta) = b$ and $\delta(\gamma) = c$. Then by Definition 4.6, $R_P\delta(\alpha)\delta(\beta)\delta(\gamma)$, that is to say, R_Pabc. $\Box$

Theorem 4.8 *Let $\mathcal{F} = \langle S, R, 0\rangle$ be a* **KR**-*frame, $P \subseteq S \setminus \{0\}$ and $\langle S(P), ^*\rangle$ the $*$-structure determined by S and P.*

1. *$\mathcal{F}(P)$ is an order-trivial totally symmetric* **R**-*frame such that*

(a) $\mathcal{F}(P)$ is an expansion of the $*$-structure $\langle S(P), ^*\rangle$;

(b) $\mathcal{F}$ is a homomorphic image of $\mathcal{F}(P)$.

2. $\mathcal{F}(P)$ is the largest order-trivial **R***-frame satisfying these two conditions.*

Proof. Part 1(a). Let a be in $S(P)$, δ a choice function on $r(S)$ where $\delta(r(a)) = a$. Since $R^r 0r(a)r(a)$, it follows that $R_P 0aa$. Conversely, assume that $R_P 0ab$, for $a, b \in S(P)$. By Definition 4.6, there is a choice function δ on $r(S)$ and α, β in $r(S)$ so that $a = \delta(\alpha)$, $b = \delta(\beta)$ and $R\{0\}\alpha\beta$. By Lemma 4.5, $\mathcal{F}^r$ is order-trivial, so that $\alpha = \beta$; it follows that $a = b$, showing that $\mathcal{F}(P)$ is also order-trivial.

If $a \in S(P)$, then by Lemma 4.5, $R^r(a)r(a)r(a)$; setting $\delta(r(a)) = a$, we have $Raaa$.

For Pasch's Law, assume for $a, b, c, d, e \in S(P)$, that $R_P abc$ and $R_P cde$, so that for $\alpha, \beta, \gamma, \delta, \epsilon \in r(S)$, and κ a choice function on $S(P)$, we have $R^r\alpha\beta\gamma$, $a = \kappa(\alpha)$, $b = \kappa(\beta)$ and $c = \kappa(\gamma)$, and in addition, $R^r\gamma\delta\epsilon$, $d = \kappa(\gamma)$, $e = \kappa(\delta)$ and $c = \kappa(\epsilon)$. By Lemma 4.5, there is a $\zeta \in r(S)$, so that $R^r\alpha\delta\zeta$ and $R^r\zeta\beta\epsilon$. Now let λ be a choice function on $r(S)$ that agrees with κ on $\{\alpha, \beta, \delta, \epsilon\}$, and set $f = \lambda(\zeta)$. Then we have by Lemma 4.7 that $R_P adf$ and $R_P fbe$, completing the proof of Pasch's Law.

The remaining postulates for an **R**-frame are easy to verify. For Postulate **P 4**, if $R_P 0da$ and $R_P abc$, then $d = a$, by order-triviality, showing that $R_P dbc$. For Postulate **P 5**, if $R_P abc$, then $R^r\alpha\beta\gamma$, where $a \in \alpha$, $b \in \beta$, $c \in \gamma$, so that $R_P r(a)r(c)r(b)$, by Lemma 4.5. Since $b^* \in \beta$ and $c^* \in \gamma$, by Lemma 4.7, $R_P ac^*b^*$. Postulate **P 6** follows by construction.

Finally, the fact that R_P is totally symmetric follows from the total symmetry of $\mathcal{F}^r$ and Lemma 4.5.

Part1(b). The relational structure $[\mathcal{F}(P)]$ is a homomorphic image of $\mathcal{F}(P)$, by Theorem 4.3. By Definitions 4.2 and 4.6,

$$[\mathcal{F}(P)] = \langle [S(P)], [R_P], \{0\}\rangle.$$

With $r(x) = \{x, x^*\}$, $[S(P)] = r(S)$ and $[R_P] = R^r$. Hence, $[\mathcal{F}(P)] = \mathcal{F}^r$. Lemma 4.5 shows that $\mathcal{F}$ is isomorphic to $\mathcal{F}^r$, and hence that $\mathcal{F}$ is a homomorphic image of $\mathcal{F}(P)$.

Part 2. Let $\mathcal{G} = \langle 0, S(P), R, ^*\rangle$ be an order-trivial **R**-frame satisfying the conditions (a) and (b) in the statement of the theorem, and assume $Rabc$. Then $[R]\varphi(a)\varphi(b)\varphi(c)$ holds in $[\mathcal{G}]$. Consequently, $[R]^r r(\varphi(a))r(\varphi(b))r(\varphi(c))$ holds in $[\mathcal{G}]^r$. Let δ be a choice function on $r(S)$ so that $\delta(r(\varphi(a))) = a$, $\delta(r(\varphi(b))) = b$, $\delta(r(\varphi(c))) = c$. Then by Definition 4.6, $R_P abc$. □

Theorem 4.8 provides (in a weak sense) a complete description of the **R**-frames satisfying the two conditions of Theorem 4.8. Start with the **R**-frame $\mathcal{F}(P) = \langle 0, S(P), R_P, ^*\rangle$ and a subset T of R_P containing $\{(0, a, a) : a \in S(P)\}$ and $\{(a, a, a) : a \in S(P)\}$ that is closed under the map $(a, b, c) \longmapsto (a, c^*, b^*)$. Then the structure $\langle S(P), T, ^*\rangle$ satisfies all of the postulates for an **R**-frame, with the possible exception of **P 3** (Pasch's Law). It is this last postulate that makes the problem of describing such extensions difficult. We have to "thin out" the relation R_P while preserving the witnesses for Pasch's Law, and it is not clear how to give a general analysis of this process.

The **R**-frame $\mathcal{F}(P)$ constructed in Theorem 4.8, provided that $P \neq \emptyset$, does not validate *ex falso quodlibet* $(A \wedge \neg A) \to B$. On the other hand, since the relation R_P is totally symmetric, the condition $R_P aa0$ holds for any $a \in S(P)$, so that the rather odd-looking $(A \circ A) \vee (A \to B)$ is valid in the constructed **R**-frame $\mathcal{F}(P)$. This is a close cousin of the irrelevant classical tautology $A \vee (A \to B)$. For further results involving this principle as well as closely related formulas, the reader can consult §54 of Volume II of *Entailment* [1]. In particular, that section of the book contains a completeness proof for $\mathbf{KR}_{\to \& \circ \mathbf{t}}$, that is, the fragment of **KR** based on the connectives $\to, \&, \circ$ and the constant $\mathbf{t}$.

5 Concluding Remarks

The main construction of this paper is in a sense more natural than the construction based on projective spaces [6], since the relation of betweenness in an ordered geometry is symmetric in only two of its three places, a property shared by the ternary relation in models of **R**. On the other hand, from the purely mathematical point of view, the construction is much less elegant, since it involves a plethora of case distinctions, as we saw in §3.2.

Another drawback is the fact that the method does not allow us to construct finite models, in contrast to the projective case, where we can build finite models for the logic **KR** from finite projective spaces. In the proof of Lemma 3.3, the fact that the betweenness ordering is dense (Theorem 6 of Coxeter's axiom system) is used repeatedly. This means that the construction of the lemma can only produce infinite models.

To conclude, here is a natural open problem.

Problem 5.1 *Consider the* $\mathbf{R}^+$*-frame constructed from the betweenness relation defined on the real ordered plane* $\mathbb{R}^2$*. Can this logic be axiomatized?*

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