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The Moral Law and The Good in Temporal Modal Deontic Logic with Propositional Quantifiers

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Abstract

The Moral Law is fulfilled (in a possible world ω at a time τ) iff (if and only if) everything that ought to be the case is the case (in ω at τ), and The Good (or The Highest Possible Good) is realised in a possible world ω at a time τ iff ω is deontically accessible from ω at τ. In this paper, I will introduce a set of temporal modal deontic systems with propositional quantifiers that can be used to prove some interesting theorems about The Moral Law and The Good. First, I will describe a set of systems without any propositional quantifiers. Then, I will show how these systems can be extended by a couple of propositional quantifiers. I will use a kind of $T \times W$ semantics to describe the systems semantically and semantic tableaux to describe them syntactically. Every system will include a constant $\bullet$ that stands for The Good. '$\bullet$' is read as 'The Good is realised'. All systems that contain the propositional quantifiers will also include a constant $\star$ that stands for The Moral Law. '$\star$' is read as 'The Moral Law is fulfilled'. I will prove that all systems (without the propositional quantifiers) are sound and complete with respect to their semantics and that all systems (including the extended systems) are sound with respect to their semantics. It is left as an open question whether or not the extended systems are complete.

Keywords: The Moral Law, The Good, Temporal Modal Deontic Logic, Propositional Quantifiers, Semantic Tableaux.

1 Introduction

In this paper, I will introduce a set of temporal modal deontic tableaux systems. First, I will describe a set of systems without any propositional quantifiers. Then, I will show how these systems can be extended by a couple of propositional quantifiers. This technical apparatus will be used to say something about The Moral Law and The Good and the relationship between these concepts. Semantically, the systems are described by using a kind of $T \times W$ models, where truth is relativised to world-moment pairs. I will prove that all systems (without the propositional quantifiers) are sound and complete with respect to their semantics and that all systems (including the extended systems) are sound with respect to their semantics. It is left as an open question whether or not the extended systems are complete.

Every system in this paper includes four parts: a temporal part, a modal part (an alethic part), a deontic part and an axiological part.

Temporal logic deals with temporal concepts, such as *always* and *sometimes*, temporal propositions, arguments and systems. Every logic in this paper includes several temporal operators, for example, $\mathbb{A}$ (always), $\mathbb{S}$ (sometimes), $\mathbb{G}$ (always in the future), $\mathbb{F}$ (sometime in the future), $\mathbb{H}$ (always in the past) and $\mathbb{P}$ (sometime in the past). For more on this branch of logic, see, for example, [16], [27], [36], [45], [59], [66], [68] and [64].

Modal logic investigates modal concepts, such as *necessity*, *possibility* and *contingency*, and the logical relationships between propositions that include such concepts. Modal logicians also study various modal principles, arguments and systems. The modal (or alethic) part of our systems includes two types of operators for absolute and historical necessity and possibility, respectively: $\mathbf{U}$ (absolute necessity), $\mathbf{M}$ (absolute possibility), $\Box$ (historical necessity) and $\Diamond$ (historical possibility). For introductions to (alethic) modal logic, see, for example, [18], [19], [30], [38], [43], [52], [53], [57], [60], [74], [81] and [87].

Deontic logic is the logic of norms. It investigates normative concepts, such as the concept of an *obligation* or the concept of a *permission*, normative propositions, arguments and systems. Every logic in this paper includes two deontic operators $\mathbf{O}$ (ought) and $\mathbf{P}$ (permitted) that can be used to symbolise various normative propositions; the operator $\mathbf{F}$ (forbidden) can be defined in terms of $\mathbf{P}$ (see Definition 3). For some introductions to deontic logic, see, for example, [6], [41], [44], [47], [48], [61] and [84]. See also [69].

Every system in this paper also includes an axiological part. Axiology has to do with values. Accordingly, every system will include a constant $\bullet$ that stands for The Good. '$\bullet$' is read as 'The Good is realised'. The Good is realised in a possible world ω at a moment in time τ iff ω is deontically accessible from ω at τ. If ω' is deontically accessible from ω at τ iff ω' is one of the best possible worlds (all things considered) in ω at τ, then The Good (or The Highest Possible Good) is realised in ω at τ iff ω is one of the best possible worlds (all things considered) in ω at τ. If we assume the semantic condition $C - \Box\mathbf{O}$ (see Table 4), we can say that The Good (or The Highest Possible Good) is realised in ω at τ iff ω is one of the best possible worlds (all things considered) that are still historically accessible in ω at τ. Similar constants have been discussed before in 'pure' alethic deontic logic (see [49] for an overview). However, as far as I know, an axiological element of this kind has not been introduced into any temporal modal deontic system before and it has never been included in any tableau system. In earlier systems, the corresponding constant is usually taken to hold in a possible world ω iff ω is one of the optimal worlds. In our systems, The Good might be realised in a possible world ω at a particular moment in time τ even though The Good is not realised in ω at some other time τ' and The Good might be realised in a possible world ω at a particular moment in time τ even though The Good is not realised in another possible world ω' at the same time τ.

I will explore some possible relationships between these different parts of our systems, and consider some principles that include more than one type of logical expression. Some interactions of this kind have been investigated before. Logicians have, for example, introduced systems that combine modal and deontic logic, and temporal, modal and deontic logic. Some of the first attempts to combine deontic logic and alethic modal logic can be found in a series of essays by A. R. Anderson (see [1], [2], [3], [4]). Another early contribution is [37]; see also [55].

Several philosophers and logicians have constructed logical systems that include temporal, modal and deontic elements, see, for example, [70], [29], [10], [11], [12], [13], [82], [78], [79], [9] and [8]. Chellas ([29]) also includes a modal logic of action. For more ideas on how to combine deontic logic with temporal logic, see, for example, [20], [22], [23], [24], [25], [28], [30], Chapter 6, [34], [50], [51], and [7]. See also [14], [15], [21], [46] and [72].

For more on how to combine modal and temporal logic, see, for example, [31], [33], [80] and [88]. See [63] for an early attempt to combine various branches of logic.

Some systems in this paper will include a couple of propositional quantifiers: $\forall$ and $\exists$. All systems that contain the propositional quantifiers are called ‘extended’ and will include a constant $\star$ that stands for The Moral Law. ‘$\star$’ is read as ‘The Moral Law is fulfilled’. We shall say that The Moral Law is fulfilled (in a possible world ω at a time τ) iff everything that ought to be the case is the case (in ω at τ) (see Definition 5). As far as I know, there are no systems in the literature that combine propositional quantifiers with temporal, modal, deontic and axiological elements. Hence, this part of the paper is entirely new. We will be particularly interested in how this component interacts with other parts of the systems. For more information on propositional quantifiers, see, for example, [73], [60], [58], [26], [35], [56], [39] and [42].

Since many systems (including all extended systems) in this paper are new, there are good reasons related to logic to be interested in our results. Let us also mention some philosophically interesting reasons. In languages with propositional quantifiers we can express many theses that cannot be expressed in any quantifier-free normal modal systems. We can, for example, say that there is something that is optional ($\exists A \mathbb{K}A = \exists A(\mathbb{P}A \wedge \mathbb{P}\neg A)$), that some contingent 'proposition' is obligatory ($\exists A(\triangledown A \wedge \mathbb{O}A)$), that some contingent 'things' are forbidden ($\exists A(\triangledown A \wedge \mathbb{F}A)$), and that everything that ought to be the case is possible ($\forall A(\mathbb{O}A \rightarrow \Diamond A)$). In our extended systems we can explore some interesting relationships between the normative, axiological and alethic (modal) elements. We can, for example, prove that all normative operators, in principle, are definable in terms of $\bullet$ and the modal operators in some systems (see Theorem 35). We can use our systems to explore various (normative) classifications of 'states of affairs'. We can, for example, prove that everything is obligatory, optional or forbidden ($\forall A(\mathbb{O}A \vee \mathbb{K}A \vee \mathbb{F}A)$) according to every (extended) system in this paper (see Theorem 15) and that these categories are mutually exclusive in every system that includes the tableau rule $T - dD$ (see Table 18); that is, nothing is both obligatory and optional ($\neg\exists A(\mathbb{O}A \wedge \mathbb{K}A)$), nothing is both forbidden and optional ($\neg\exists A(\mathbb{F}A \wedge \mathbb{K}A)$) and nothing is both obligatory and forbidden ($\neg\exists A(\mathbb{O}A \wedge \mathbb{F}A)$) according to those systems (see Table 24). These results are intuitively very plausible.

We can use the systems in this paper to describe the purpose, aim or goal of morality in a succinct way. Why should we be moral? Why should we do the things that we ought to do? If the approach in this paper is right, we can say that the purpose, aim or goal of morality is The (Highest Possible) Good. We should do the things that we ought to do because doing the things that we ought to do is a necessary condition for The (Highest Possible) Good. If we do not do the things that we ought to do, The (Highest Possible) Good cannot be realised. And The (Highest Possible) Good ought to be realised (Theorem 29).[1] In other words, the purpose or aim or goal of morality is to create a possible world in which The (Highest Possible) Good is realised. Likewise, we can ask why The Moral Law should be fulfilled. If the approach in this paper is correct, The Moral Law is a necessary (and according to some systems sufficient) condition for The (Highest Possible) Good (Theorem 34). Hence, it is necessary that if The Moral Law is not fulfilled, The (Highest Possible) Good is not realised. This fact explains why it is reasonable to use 'must' both in an alethic and in a normative sense. We really *must* do the things morality requires, not in the sense that we cannot act differently, but in the sense that it is necessary that we do the things that we ought to do if we are going to realise the goal of morality.

So, The (Highest Possible) Good ought to be realised. However, it is not only true that The (Highest Possible) Good ought to be realised, it is also true that it ought to be the case that it is always going to be the case that The (Highest Pos-

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[1] In a strict sense, we cannot prove that The (Highest Possible) Good ought to be realised in every system. We can only prove this proposition in every system that includes $T - \mathbb{O}dT$ (see Table 18), but this rule seems reasonable to me.

sible) Good is realised (Theorem 29).[2] The (Highest Possible) Good can be realised in a possible world ω at one particular moment in time τ, even though it is not realised in ω at any moment, τ', later than τ (according to some [but not all] systems). Consequently, the goal of morality is not only to realise The (Highest Possible) Good, but to realise The (Highest Possible) Good forever after. In this sense, realising The (Highest Possible) Good is an ongoing, never ending process.[3]

The idea that the purpose, aim or goal of morality is (to realise) The (Highest Possible) Good is compatible with many different value theories (and metaethical theories). A hedonist can say that The (Highest Possible) Good is realised in a possible world ω at a moment in time τ iff ω at τ includes a maximum amount of pleasure over pain. Some perfectionists can say that The (Highest Possible) Good is realised in a possible world ω at a moment in time τ iff ω at τ includes a maximum amount of perfection. A pluralist can say that The (Highest Possible) Good is realised in a possible world ω at a moment in time τ iff ω at τ includes a maximum amount of value. An ideal observer theorist can say that The (Highest Possible) Good is realised in a possible world ω at a moment in time τ iff an ideal observer does not prefer any other possible world in ω at τ. A decision theorist can say that The (Highest Possible) Good is realised in a possible world ω at a moment in time τ iff ω at τ has a maximum amount of *expected* value, etc.[4] The last proposition shows that we can avoid a potential problem with this approach. In some cases, the probability of obtaining a desired effect might be very small. In those cases, it might seem to be unreasonable to do the things that are *necessary* to realise The (Highest Possible) Good. But we can take such factors into account

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[2] Again, in a strict sense, this is not a theorem in every system in this paper. We can only prove that it ought to be the case that it is always going to be the case that The (Highest Possible) Good is realised in every system that includes $T - \mathbb{O}\mathbb{G}dT$ (see Table 21), which also seems reasonable to me.

[3] . . . at least if there is no last moment in time. If there is a last moment in time, then in the last moment of time it is vacuously true that it is always going to be the case that The Good is realised. This also shows the 'need' for combining all the different elements of our systems. Note that $C - \mathbb{O}\mathbb{G}dT$ (Table 6) does not follow from $C - \mathbb{O}dT$ (Table 3). To be able to prove that it ought to be the case that The Good is realised and that it is always going to be the case that The Good is realised, we should include both $T - \mathbb{O}dT$ and $T - \mathbb{O}\mathbb{G}dT$ in our systems.

[4] All of these 'definitions' can be restricted to the possible worlds that are alethically accessible in ω at τ. A potential problem with the explications in this paragraph is that there perhaps are no best possible worlds. If there are infinitely many possible worlds, and among them an unending series of better and better worlds at some moment in time, then no possible world will be deontically accessible from a possible world at this time (some 'satisficing' theories (see next paragraph) avoids this particular problem). However, in our ordinary lives, we are usually not interested in such purely logically possible worlds; when we try to decide what to do in a particular situation at a particular moment in time, we concentrate on the worlds that are still historically accessible at this time (and that we can still bring about). It is reasonable to think that there is no such infinite series of better and better worlds that are accessible in such situations. So, in practice, it seems that we do not have to worry about this problem. In any case, I will not say anything more about it in the present paper.

when we decide what it means for a possible world to be deontically accessible from a possible world at a time. Hence, this is not necessarily a decisive argument against the approach in this paper. I will not try to defend this view against all possible counterarguments. From now on, I will assume that the basic approach is reasonable. In any case, it is certainly interesting enough to be worth exploring further.

I have suggested that we read '$\bullet$' as 'The (Highest Possible) Good'. But we could also read '$\bullet$' as 'The Good Enough'. If we do that, the systems in this paper are also compatible with various 'satisficing' theories and not only with various 'maximising' theories (see [85] for more on these concepts). For our purposes in this paper, we do not have to decide which of these theories (if any) is the correct one.

Another good reason to be interested in the systems in this paper is that we seem to need them to explain the validity of certain arguments that are intuitively valid. Consider, for example, the following deductions:

Argument 1
(1) It is forbidden that you torture this innocent child just for the fun of it. $\mathbb{F}T$
Hence, (2) There is something that is forbidden [From (1)]. $\exists X \mathbb{F} X$

Argument 2
(3) Nihilism is true only if everything is permitted. $Q \rightarrow \forall X \mathbb{P} X$
Hence, (4) Nihilism is not true [from (2) and (3)]. $\neg Q$

Argument 3
(4) It is permitted that you drink this glass of water. $\mathbb{P}R$
(5) It is permitted that you do not drink this glass of water. $\mathbb{P}\neg R$
Hence, (6) There is something that is optional [From (4) and (5)]. $\exists X \mathbb{K} X$

Argument 4
(7) Rigorism is true iff everything is obligatory or forbidden. $S \leftrightarrow \forall X(\mathbb{O}X \vee \mathbb{F}X)$
Hence, (8) Rigorism is not true [From (6) and (7)]. $\neg S$

All of these arguments seem to be valid. In every case, the conclusion seems to follow from the premises. But it is hard to prove this in any existing systems in the literature. To show that they are valid, we appear to need some system with propositional quantifiers.[5] In all extended systems in this paper, we can show that

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[5] It might be possible to translate the sentences in the arguments somehow and then use predicate

the arguments above are valid (if they are symbolised as above). In every case, the conclusion is derivable from the premises.

I conclude that we have good technical, as well as philosophical, reasons to be interested in the systems in this article.

The paper consists of five main sections. Section 2 is about the syntax and Section 3 about the semantics of our systems. In Section 4, I explore the proof theory of our logics; I will also consider some interesting theorems that can be established in various systems. Section 5 contains soundness results for *every* system in this paper and soundness *and* completeness proofs for every *non-extended* system.

2 Syntax

First, we introduce a quantifier-free language. Then we extend this language with propositional quantifiers.

Definition 1 ***(Alphabet).***

(i) Propositional variables: P, Q, R, S, T, X, Y, Z, W, P_1, Q_1, R_1, S_1, T_1, X_1, Y_1, Z_1, W_1, P_2, Q_2, R_2, S_2, T_2, X_2, Y_2, Z_2, W_2, $\dots$.

(ii) Constants: $\bullet$ *(The Good),* $\bot$ *(Falsum),* $\top$ *(Verum).*

(iii) Primitive truth-functional connectives: $\neg$ *(negation),* $\wedge$ *(conjunction),* $\vee$ *(disjunction),* $\rightarrow$ *(material implication), and* $\leftrightarrow$ *(material equivalence).*

(iv) Temporal operators $\mathbb{A}$, $\mathbb{S}$, $\mathbb{G}$, $\mathbb{F}$, $\mathbb{H}$ *and* $\mathbb{P}$.

(v) Modal (alethic) operators $\mathbb{U}$, $\mathbb{M}$, $\Box$ *and* $\Diamond$.

(vi) Deontic operators $\mathbb{O}$ *and* $\mathbb{P}$.

(vii) Brackets: $)$, $($.

Definition 2 ***(The language*** $\mathcal{L}$***).*** *The language* $\mathcal{L}$ *is defined in the following way:*

(i) Propositional variables, $\bullet$, $\bot$ *and* $\top$ *are (atomic) formulas.*

(ii) If A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ *and* $(A \leftrightarrow B)$.

(iii) If B is a formula, then $\mathbb{A}B$ *(it is always the case that B),* $\mathbb{S}B$ *(it is sometimes the case that B),* $\mathbb{G}B$ *(it is always going to be the case that B),* $\mathbb{F}B$ *(it will some time [in the future] be the case that B),* $\mathbb{H}B$ *(it has always been the case that B) and* $\mathbb{P}B$ *(it was some time [in the past] the case that B) are formulas.*

(iv) If A is a formula, then $\mathbb{U}A$ *('it is universally [or absolutely] necessary that A'),* $\mathbb{M}A$ *('it is universally [or absolutely] possible that A'),* $\Box A$ *('it is [historically] necessary that A') and* $\Diamond A$ *('it is [historically] possible that A') are formulas.*

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logic or a combination of predicate logic and deontic logic to symbolise them. But this approach seems to be less natural.

(v) If A is a formula, then $\mathbb{O}A$ ('it ought to be the case that A') and $\mathbb{P}A$ ('it is permitted that A') are formulas.

(vi) Nothing else is a formula.

Definition 3 ***(Some operators in $\mathcal{L}$).*** *All the following definitions are added to $\mathcal{L}$: $\mathbb{F}A$ (it is forbidden that A) $=_{df}$ $\neg\mathbb{P}A$; $\mathbb{K}A$ (it is optional that A) $=_{df}$ $\mathbb{P}A \wedge \mathbb{P}\neg A$; $\mathbb{N}A$ (it is nonoptional that A) $=_{df}$ $\neg\mathbb{K}A = \neg(\mathbb{P}A \wedge \mathbb{P}\neg A)$ (or $\mathbb{O}A \vee \mathbb{O}\neg A$); $\diamondminus A$ (it is impossible that A) $=_{df}\neg \Diamond A$ (or $\Box\neg A$); $\boxminus A$ (it is non-necessary that A) $=_{df}\neg \Box A$; $\triangledown A$ (it is contingent that A) $=_{df}(\Diamond A \wedge \Diamond\neg A)$; $\vartriangle A$ (it is noncontingent that A) $=_{df}\neg \triangledown A = \neg(\Diamond A \wedge \Diamond\neg A)$ (or $(\Box A \vee \Box\neg A)$); $(A \bigcirc B)$ (A is consistent with B) $=_{df}\Diamond(A{\wedge}B)$; $(A{\ominus}B)$ (A is inconsistent with B) $=_{df}\neg(A{\bigcirc}B)$ ($\neg\Diamond(A{\wedge}B)$, $\diamondminus(A{\wedge}B)$, or $\Box\neg(A \wedge B)$); $(A \Rightarrow B)$ (A strictly implies B) $=_{df}\Box(A \rightarrow B)$; $(A \Leftrightarrow B)$ (A is strictly equivalent with B) $=_{df}((A \Rightarrow B) \wedge (B \Rightarrow A))$ ($((\Box(A \rightarrow B) \wedge \Box(B \rightarrow A))$, or $\Box(A \leftrightarrow B)$); $[\mathbb{G}]A$ ('it is and it is always going to be the case that A') $=_{df}$ $(A \wedge \mathbb{G}A)$; $[\mathbb{H}]A$ ('it is and it has always been the case that A') $=_{df}$ $(A \wedge \mathbb{H}A)$; $\langle\mathbb{F}\rangle A$ ('it is or it will some time in the future be the case that A') $=_{df}$ $(A \vee \mathbb{F}A)$; $\langle\mathbb{P}\rangle A$ ('it is or it has some time in the past been the case that A') $=_{df}$ $(A \vee \mathbb{P}A)$.*

Definition 4 ***(The language $\mathcal{L}_{Ext}$).*** *To obtain the language $\mathcal{L}_{Ext}$, we extend $\mathcal{L}$ by adding two propositional quantifiers $\forall$ (everything/for all) and $\exists$ (something/for some) in the usual way. Hence, if A is any formula and X is any propositional variable, then $\forall XA$ and $\exists XA$ are formulas. We will call this extended language $\mathcal{L}_{Ext}$.*

Definition 5 ***(Definition of $\star$ in $\mathcal{L}_{Ext}$).*** *The following definition is added to $\mathcal{L}_{Ext}$: $\star =_{df} \forall A(\mathbb{O}A \rightarrow A)$. '$\star$' is read as 'The Moral Law is fulfilled'. This means that The Moral Law is fulfilled (in a possible world ω at a time τ) iff everything that ought to be the case is the case (in ω at τ).*

Parentheses in formulas are usually dropped when it does not lead to any ambiguity.

The concept of a free variable X in a formula A is defined in the usual way. A variable X is free in A iff it has a free occurrence in A. Intuitively, an occurrence of a variable is free in a formula just in case it is not bound by any quantifier. If A does not contain $\forall$ or $\exists$, then every occurrence of X in A is free. An occurrence of X is free in $\otimes B$ (where $\otimes = \neg$ or some monadic operator in our language) iff the corresponding occurrence of X is free in B, and an occurrence of X in $A \wedge B$ is free iff the corresponding occurrence of X in A or B is free, etc. Finally, an occurrence of X is free in $\forall YB$ ($\exists YB$) iff the corresponding occurrence of X is free in B and X is distinct from Y. Any variable occurrences in a formula that are not free are said to be bound. Every free occurrence of X in B is bound by $\forall$ ($\exists$) in $\forall XB$ ($\exists XB$).

Let $(A)[B_1/X_1, \ldots, B_n/X_n]$ be the formula that results by simultaneously replacing all free occurrences of the variable X_1 in A by B_1, …, and all free occurrences of the variable X_n in A by B_n. If there are no free occurrences of $X_1, \ldots, X_n$ in A, then $(A)[B_1/X_1, \ldots, B_n/X_n] = A$. We say that B is substitutable for X in A just in case no variable in B gets bound by a quantifier when B is substituted for X in A. In other words, B is substitutable for X in A just in case, for every variable Y, if an occurrence of Y is free in B, then the corresponding occurrence of Y is free when X is replaced by B in A.

3 Semantics

Definition 6 ***(Models)*** *A model $\mathcal{M}$ is a relational structure $\langle \mathfrak{W}, \mathfrak{T}, <, \mathfrak{R}, \mathfrak{S}, \mathfrak{v} \rangle$, where $\mathfrak{W}$ is a non-empty set of possible worlds, $\mathfrak{T}$ is a non-empty set of times, $<$ is a binary relation on $\mathfrak{T}$ ($<$ is a subset of $\mathfrak{T} \times \mathfrak{T}$), $\mathfrak{R}$ is a ternary alethic accessibility relation ($\mathfrak{R}$ is a subset of $\mathfrak{W} \times \mathfrak{W} \times \mathfrak{T}$), $\mathfrak{S}$ is a ternary deontic accessibility relation ($\mathfrak{S}$ is a subset of $\mathfrak{W} \times \mathfrak{W} \times \mathfrak{T}$), and $\mathfrak{v}$ is an interpretation function.*

Let X be a propositional variable. Then $\mathfrak{v}(X)$ is a subset of $\mathfrak{W} \times \mathfrak{T}$. Intuitively, $\mathfrak{v}(X)$ is the proposition that X expresses or the set of world-moment pairs in which X is true. In other words, if X is a propositional variable, then X is true in the possible world ω at the time τ iff $\langle \omega, \tau \rangle$ is in $\mathfrak{v}(X)$.

$\mathfrak{R}$ is used in the definition of the truth conditions for sentences that begin with the alethic operators $\Box$ and $\Diamond$, $\mathfrak{S}$ is used in the definition of the truth conditions for sentences that begin with the deontic operators $\mathbb{O}$ and $\mathbb{P}$, and $<$ is used to define the truth conditions for sentences that begin with the temporal operators. Intuitively, $\tau < \tau'$ says that the time τ is before the time τ', $\mathfrak{R}\omega\omega'\tau$ says that the possible world ω' is alethically (historically) accessible from the possible world ω at the time τ, $\mathfrak{S}\omega\omega'\tau$ says that the possible world ω' is deontically accessible from the possible world ω at the time τ. We shall also say that $\mathfrak{S}\omega\omega'\tau$ just in case ω' is one of the best possible worlds (all things considered) in ω at τ. If we assume condition $C - \Box\mathbb{O}$ (see Table 4), we can say that $\mathfrak{S}\omega\omega'\tau$ iff ω' is one of the best possible worlds (all things considered) that are historically accessible from ω at τ.[6]

Definition 7 ***(Truth conditions for sentences in $\mathcal{L}$).***

Let $\mathcal{M}$ be any model $\langle \mathfrak{W}, \mathfrak{T}, <, \mathfrak{R}, \mathfrak{S}, \mathfrak{v} \rangle$. Let $\omega \in \mathfrak{W}$, $\tau \in \mathfrak{T}$ and let A be a well-formed sentence in $\mathcal{L}$. Then $\mathcal{M}, \omega, \tau \Vdash A$ is an abbreviation of 'A is true in ω at τ in $\mathcal{M}$' (or 'A is true in the pair $\langle \omega, \tau \rangle$ in $\mathcal{M}$'). $\mathcal{M}, \omega, \tau \nVdash A$ just in case it is

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[6] As I mentioned in the introduction, this basic approach is consistent with many different value theories. It is even, in principle, consistent with various satisficing theories; even though the reading of $\mathfrak{S}$ above suggests some kind of maximising theory this reading is not strictly necessary.

not true that $\mathcal{M}, \omega, \tau \Vdash A$. Note that $\mathcal{M}, \omega, \tau \nVdash A$ iff $\mathcal{M}, \omega, \tau \Vdash \neg A$. '$\forall \omega' \in \mathfrak{W}$' is read as 'for all possible worlds ω' in $\mathfrak{W}$'; '$\exists \omega' \in \mathfrak{W}$' is read as 'for some possible world ω' in $\mathfrak{W}$', etc. The truth conditions for various sentences in $\mathcal{L}$ can now be defined in the following way (the truth conditions for the omitted sentences are straightforward; Verum is true in every world-moment pair in every model and Falsum is false in every world-moment pair in every model):

(i) If P is a propositional variable, then $\mathcal{M}, \omega, \tau \Vdash P$ iff $\langle \omega, \tau \rangle \in \mathfrak{v}(P)$.

(ii) $\mathcal{M}, \omega, \tau \Vdash \bullet$ iff $\mathfrak{S}\omega\omega\tau$.

(iii) $\mathcal{M}, \omega, \tau \Vdash A \wedge B$ iff $\mathcal{M}, \omega, \tau \Vdash A$ and $\mathcal{M}, \omega, \tau \Vdash B$.

(iv) $\mathcal{M}, \omega, \tau \Vdash \mathbb{A}B$ iff $\forall \tau' \in \mathfrak{T}$: $\mathcal{M}, \omega, \tau' \Vdash B$.

(v) $\mathcal{M}, \omega, \tau \Vdash \mathbb{S}B$ iff $\exists \tau' \in \mathfrak{T}$: $\mathcal{M}, \omega, \tau' \Vdash B$.

(vi) $\mathcal{M}, \omega, \tau \Vdash \mathbb{G}B$ iff $\forall \tau' \in \mathfrak{T}$ s.t. $\tau < \tau'$: $\mathcal{M}, \omega, \tau' \Vdash B$.

(vii) $\mathcal{M}, \omega, \tau \Vdash \mathbb{F}B$ iff $\exists \tau' \in \mathfrak{T}$ s.t. $\tau < \tau'$: $\mathcal{M}, \omega, \tau' \Vdash B$.

(viii) $\mathcal{M}, \omega, \tau \Vdash \mathbb{H}B$ iff $\forall \tau' \in \mathfrak{T}$ s.t. $\tau' < \tau$: $\mathcal{M}, \omega, \tau' \Vdash B$.

(ix) $\mathcal{M}, \omega, \tau \Vdash \mathbb{P}B$ iff $\exists \tau' \in \mathfrak{T}$ s.t. $\tau' < \tau$: $\mathcal{M}, \omega, \tau' \Vdash B$.

(x) $\mathcal{M}, \omega, \tau \Vdash \mathbb{U}A$ iff $\forall \omega' \in \mathfrak{W}$ and $\forall \tau' \in \mathfrak{T}$: $\mathcal{M}, \omega', \tau' \Vdash A$.

(xi) $\mathcal{M}, \omega, \tau \Vdash \mathbb{M}A$ iff $\exists \omega' \in \mathfrak{W}$ and $\exists \tau' \in \mathfrak{T}$: $\mathcal{M}, \omega', \tau' \Vdash A$.

(xii) $\mathcal{M}, \omega, \tau \Vdash \Box A$ iff $\forall \omega' \in \mathfrak{W}$ s.t. $\mathfrak{R}\omega\omega'\tau$: $\mathcal{M}, \omega', \tau \Vdash A$.

(xiii) $\mathcal{M}, \omega, \tau \Vdash \Diamond A$ iff $\exists \omega' \in \mathfrak{W}$ s.t. $\mathfrak{R}\omega\omega'\tau$: $\mathcal{M}, \omega', \tau \Vdash A$.

(xiv) $\mathcal{M}, \omega, \tau \Vdash \mathbb{O}A$ iff $\forall \omega' \in \mathfrak{W}$ s.t. $\mathfrak{S}\omega\omega'\tau$: $\mathcal{M}, \omega', \tau \Vdash A$.

(xv) $\mathcal{M}, \omega, \tau \Vdash \mathbb{P}A$ iff $\exists \omega' \in \mathfrak{W}$ s.t. $\mathfrak{S}\omega\omega'\tau$: $\mathcal{M}, \omega', \tau \Vdash A$.

Now we have to consider the truth-conditions for the new sentences in $\mathcal{L}_{Ext}$. Intuitively, $\forall XA$ is true in a possible world ω at a moment in time τ iff $A[B/X]$ is true in ω at τ for every sentence B in $\mathcal{L}$, and $\exists XA$ is true in ω at τ iff $A[B/X]$ is true in ω at τ for some sentence B in $\mathcal{L}$. Hence, the quantifiers are 'substitutional' in this paper rather than 'objectual'. For example, they do not vary directly over (sets of) world-moment pairs (the range is not a (the) set of (all) subsets of (the set of all) world-moment pairs). To avoid circularity, we only use formulas from $\mathcal{L}$ in our substitutions. To see the potential problem, let $A = \forall XX$ and assume that our substitutions can include any formula whatsoever. Then $A[A/X] = A$, for $\forall XX[\forall XX/X] = \forall XX$. More precisely, the truth conditions for the new sentences in $\mathcal{L}_{Ext}$ are defined in the following way:

Definition 8 ***(Truth conditions for sentences in $\mathcal{L}_{Ext}$).***

(xvi) $\mathcal{M}, \omega, \tau \Vdash \forall XA$ iff for every sentence B (that is substitutable for X in A) in $\mathcal{L}$, $\mathcal{M}, \omega, \tau \Vdash A[B/X]$.

(xvii) $\mathcal{M}, \omega, \tau \Vdash \exists XA$ iff there is some sentence B (that is substitutable for X in A) in $\mathcal{L}$ such that $\mathcal{M}, \omega, \tau \Vdash A[B/X]$.[7]

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[7] In [73], I develop a set of pure monomodal system that include two propositional quantifiers. In

Definition 9 ***(Some semantic concepts).***

Validity. *A sentence A is valid in a model,* $\mathcal{M} \Vdash A$*, iff A is true in every possible world* ω *at every moment in time* τ *in* $\mathcal{M}$*. Let* $\mathfrak{M}$ *be a class of models. Then A is valid in* $\mathfrak{M}$*,* $\mathfrak{M} \Vdash A$*, iff A is valid in every model* $\mathcal{M}$ *in* $\mathfrak{M}$*, that is, iff A is true in every possible world* ω *at every moment in time* τ *in every model* $\mathcal{M}$ *in* $\mathfrak{M}$*.*

Logical consequence. *Let A be a sentence, let* Γ *be a finite set of sentences and let* $\mathfrak{M}$ *be a class of models. Then, A is a logical consequence of* Γ *in* $\mathfrak{M}$*,* $\mathfrak{M}, \Gamma \Vdash A$*, iff for every model* $\mathcal{M}$ *in* $\mathfrak{M}$ *and world* ω *and every moment in time* τ *in* $\mathcal{M}$*, if all elements of* Γ *are true in* ω *at* τ *in* $\mathcal{M}$*, then A is true in* ω *at* τ *in* $\mathcal{M}$*. If* $\mathfrak{M}, \Gamma \Vdash A$*, we also say that* Γ *entails A in* $\mathfrak{M}$ *and that the argument from* Γ *to A is valid in* $\mathfrak{M}$*. An argument is invalid in* $\mathfrak{M}$ *iff it is not valid in* $\mathfrak{M}$*.*

3.1 Conditions on models

In this section, I will introduce some conditions that might be imposed on the various accessibility relations in a model.

The conditions are divided into eight classes. The first class tells us something about the formal properties of the relation <, the second about the formal properties of the relation $\mathfrak{R}$ (at a time), the third about the formal properties of the relation $\mathfrak{S}$ (at a time), the fourth about how $\mathfrak{R}$ and $\mathfrak{S}$ are related to each other (at a time), the fifth about how $\mathfrak{R}$ and < are related to each other, the sixth about how $\mathfrak{S}$ and < are related to each other, the seventh about how $\mathfrak{R}$, $\mathfrak{S}$ and < are related to each other and the eighth consists of two conditions that we may impose on the valuation function $\mathfrak{v}$ in a model.

The variables ω, ω', ω'', ω''' in tables 2–8 are taken to range over possible worlds in $\mathfrak{W}$, τ, τ', τ'' and τ''' in tables 1–8 over times in $\mathfrak{T}$, and the symbols $\wedge$, $\rightarrow$, $\forall$ and $\exists$ are used as metalogical symbols in the standard way. Let $\mathcal{M} = \langle \mathfrak{W}, \mathfrak{T}, <, \mathfrak{R}, \mathfrak{S}, \mathfrak{v} \rangle$ be a model. If $\forall \tau \forall \omega \exists \omega' \mathfrak{S} \omega \omega' \tau$, we say that $\mathfrak{S}$ satisfies or fulfills condition $C - dD$ and also that $\mathcal{M}$ satisfies or fulfills condition $C - dD$ and similarly in all other cases. $C - dD$ is called "$C - dD$" because the tableau rule $T - dD$ "corresponds" to $C - dD$ and the sentence dD ($\neg(\mathbb{O}A \wedge \mathbb{O}\neg A)$) is valid in the class of all models that satisfy condition $C - dD$ and similarly in all other cases. Let C be any of the conditions in tables 1–8. Then a C-model is a model that satisfies C.

Most of the conditions in this section are self-explanatory. Nevertheless, we will add a few comments about some of them.

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this paper, I use a pair of similar quantifiers. However, in (xvi) and (xvii) above the truth-conditions are relativised to world-moment pairs and not only to possible worlds. For more on some vaguely similar approaches to propositional quantification, see [26] and [58].

3.1.1 Conditions on the relation $<$

Condition	Formalisation of condition
$C-PD$	$\forall\tau\exists\tau'\tau' < \tau$
$C-FD$	$\forall\tau\exists\tau'\tau < \tau'$
$C-t4$	$\forall\tau\forall\tau'\forall\tau''((\tau < \tau' \wedge \tau' < \tau'') \rightarrow \tau < \tau'')$
$C-DE$	$\forall\tau\forall\tau'(\tau < \tau' \rightarrow \exists\tau''(\tau < \tau'' \wedge \tau'' < \tau'))$
$C-FC$	$\forall\tau\forall\tau'\forall\tau''((\tau < \tau' \wedge \tau < \tau'') \rightarrow (\tau' < \tau'' \vee \tau' = \tau'' \vee \tau'' < \tau'))$
$C-PC$	$\forall\tau\forall\tau'\forall\tau''((\tau' < \tau \wedge \tau'' < \tau) \rightarrow (\tau' < \tau'' \vee \tau' = \tau'' \vee \tau'' < \tau'))$
$C-C$	$\forall\tau\forall\tau'(\tau < \tau' \vee \tau = \tau' \vee \tau' < \tau)$
$C-UB$	$\forall\tau\forall\tau'\forall\tau''((\tau < \tau' \wedge \tau < \tau'') \rightarrow \exists\tau'''(\tau' < \tau''' \wedge \tau'' < \tau'''))$
$C-LB$	$\forall\tau\forall\tau'\forall\tau''((\tau' < \tau \wedge \tau'' < \tau) \rightarrow \exists\tau'''(\tau''' < \tau' \wedge \tau''' < \tau''))$

Table 1

The conditions in **Table 1** are well-known. They say something about the formal properties of the temporal relation 'earlier than', $<$. *PD* stands for 'past *D*', *FD* for 'future *D*', *DE* for 'dense', *FC* for 'future convergence', *PC* for 'past convergence', *C* for 'comparability', *UB* for 'upper bounds', and *LB* for 'lower bounds'. According to $C-FD$, for example, there is no last point in time; for every moment in time τ there is a moment in time τ' that occurs later than τ. According to $C-t4$, time is transitive, that is for every τ, τ' and τ'': if τ' occurs later than τ and τ'' occurs later than τ' then τ'' occurs later than τ. $C-DE$ says that time is dense, $C-PC$ that time doesn't branch towards the past and $C-FC$ that time doesn't branch towards the future, etc. The conditions in **Table 1** are often described in various introductions to temporal logic and require no further comments (see, for example, [16], [27], [36], [45], [59], [66], [68] and [64]).

3.1.2 Conditions on the relation $\mathfrak{R}$

Condition	Formalisation of condition
$C-aT$	$\forall\tau\forall\omega\mathfrak{R}\omega\omega\tau$
$C-aD$	$\forall\tau\forall\omega\exists\omega'\mathfrak{R}\omega\omega'\tau$
$C-aF$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{R}\omega\omega'\tau \wedge \mathfrak{R}\omega\omega''\tau) \rightarrow \omega' = \omega'')$
$C-aB$	$\forall\tau\forall\omega\forall\omega'(\mathfrak{R}\omega\omega'\tau \rightarrow \mathfrak{R}\omega'\omega\tau)$
$C-a4$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{R}\omega\omega'\tau \wedge \mathfrak{R}\omega'\omega''\tau) \rightarrow \mathfrak{R}\omega\omega''\tau)$
$C-a5$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{R}\omega\omega'\tau \wedge \mathfrak{R}\omega\omega''\tau) \rightarrow \mathfrak{R}\omega'\omega''\tau)$

Table 2

The conditions on $\mathfrak{R}$ in **Table 2** are similar to well-known conditions on the alethic accessibility relation in mono-modal alethic logic (see, for example, [30]). The only difference is that $\mathfrak{R}$ is a 3-place relation in our systems. Intuitively, this corresponds to the idea that the ordinary 2-place alethic relation is relativised to particular moments in time. So, for example, $C - aT$ says that $\mathfrak{R}$ is reflexive at every time, $C - aB$ says that $\mathfrak{R}$ is symmetric at every time, etc.

$C - aD$ says that every possible world can see at least one possible world alethically (at a particular moment in time), and $C - aF$ says that every possible world can see at most one possible world alethically (at a particular moment in time). Accordingly, if we assume $C - aD$ and $C - aF$, every world can see one and exactly one possible world alethically (at a particular moment in time). If a model satisfies $C - aF$, then $\Diamond P \rightarrow \Box P$ is valid in this model, and if it satisfies $C - aD$, then $\Box P \rightarrow \Diamond P$ is valid in this model. Consequently, if a model satisfies both $C - aD$ and $C - aF$, $\Diamond P \leftrightarrow \Box P$, $\Diamond \neg P \leftrightarrow \neg \Diamond P$ and $\boxminus P \leftrightarrow ⟠ P$ are valid in this model. Hence, the distinctions between possibility and necessity and between non-necessity and impossibility collapse.

$C - aT$ says that every possible world can see itself alethically (at a particular moment in time). Hence, if we assume $C - aT$ and $C - aF$, every possible world can see itself and nothing but itself alethically (at a particular moment in time). If a model satisfies $C - aT$ and $C - aF$, then $P \leftrightarrow \Box P$, $\neg P \leftrightarrow \neg \Diamond P$ and $\neg P \leftrightarrow ⟠ P$ are valid in this model. Consequently, the distinctions between truth and necessary truth and between falsehood and impossibility collapse. Furthermore, if $C - aT$ holds, then $C - aD$ also holds. So, it is also true that if a model satisfies $C - aT$ and $C - aF$, then $\Diamond P \leftrightarrow \Box P$, $\Diamond \neg P \leftrightarrow \neg \Diamond P$ and $\boxminus P \leftrightarrow ⟠ P$ are valid in this model. Therefore, the distinctions between what is true, possible and necessary collapse, that is, the following equivalences are valid $P \leftrightarrow \Diamond P \leftrightarrow \Box P$. Moreover, we have $\neg P \leftrightarrow \Diamond \neg P \leftrightarrow \neg \Diamond P \leftrightarrow \boxminus P \leftrightarrow ⟠ P$. Hence, the distinctions between what is false, possibly false, impossible and unnecessary also collapse. $C - aF$ is technically interesting, but intuitively problematic. (For more on this see [73]; see also [70].)

3.1.3 Conditions on the relation $\mathfrak{S}$

Condition	Formalisation of condition
$C - dD$	$\forall \tau \forall \omega \exists \omega' \mathfrak{S} \omega \omega' \tau$
$C - dF$	$\forall \tau \forall \omega \forall \omega' \forall \omega'' ((\mathfrak{S} \omega \omega' \tau \land \mathfrak{S} \omega \omega'' \tau) \rightarrow \omega' = \omega'')$
$C - d4$	$\forall \tau \forall \omega \forall \omega' \forall \omega'' ((\mathfrak{S} \omega \omega' \tau \land \mathfrak{S} \omega' \omega'' \tau) \rightarrow \mathfrak{S} \omega \omega'' \tau)$
$C - d5$	$\forall \tau \forall \omega \forall \omega' \forall \omega'' ((\mathfrak{S} \omega \omega' \tau \land \mathfrak{S} \omega \omega'' \tau) \rightarrow \mathfrak{S} \omega' \omega'' \tau)$
$C - \mathbb{O}dT$	$\forall \tau \forall \omega \forall \omega' (\mathfrak{S} \omega \omega' \tau \rightarrow \mathfrak{S} \omega' \omega' \tau)$
$C - \mathbb{O}dB$	$\forall \tau \forall \omega \forall \omega' \forall \omega'' ((\mathfrak{S} \omega \omega' \tau \land \mathfrak{S} \omega' \omega'' \tau) \rightarrow \mathfrak{S} \omega'' \omega' \tau)$

Table 3

Again, the conditions on $\mathfrak{S}$ in **Table 3** are similar to well-known conditions on the deontic accessibility relation in mono-modal deontic logic (see, for example,

[6]). The only difference is that $\mathfrak{S}$ is a 3-place relation in our systems. Intuitively, this corresponds to the idea that the ordinary 2-place deontic relation is relativised to particular moments in time. Consequently, $C - dD$ says that $\mathfrak{S}$ is serial at every time, $C - d4$ says that $\mathfrak{S}$ is transitive at every time, etc. (see [70]).

Nothing like $C - dF$ is usually mentioned in deontic logic. But the condition is technically interesting and it also has some interesting consequences. In every model that satisfies $C - dF$, the following sentences are valid: $\mathbb{P}A \to \mathbb{O}A$, $\mathbb{O}A \vee \mathbb{O}\neg A$, $\mathbb{O}A \vee \mathbb{F}A$, $\neg(\mathbb{P}A \wedge \mathbb{P}\neg A)$, $\neg\mathbb{K}A$, $\mathbb{N}A$. Hence, if we assume this condition, nothing is optional, there is no A such that both A and not-A are permitted, and everything is either obligatory or forbidden. If we accept this condition, we have to accept a kind of moral rigorism (for more on moral rigorism, see Section 4.3). If a model satisfies $C - dF$ and $C - dD$ then the following equivalence holds: $\mathbb{P}A \leftrightarrow \mathbb{O}A$, that is, the distinction between what is obligatory and what is permitted collapses. Most people are probably inclined to reject those consequences, but some seem to accept them. Hence, this condition is worth mentioning, even though it is intuitively problematic.

3.1.4 Conditions concerning the relation between $\mathfrak{R}$ and $\mathfrak{S}$

Condition	Formalisation of condition
$C - \Box\mathbb{O}$	$\forall\tau\forall\omega\forall\omega'(\mathfrak{S}\omega\omega'\tau \to \mathfrak{R}\omega\omega'\tau)$
$C - \mathbb{O}\Diamond$	$\forall\tau\forall\omega\exists\omega'(\mathfrak{S}\omega\omega'\tau \wedge \mathfrak{R}\omega\omega'\tau)$
$C - \mathbb{O}\Box\mathbb{O}$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{S}\omega\omega'\tau \wedge \mathfrak{S}\omega'\omega''\tau) \to \mathfrak{R}\omega'\omega''\tau)$
$C - \mathbb{O}\mathbb{O}\Diamond$	$\forall\tau\forall\omega\forall\omega'(\mathfrak{S}\omega\omega'\tau \to \exists\omega''(\mathfrak{S}\omega'\omega''\tau \wedge \mathfrak{R}\omega'\omega''\tau))$
$C - ad4$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{R}\omega\omega'\tau \wedge \mathfrak{S}\omega'\omega''\tau) \to \mathfrak{S}\omega\omega''\tau)$
$C - ad5$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{R}\omega\omega'\tau \wedge \mathfrak{S}\omega\omega''\tau) \to \mathfrak{S}\omega'\omega''\tau)$
$C - \mathbb{P}\Box P$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{S}\omega\omega'\tau \wedge \mathfrak{R}\omega\omega''\tau) \to \exists\omega'''(\mathfrak{R}\omega'\omega'''\tau \wedge \mathfrak{S}\omega''\omega'''\tau))$
$C - \mathbb{O}\Box P$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{R}\omega\omega'\tau \wedge \mathfrak{S}\omega'\omega''\tau) \to \exists\omega'''(\mathfrak{S}\omega\omega'''\tau \wedge \mathfrak{R}\omega'''\omega''\tau))$
$C - \Box\mathbb{O}P$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{S}\omega\omega'\tau \wedge \mathfrak{R}\omega'\omega''\tau) \to \exists\omega'''(\mathfrak{R}\omega\omega'''\tau \wedge \mathfrak{S}\omega'''\omega''\tau))$

Table 4

The conditions in **Table 4** deal with some possible relations between the alethic and the deontic accessibility relations. In every model that satisfies $C - \Box\mathbb{O}$, $\Box A \to \mathbb{O}A$ (the necessity-ought or must-ought principle) is valid, and in every model that satisfies $C - \mathbb{O}\Diamond$, $\mathbb{O}A \to \Diamond A$ (the ought-possibility or ought-can principle) is valid. $C - \Box\mathbb{O}$ is stronger than $C - \mathbb{O}\Box\mathbb{O}$ and $C - \mathbb{O}\Diamond$ is stronger than $C - \mathbb{O}\mathbb{O}\Diamond$. In every model that satisfies $C - \mathbb{O}\Box\mathbb{O}$, $\mathbb{O}(\Box A \to \mathbb{O}A)$ is valid, and in every model that satisfies $C - \mathbb{O}\mathbb{O}\Diamond$, $\mathbb{O}(\mathbb{O}A \to \Diamond A)$ is valid.

$\mathbb{O}A \to \Box\mathbb{O}A$ is valid in every model that satisfies $C - ad4$ and $\mathbb{P}A \to \Box\mathbb{P}A$ is valid in every model that satisfies $C - ad5$. In every model that satisfies $C - \mathbb{P}\Box P$,

$\mathbb{P} \Box A \to \Box\mathbb{P}A$ is valid; in every model that satisfies $C - \mathbb{O} \Box P$, $\mathbb{O} \Box A \to \Box\mathbb{O}A$ is valid; and in every model that satisfies $C - \Box\mathbb{O}P$, $\Box\mathbb{O}A \to \mathbb{O} \Box A$ is valid.

I have described these conditions before, see [70]; see also [71]. For some general ideas about how to combine two or more modal systems, see, for example, [57] and [40].

3.1.5 Temporal alethic interactions: Conditions concerning the relation between $\mathfrak{R}$ and $<$

Condition	Formalisation of condition
$C - ASP$	$\forall\tau\forall\tau'\forall\omega\forall\omega'((\tau < \tau' \land \mathfrak{R}\omega\omega'\tau') \to \mathfrak{R}\omega\omega'\tau)$
$C - AR$	$\forall\tau\forall\tau'\forall\omega\forall\omega'\forall\omega''((\tau < \tau' \land \mathfrak{R}\omega\omega'\tau \land \mathfrak{R}\omega'\omega''\tau') \to \mathfrak{R}\omega\omega''\tau)$

Table 5

The conditions in **Table 5** are concerned with some possible interactions between $\mathfrak{R}$ and $<$. 'ASP' is an abbreviation of 'alethic shared past' and 'AR' of 'alethic ramification'.

The condition $C - ASP$ says that it is true that if a world ω' is alethically accessible from a world ω at time τ', then ω' is alethically accessible from ω at every moment in time τ that is earlier than τ'. This condition is plausible if we represent reality as a tree-like structure that branches towards the future and not as a set of entirely unrelated possible worlds and moments in time. We can think of the possible worlds in $\mathfrak{W}$ as possible histories of one and the same world (reality) rather than as distinct worlds.

$C - AR$ follows from $C - ASP$ and $C - a4$. Therefore, $C - AR$ too is plausible if we represent the world as a tree-like structure. If a model satisfies $C - ASP$, we can show that the following sentences are valid in this model: $\mathbb{H} \Box A \to \Box\mathbb{H}A$, $\mathbb{P} \Box A \to \Box\mathbb{P}A$, $\Box\mathbb{G}A \to \mathbb{G} \Box A$ and $\Box A \to \mathbb{G} \Box \mathbb{P}A$. If a model satisfies $C - AR$, we can show that $\Box\mathbb{G}A \to \Box\mathbb{G} \Box A$ is valid in this model.

3.1.6 Temporal deontic interactions: Conditions concerning the relation between $\mathfrak{S}$ and $<$

Condition	Formalisation of condition
$C - \mathbb{O}\mathbb{G}dT$	$\forall\tau\forall\tau'\forall\omega\forall\omega'((\tau < \tau' \land \mathfrak{S}\omega\omega'\tau) \to \mathfrak{S}\omega'\omega'\tau')$
$C - \mathbb{O}\mathbb{G}dB$	$\forall\tau\forall\tau'\forall\omega\forall\omega'\forall\omega''((\tau < \tau' \land \mathfrak{S}\omega\omega'\tau \land \mathfrak{S}\omega'\omega''\tau') \to \mathfrak{S}\omega''\omega'\tau')$
$C - DR$	$\forall\tau\forall\tau'\forall\omega\forall\omega'\forall\omega''((\tau < \tau' \land \mathfrak{S}\omega\omega'\tau \land \mathfrak{S}\omega'\omega''\tau') \to \mathfrak{S}\omega\omega''\tau)$

Table 6

The conditions in **Table 6** deal with some possible relations between $\mathfrak{S}$ and $<$.

If a model satisfies $C - \mathbb{O}\mathbb{G}dT$, then $\mathbb{O}\mathbb{G}(\mathbb{O}A \to A)$, $\bullet \to \mathbb{G}\bullet$ and $\bullet \to [\mathbb{G}]\bullet$ are valid in this model, and if it satisfies $C - \mathbb{O}\mathbb{G}dB$, then $\mathbb{O}\mathbb{G}(A \to \mathbb{O}\mathbb{P}A)$ is valid in this model. In every model that satisfies $C - DR$ ('deontic ramification'), $\mathbb{O}\mathbb{G}A \to \mathbb{O}\mathbb{G}\mathbb{O}A$ is valid. According to $\mathbb{O}\mathbb{G}(\mathbb{O}A \to A)$, it ought to be that it is always going to be that if A ought to be then A is true, and according to $\bullet \to \mathbb{G}\bullet$, The Good is realised only if it is always going to be the case that The Good is realised. $\mathbb{O}\mathbb{G}(A \to \mathbb{O}\mathbb{P}A)$ says that it ought to be that it is always going to be that if A is true then it ought to be permitted that A.

Note that if a model satisfies both $C - \mathbb{O}\mathbb{G}dT$ and $C - \mathbb{O}dT$, then $\mathbb{O}[\mathbb{G}](\mathbb{O}A \to A)$ is valid in this model, that is, then it is true that it ought to be that it is and that it is always going to be the case that if it ought to be the case that A then A. $\mathbb{O}[\mathbb{G}](\mathbb{O}A \to A)$ is by definition equivalent with $\mathbb{O}((\mathbb{O}A \to A) \wedge \mathbb{G}(\mathbb{O}A \to A))$. Similarly, if a model satisfies both $C - \mathbb{O}\mathbb{G}dB$ and $C - \mathbb{O}dB$, then $\mathbb{O}[\mathbb{G}](A \to \mathbb{O}\mathbb{P}A)$ is valid in this model. $\mathbb{O}[\mathbb{G}](A \to \mathbb{O}\mathbb{P}A)$ is by definition equivalent with $\mathbb{O}((A \to \mathbb{O}\mathbb{P}A) \wedge \mathbb{G}(A \to \mathbb{O}\mathbb{P}A))$.

3.1.7 Temporal alethic deontic interactions: Conditions concerning the relation between $\mathfrak{R}$, $\mathfrak{S}$ and $<$

Condition	Formalisation of condition
$C - \mathbb{O}\mathbb{G}\Box\mathbb{O}$	$\forall\tau\forall\tau'\forall\omega\forall\omega'\forall\omega''((\tau < \tau' \wedge \mathfrak{S}\omega\omega'\tau \wedge \mathfrak{S}\omega'\omega''\tau') \to \mathfrak{R}\omega'\omega''\tau')$
$C - \mathbb{O}\mathbb{G}\mathbb{O}\Diamond$	$\forall\tau\forall\tau'\forall\omega\forall\omega'((\tau < \tau' \wedge \mathfrak{S}\omega\omega'\tau) \to \exists\omega''(\mathfrak{S}\omega'\omega''\tau' \wedge \mathfrak{R}\omega'\omega''\tau'))$

Table 7

The conditions in **Table 7** are concerned with some possible relations between $\mathfrak{R}$, $\mathfrak{S}$ and $<$. In every model that satisfies $C - \mathbb{O}\mathbb{G}\Box\mathbb{O}$, $\mathbb{O}\mathbb{G}(\Box A \to \mathbb{O}A)$ is valid, and in every model that satisfies $C - \mathbb{O}\mathbb{G}\mathbb{O}\Diamond$, $\mathbb{O}\mathbb{G}(\mathbb{O}A \to \Diamond A)$ is valid. If a model satisfies $C - \mathbb{O}\mathbb{G}\Box\mathbb{O}$ and $C - \mathbb{O}\Box\mathbb{O}$, $\mathbb{O}[\mathbb{G}](\Box A \to \mathbb{O}A)$ is valid in this model, and if a model satisfies $C - \mathbb{O}\mathbb{G}\mathbb{O}\Diamond$ and $C - \mathbb{O}\mathbb{O}\Diamond$, $\mathbb{O}[\mathbb{G}](\mathbb{O}A \to \Diamond A)$ is valid in this model.

3.1.8 Conditions on the valuation function $\mathfrak{v}$ in a model

Condition	Formalisation of condition
$C - FT$	If $\mathfrak{R}\omega\omega'\tau$ and $\langle\omega,\tau\rangle \in \mathfrak{v}(P)$, then $\langle\omega',\tau\rangle \in \mathfrak{v}(P)$ for all propositional variables P, for all τ in $\mathfrak{T}$ and ω and ω' in $\mathfrak{W}$.
$C - BT$	If $\mathfrak{R}\omega\omega'\tau$ and $\langle\omega',\tau\rangle \in \mathfrak{v}(P)$, then $\langle\omega,\tau\rangle \in \mathfrak{v}(P)$ for all propositional variables P, for all τ in $\mathfrak{T}$ and ω and ω' in $\mathfrak{W}$.

Table 8

'FT' in '$C - FT$' is an abbreviation of 'Forward Transfer', and 'BT' in '$C - BT$' of 'Backward Transfer'. If a model satisfies $C - FT$ and P is a propositional variable, then $P \to \Box P$ is valid in this model. If a model satisfies $C - BT$ and P is a propositional variable, then $\Diamond P \to P$ is valid in this model. If a model satisfies $C - FT$, $C - BT$ and $C - aT$, we can show that the distinctions between what is true, necessarily true and possibly true collapse for propositional variables (and historical modalities), that is, the following equivalences hold: $P \leftrightarrow \Box P \leftrightarrow \Diamond P$. However, note that this does not entail that similar results hold for every P (if P is not a propositional variable) and we cannot show that $P \to \mathbb{U}P$ or $\mathbb{M}P \to P$ are valid. $C - FT$ and $C - BT$ seem to be reasonable conditions if we think of reality as a tree-like structure that branches towards the future.[8]

3.2 The system of a class of models

The conditions mentioned in Section 3.1 can be used to obtain a categorisation of the set of all models into various kinds. We shall say that $\mathfrak{M}(C_1, \dots, C_n)$ is the class of all models that satisfy the conditions $C_1, \dots, C_n$. For example, $\mathfrak{M}(C - aT, C - aB, C - a4, C - dD)$ is the class of all models that satisfy the conditions $C - aT$, $C - aB$, $C - a4$ and $C - dD$. By using this classification of model classes we can define a large set of systems.

Definition 10 *The set of all sentences in a language that are valid in a class of models $\mathfrak{M}$ is the (logical) system of $\mathfrak{M}$, $S(\mathfrak{M})$.*

For example, $S(\mathfrak{M}(C - aT, C - aB, C - a4, C - dD))$ (the system of $\mathfrak{M}(C - aT, C - aB, C - a4, C - dD))$ is the class of sentences (in our language) that are valid in the class of models that satisfy the conditions $C - aT$, $C - aB$, $C - a4$ and $C - dD$.

4 Proof theory

In this section, I will develop a set of tableau systems. The propositional part of these systems is similar to systems introduced by [77] and [54], and the modal part is similar to systems discussed by [65]. The rules for the propositional quantifiers, the axiological rules, and some of the rules that are concerned with the interactions between $<$, R and S are new.[9] For more information about the tableau method and various kinds of tableau systems, see, for example, [32] and [38].

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[8] Note that this does not entail that time itself branches towards the future. For more on $C - FT$ and $C - BT$, see [70].

[9] However, see my [70], [71], [72] and [73].

4.1 Tableau rules

4.1.1 Propositional rules

$\neg\neg$	$\wedge$	$\neg\wedge$
$\neg\neg A, w_it_j$ $\downarrow$ A, w_it_j	$A \wedge B, w_it_j$ $\downarrow$ A, w_it_j B, w_it_j	$\neg(A \wedge B), w_it_j$ $\swarrow \searrow$ $\neg A, w_it_j \quad \neg B, w_it_j$
$\vee$	$\neg\vee$	$\rightarrow$
$A \vee B, w_it_j$ $\swarrow \searrow$ $A, w_it_j \quad B, w_it_j$	$\neg(A \vee B), w_it_j$ $\downarrow$ $\neg A, w_it_j$ $\neg B, w_it_j$	$A \rightarrow B, w_it_j$ $\swarrow \searrow$ $\neg A, w_it_j \quad B, w_it_j$
$\neg \rightarrow$	$\leftrightarrow$	$\neg \leftrightarrow$
$\neg(A \rightarrow B), w_it_j$ $\downarrow$ A, w_it_j $\neg B, w_it_j$	$A \leftrightarrow B, w_it_j$ $\swarrow \searrow$ $A, w_it_j \quad \neg A, w_it_j$ $B, w_it_j \quad \neg B, w_it_j$	$\neg(A \leftrightarrow B), w_it_j$ $\swarrow \searrow$ $A, w_it_j \quad \neg A, w_it_j$ $\neg B, w_it_j \quad B, w_it_j$

Table 9: Propositional rules

4.1.2 Basic temporal rules (b t-rules)

$\mathbb{A}$	$\neg\mathbb{A}$	$\mathbb{S}$	$\neg\mathbb{S}$
$\mathbb{A}A, w_it_j$ $\downarrow$ A, w_it_k for every t_k on the branch	$\neg\mathbb{A}A, w_it_j$ $\downarrow$ $\mathbb{S}\neg A, w_it_j$	$\mathbb{S}A, w_it_j$ $\downarrow$ A, w_it_k where t_k is new to the branch	$\neg\mathbb{S}A, w_it_j$ $\downarrow$ $\mathbb{A}\neg A, w_it_j$
$\mathbb{G}$	$\neg\mathbb{G}$	$\mathbb{F}$	$\neg\mathbb{F}$
$\mathbb{G}A, w_it_j$ $t_j < t_k$ $\downarrow$ A, w_it_k	$\neg\mathbb{G}A, w_it_j$ $\downarrow$ $\mathbb{F}\neg A, w_it_j$	$\mathbb{F}A, w_it_j$ $\downarrow$ $t_j < t_k$ A, w_it_k where t_k is new	$\neg\mathbb{F}A, w_it_j$ $\downarrow$ $\mathbb{G}\neg A, w_it_j$
$\mathbb{H}$	$\neg\mathbb{H}$	$\mathbb{P}$	$\neg\mathbb{P}$
$\mathbb{H}A, w_it_j$ $t_k < t_j$ $\downarrow$ A, w_it_k	$\neg\mathbb{H}A, w_it_j$ $\downarrow$ $\mathbb{P}\neg A, w_it_j$	$\mathbb{P}A, w_it_j$ $\downarrow$ $t_k < t_j$ A, w_it_k where t_k is new	$\neg\mathbb{P}A, w_it_j$ $\downarrow$ $\mathbb{H}\neg A, w_it_j$

Table 10: Basic temporal rules

4.1.3 Basic modal rules

$\mathbb{U}$	$\mathbb{M}$	$\Box$	$\Diamond$
$\mathbb{U}A, w_it_k$ $\downarrow$ A, w_jt_l for any w_j and t_l	$\mathbb{M}A, w_it_k$ $\downarrow$ A, w_jt_l where w_j and t_l are new	$\Box A, w_it_k$ $Rw_iw_jt_k$ $\downarrow$ A, w_jt_k	$\Diamond A, w_it_k$ $\downarrow$ $Rw_iw_jt_k$ A, w_jt_k where w_j is new
$\neg\mathbb{U}$	$\neg\mathbb{M}$	$\neg\Box$	$\neg\Diamond$
$\neg\mathbb{U}A, w_it_k$ $\downarrow$ $\mathbb{M}\neg A, w_it_k$	$\neg\mathbb{M}A, w_it_k$ $\downarrow$ $\mathbb{U}\neg A, w_it_k$	$\neg \Box A, w_it_k$ $\downarrow$ $\Diamond\neg A, w_it_k$	$\neg \Diamond A, w_it_k$ $\downarrow$ $\Box\neg A, w_it_k$

Table 11: Basic modal rules

4.1.4 Basic deontic rules (b d-rules)

$\mathbb{O}$	$\mathbb{P}$	$\neg\mathbb{O}$	$\neg\mathbb{P}$
$\mathbb{O}B, w_it_k$ $Sw_iw_jt_k$ $\downarrow$ B, w_jt_k	$\mathbb{P}B, w_it_k$ $\downarrow$ $Sw_iw_jt_k$ B, w_jt_k where w_j is new	$\neg\mathbb{O}B, w_it_k$ $\downarrow$ $\mathbb{P}\neg B, w_it_k$	$\neg\mathbb{P}B, w_it_k$ $\downarrow$ $\mathbb{O}\neg B, w_it_k$

Table 12: Basic deontic rules

4.1.5 Quantifier rules

$\forall$	$\exists$	$\neg\forall$	$\neg\exists$
$\forall XA, w_it_j$ $\downarrow$ $A[B/X], w_it_j$	$\exists XA, w_it_j$ $\downarrow$ $A[Y/X], w_it_j$	$\neg\forall XA, w_it_j$ $\downarrow$ $\exists X\neg A, w_it_j$	$\neg\exists XA, w_it_j$ $\downarrow$ $\forall X\neg A, w_it_j$

Table 13: Propositional quantifier rules

Note that in ($\forall$), B is any (quantifier-free) formula (in $\mathcal{L}$) that is substitutable for X in A; and in ($\exists$), Y is a propositional variable that is new to the branch.

4.1.6 The CUT-rule and the identity-rules

CUT	$T-TIi$	$T-TIii$	$T-AIi$	$T-AIii$
w_i, t_k	$A(t_i)$	$A(t_i)$	$A(w_i)$	$A(w_i)$
$\swarrow \quad \searrow$	$t_i = t_j$	$t_j = t_i$	$w_i = w_j$	$w_j = w_i$
$A, w_it_k \quad \neg A, w_it_k$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
for every A, w_i and t_k	$A(t_j)$	$A(t_j)$	$A(w_j)$	$A(w_j)$

Table 14: The CUT-rule and the identity-rules

The CUT rule (**Table 14**) is often useful to produce elegant proofs and to derive new rules. However, for our purposes in this paper it is not strictly needed. Note that A can be replaced by any sentence in CUT. 'w_i, t_k' 'means' that both 'w_i' and 't_k' are on the branch. Similar constructions in other rules in this section (Section 4.1) are interpreted similarly.

$T-Ii$ and $T-Iii$ are redundant in any system that does not include $T-FC$, $T-PC$ or $T-C$ (**Table 16**), and $T-AIi$ and $T-AIii$ are redundant in any system that does not include $T-aF$ (**Table 17**) or $T-dF$ (**Table 18**).

($T-TIi$) is interpreted in the following way. $A(t_i)$ is a line in a tableau that includes 't_i', and $A(t_j)$ is like $A(t_i)$ except that 't_i' is replaced by 't_j'. That is, if $A(t_i)$ is A, w_kt_i, then $A(t_j)$ is A, w_kt_j; if $A(t_i)$ is $Rw_kw_lt_i$, then $A(t_j)$ is $Rw_kw_lt_j$; if $A(t_i)$ is $t_i = t_k$, then $A(t_j)$ is $t_j = t_k$, etc. If $A(t_i)$ is A, w_kt_i, we only apply the rule when A is atomic. $T-TIii$, $T-AIi$ and $T-AIii$ are interpreted similarly.

4.1.7 Basic axiological rules

$\bullet E$	$\bullet I$
$\bullet, w_it_j$	$Sw_iw_it_j$
$\downarrow$	$\downarrow$
$Sw_iw_it_j$	$\bullet, w_it_j$

Table 15: $\bullet$-rules

'$\bullet E$' is an abbreviation of '$\bullet$-elimination' and '$\bullet I$' of '$\bullet$-introduction'. Intuitively, '$\bullet E$' says that if '$\bullet$' is true in the possible world denoted by 'w_i' at the time denoted by 't_j', then the world denoted by 'w_i' is deontically accessible from itself at the time denoted by 't_j'; and '$\bullet I$' says that if the world denoted by 'w_i' is deontically accessible from itself at the time denoted by 't_j', then '$\bullet$' is true in the world denoted by 'w_i' at the time denoted by 't_j'.

4.1.8 Temporal accessibility rules (t-rules)

$T - t4$	$T - PD$	$T - FD$
$t_i < t_j$ $t_j < t_k$ $\downarrow$ $t_i < t_k$	t_j $\downarrow$ $t_k < t_j$ where t_k is new	t_j $\downarrow$ $t_j < t_k$ where t_k is new
$T - DE$	$T - FC$	$T - PC$
$t_i < t_j$ $\downarrow$ $t_i < t_k$ $t_k < t_j$ where t_k is new	$t_i < t_j$ $t_i < t_k$ $\swarrow\downarrow\searrow$ $t_j < t_k \; t_j = t_k \; t_k < t_j$	$t_j < t_i$ $t_k < t_i$ $\swarrow\downarrow\searrow$ $t_j < t_k \; t_j = t_k \; t_k < t_j$
$T - C$	$T - UB$	$T - LB$
t_i, t_j $\swarrow \quad \downarrow \quad \searrow$ $t_i < t_j \quad t_i = t_j \quad t_j < t_i$	$t_i < t_j$ $t_i < t_k$ $\downarrow$ $t_j < t_l$ $t_k < t_l$ where t_l is new to the branch	$t_j < t_i$ $t_k < t_i$ $\downarrow$ $t_l < t_j$ $t_l < t_k$ where t_l is new to the branch

Table 16: Temporal accessibility rules

4.1.9 Alethic accessibility rules (a-rules)

$T - aD$	$T - aT$	$T - aB$	$T - aF$	$T - a4$	$T - a5$
w_i, t_k $\downarrow$ $Rw_iw_jt_k$ where w_j is new	w_i, t_k $\downarrow$ $Rw_iw_it_k$	$Rw_iw_jt_k$ $\downarrow$ $Rw_jw_it_k$	$Rw_iw_jt_l$ $Rw_iw_kt_l$ $\downarrow$ $w_j = w_k$	$Rw_iw_jt_l$ $Rw_jw_kt_l$ $\downarrow$ $Rw_iw_kt_l$	$Rw_iw_jt_l$ $Rw_iw_kt_l$ $\downarrow$ $Rw_jw_kt_l$

Table 17: Alethic accessibility rules

4.1.10 Deontic accessibility rules (d-rules)

$T - dD$	$T - dF$	$T - d4$	$T - d5$	$T - \mathbb{O}dT$	$T - \mathbb{O}dB$
w_i, t_k $\downarrow$ $Sw_iw_jt_k$ where w_j is new	$Sw_iw_jt_l$ $Sw_iw_kt_l$ $\downarrow$ $w_j = w_k$	$Sw_iw_jt_l$ $Sw_jw_kt_l$ $\downarrow$ $Sw_iw_kt_l$	$Sw_iw_jt_l$ $Sw_iw_kt_l$ $\downarrow$ $Sw_jw_kt_l$	$Sw_iw_jt_l$ $\downarrow$ $Sw_jw_jt_l$	$Sw_iw_jt_l$ $Sw_jw_kt_l$ $\downarrow$ $Sw_kw_jt_l$

Table 18: Deontic accessibility rules

4.1.11 Alethic-deontic accessibility rules (ad-rules)

$T - \Box\mathbb{O}$	$T - \mathbb{O}\Box\mathbb{O}$	$T - \mathbb{O}\Diamond$	$T - \mathbb{O}\mathbb{O}\Diamond$	
$Sw_iw_jt_k$ $\downarrow$ $Rw_iw_jt_k$	$Sw_iw_jt_l$ $Sw_jw_kt_l$ $\downarrow$ $Rw_jw_kt_l$	w_i, t_k $\downarrow$ $Sw_iw_jt_k$ $Rw_iw_jt_k$ where w_j is new	$Sw_iw_jt_l$ $\downarrow$ $Rw_jw_kt_l$ $Sw_jw_kt_l$ where w_k is new	
$T - ad4$	$T - ad5$	$T - \mathbb{P}\Box P$	$T - \mathbb{O}\Box P$	$T - \Box\mathbb{O}P$
$Rw_iw_jt_l$ $Sw_jw_kt_l$ $\downarrow$ $Sw_iw_kt_l$	$Rw_iw_jt_l$ $Sw_iw_kt_l$ $\downarrow$ $Sw_jw_kt_l$	$Sw_iw_jt_m$ $Rw_iw_kt_m$ $\downarrow$ $Rw_jw_lt_m$ $Sw_kw_lt_m$ where w_l is new	$Rw_iw_jt_m$ $Sw_jw_kt_m$ $\downarrow$ $Sw_iw_lt_m$ $Rw_lw_kt_m$ where w_l is new	$Sw_iw_jt_m$ $Rw_jw_kt_m$ $\downarrow$ $Rw_iw_lt_m$ $Sw_lw_kt_m$ where w_l is new

Table 19: Alethic-deontic accessibility rules

4.1.12 Temporal-alethic accessibility rules

$T - ASP$	$T - AR$
$Rw_iw_jt_l$ $t_k < t_l$ $\downarrow$ $Rw_iw_jt_k$	$Rw_iw_jt_l$ $t_l < t_m$ $Rw_jw_kt_m$ $\downarrow$ $Rw_iw_kt_l$

Table 20: Temporal-alethic accessibility rules

4.1.13 Temporal-deontic accessibility rules

$T - \mathbb{O}\mathbb{G}dT$	$T - \mathbb{O}\mathbb{G}dB$	$T - DR$
$Sw_iw_jt_l$ $t_l < t_m$ $\downarrow$ $Sw_jw_jt_m$	$Sw_iw_jt_l$ $t_l < t_m$ $Sw_jw_kt_m$ $\downarrow$ $Sw_kw_jt_m$	$Sw_iw_jt_l$ $t_l < t_m$ $Sw_jw_kt_m$ $\downarrow$ $Sw_iw_kt_l$

Table 21: Temporal-deontic accessibility rules

4.1.14 Temporal-alethic-deontic accessibility rules

$T - \mathbb{OG}\Box\mathbb{O}$	$T - \mathbb{OGO}\Diamond$
$Sw_iw_jt_l$	$Sw_iw_jt_l$
$t_l < t_m$	$t_l < t_m$
$Sw_jw_kt_m$	$\downarrow$
$\downarrow$	$Rw_jw_kt_m$
$Rw_jw_kt_m$	$Sw_jw_kt_m$
	where w_k
	is new

Table 22: Temporal-alethic-deontic accessibility rules

4.1.15 Transfer rules

$T - FT$	$T - BT$
A, w_it_l	A, w_jt_l
$Rw_iw_jt_l$	$Rw_iw_jt_l$
$\downarrow$	$\downarrow$
A, w_jt_l	A, w_it_l
where A is a propositional variable	where A is a propositional variable

Table 23: Transfer rules

'FT' in '$T - FT$' is an abbreviation of 'Forward Transfer' and 'BT' in '$T - BT$' of 'Backward Transfer'. Note that A is a propositional variable in these rules.

4.2 Tableau systems and some basic proof-theoretical concepts

Definition 11 *A (semantic)* ***tableau*** *is a tree-like structure where the elements in the structure, the nodes in the tree, have the following form: A, w_it_j, where A is a formula in our language and i and j are in $\{0, 1, 2, 3, \ldots\}$, or something of the form $Rw_iw_jt_k$, $Sw_iw_jt_k$, $t_i < t_j$, $t_i = t_j$ or $w_i = w_j$ where i, j, k are in $\{0, 1, 2, 3, \ldots\}$. The first node in the tableau is called the* ***root****. Nodes without successors are called* ***tips*** *or* ***leaves****. A* ***branch*** *is a path from the root to a tip. A branch in a tableau is* ***closed*** *just in case both A, w_it_j and $\neg A, w_it_j$ occur on the branch (for some A, w_i and t_j) or if we have $\neg\top, w_it_j$ or $\bot, w_it_j$ on the branch (for some w_i and t_j); it is* ***open*** *just in case it is not closed. A tableau itself is* ***closed*** *iff every branch in it is closed; it is* ***open*** *iff it is not closed.*

Definition 12 ***Tableau system:*** *A tableau system is a class of tableau rules.* ***Temporal modal (alethic) deontic tableau system:*** *a temporal modal (or alethic) deontic tableau system is a tableau system that includes all propositional rules, all basic* *temporal rules, all basic alethic rules, all basic deontic rules.* ***Extended tableau system:*** *If S is a temporal alethic deontic tableau system, then S_{Ext} is the temporal alethic deontic tableau system S extended by the quantifier rules.*

To make some proofs easier, it is often useful to add the CUT-rule to our systems. $T - TIi$ and $T - TIii$ are included in every system that contains $T - FC$, $T - PC$ or $T - C$, and $T - AIi$ and $T - AIii$ are included in every system that contains $T - aF$ or $T - dF$. The smallest temporal modal deontic tableau system without any accessibility rules is called $\mathcal{S}$. By adding various accessibility rules to $\mathcal{S}$, we obtain a large class of stronger systems.

Let $T_1, \ldots, T_n$ be the (temporal modal deontic) tableau system that includes the tableau rules $T_1, \ldots, T_n$. The initial T may be omitted if it is clear that we are talking about a tableau system. Then, $aTaBa4dD$ is the (temporal modal deontic) tableau system that includes the tableau rules $T - aT$, $T - aB$, $T - a4$, and $T - dD$, etc.

Let S be any system in this paper and let an S-tableau be a tableau generated in accordance with the rules in S. Moreover, let A be a sentence and let Γ be a finite set of sentences. '$\vdash_S A$' says that A is a theorem in S and '$\Gamma \vdash_S A$' says that A is derivable from Γ in S. Then we can define some important proof theoretical concepts in the following way:

Definition 13 ***Proof in a system.*** *A proof of A in S is a closed S-tableau that begins with $\neg A, w_0t_0$.*

Theorem in a system. *A is a theorem in S, $\vdash_S A$, iff there is a proof of A in S, that is, iff there is a closed S-tableau that begins with $\neg A, w_0t_0$. A is a theorem in S iff A is provable in S.*

Derivation in a system. *A derivation of A from Γ in S, is a closed S-tableau whose initial list comprises B, w_0t_0 for every B in Γ and $\neg A, w_0t_0$. The sentences in Γ are the premises and A the conclusion of the derivation. The initial list of a tableau consists of the first nodes in this tableau whose 'satisfiability' we are exploring.*

Proof-theoretic consequence in a system. *A is a proof-theoretic consequence of Γ in S or A is derivable from Γ in S, $\Gamma \vdash_S A$, iff there is a derivation of A from Γ in S, that is, iff there is a closed S-tableau whose initial list comprises B, w_0t_0 for every B in Γ and $\neg A, w_0t_0$.*

Definition 14 ***(The logic of a tableau system)*** *The logic $L(S)$ of a tableau system S is the class of all sentences in $\mathcal{L}$ ($\mathcal{L}_{Ext}$) that are provable in this system.*

For example, $L(aTaBa4dD)$, the logic of $aTaBa4dD$, is the class of all sentences in $\mathcal{L}$ that are provable in $aTaBa4dD$, that is, in the temporal alethic deontic tableau system that includes all the rules that every system of this kind contains and the rules $T-aT$, $T-aB$, $T-a4$ and $T-dD$.

4.3 Some theorems

We are now in a position to prove some theorems. I will focus on some propositions that tell us something about the interactions between the propositional quantifiers, the deontic operators and the constants $\bullet$ and $\star$. However, first we will consider some other interesting theorems.[10]

Theorem 15 ***(The (normative) classification of 'states of affairs')*** *(i)* $\forall A(\mathbb{O}A \vee \mathbb{K}A \vee \mathbb{F}A) = \forall A(\mathbb{O}A \vee (\mathbb{P}A \wedge \mathbb{P}\neg A) \vee \mathbb{F}A)$ *(Everything is either obligatory, optional or forbidden) is a theorem in every system in this paper. (ii)* $\neg\exists A(\mathbb{O}A \wedge \mathbb{K}A) = \neg\exists A(\mathbb{O}A \wedge (\mathbb{P}A \wedge \mathbb{P}\neg A))$ *(nothing is both obligatory and optional) and* $\neg\exists A(\mathbb{F}A \wedge \mathbb{K}A) = \neg\exists A(\mathbb{F}A \wedge (\mathbb{P}A \wedge \mathbb{P}\neg A))$ *(nothing is both forbidden and optional) are theorems in every system in this paper. (iii)* $\neg\exists A(\mathbb{O}A \wedge \mathbb{F}A)$ *(nothing is both obligatory and forbidden) is a theorem in every system that includes dD. (iv) All the following sentences are theorems in every system in this paper:* $\exists X(\mathbb{O}X \wedge \mathbb{F}X) \rightarrow \forall Y\mathbb{O}Y$ *(If something is both obligatory and forbidden, then everything is obligatory),* $\exists X(\mathbb{O}X \wedge \mathbb{F}X) \rightarrow \forall Y\mathbb{F}Y$ *(If something is both obligatory and forbidden, then everything is forbidden) and* $\exists X(\mathbb{O}X \wedge \mathbb{F}X) \rightarrow \forall Y(\mathbb{O}Y \wedge \mathbb{F}Y)$ *(If something is obligatory and forbidden, everything is obligatory and forbidden). (v)* $\forall A((\mathbb{O}A \wedge \Box A) \vee (\mathbb{O}A \wedge \neg\Box A) \vee (\mathbb{P}A \wedge \mathbb{P}\neg A) \vee (\mathbb{F}A \wedge \neg ⟠ A) \vee (\mathbb{F}A \wedge ⟠ A))$ *is a theorem in every system in this paper. (vi) In every system that includes dD, the following sentences are 'mutually exclusive':* $(\mathbb{O}A \wedge \Box A)$*,* $(\mathbb{O}A \wedge \neg\Box A)$*,* $(\mathbb{P}A \wedge \mathbb{P}\neg A)$*,* $(\mathbb{F}A \wedge \neg ⟠ A)$ *and* $(\mathbb{F}A \wedge ⟠ A)$*. (vii) In those systems, we can prove the following sentences:* $\neg\exists A((\mathbb{O}A \wedge \Box A) \wedge (\mathbb{O}A \wedge \neg\Box A))$ *(Nothing is both obligatory and necessary and obligatory and non-necessary),* $\neg\exists A((\mathbb{O}A \wedge \neg\Box A) \wedge (\mathbb{P}A \wedge \mathbb{P}\neg A))$ *(Nothing is both obligatory and non-necessary and optional),* $\neg\exists A((\mathbb{P}A \wedge \mathbb{P}\neg A) \wedge (\mathbb{F}A \wedge \neg ⟠ A))$ *(Nothing is both optional and forbidden and non-impossible (possible)), etc. Intuitively, this means that every 'state of affairs' belongs to one and only one box in Table 24.*

Proof. Straightforward. ■

1	2	3	4	5
$\mathbb{O}A \wedge \Box A$	$\mathbb{O}A \wedge \neg\Box A$	$\mathbb{P}A \wedge \mathbb{P}\neg A$	$\mathbb{F}A \wedge \neg ⟠ A$	$\mathbb{F}A \wedge ⟠ A$

Table 24

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[10] In this section 'system' can mean either 'system' or 'extended system' depending on the context.

We have now considered a possible (normative) classification of all ‘states of affairs’. However, we have not seen if there are things that belong in the various boxes in Table 24. So far, our results are compatible with the proposition that they are empty. Is there anything that is obligatory and necessary? Is there anything that is obligatory but non-necessary? Is there anything that is optional? Etc. The following theorem shows that we can prove that something is obligatory and that something is forbidden (in every system in this paper), and that something is permitted (in every system that includes *dD*). This entails that at least some boxes in Table 24 are non-empty.

Theorem 16 ***(The existence of obligations, prohibitions and permissions)***

(i) $\exists A \mathbb{O} A$ *(Something is obligatory), (ii)* $\exists A \mathbb{F} A$ *(Something is forbidden), and (iii)* $\neg \forall A \mathbb{P} A$ *(Not everything is permitted) are theorems in every system in this paper. All the following sentences are theorems in every system that includes dD: (iv)* $\neg \forall A \mathbb{O} A$ *(Not everything is obligatory), (v)* $\neg \forall A \mathbb{F} A$ *(Not everything is forbidden) and (vi)* $\exists A \mathbb{P} A$ *(Something is permitted). Obviously, the following equivalences can be proved in every system in this paper: (vii)* $\exists A \mathbb{F} A \leftrightarrow \neg \forall A \mathbb{P} A$ *(Something is forbidden iff not everything is permitted), (viii)* $\exists A \mathbb{P} A \leftrightarrow \neg \forall A \mathbb{F} A$ *(Something is permitted iff not everything is forbidden), (ix)* $\neg \forall A \mathbb{O} A \leftrightarrow \exists A \neg \mathbb{O} A$ *(Not everything is obligatory iff something is not obligatory), (x)* $\forall X \mathbb{F} X \leftrightarrow \neg \exists X \mathbb{P} X$ *(Everything is forbidden iff nothing is permitted), (xi)* $\forall X \mathbb{P} X \leftrightarrow \neg \exists X \mathbb{F} X$ *(Everything is permitted iff nothing is forbidden). The following sentences are theorems in every system in this paper: (xii)* $\exists X(\mathbb{O} X \wedge \mathbb{F} X) \rightarrow \neg \exists A \mathbb{P} A$ *(If something is both obligatory and forbidden, then nothing is permitted), (xiii)* $\exists A \mathbb{P} A \rightarrow \neg \exists X(\mathbb{O} X \wedge \mathbb{F} X)$ *(If something is permitted, then nothing is both obligatory and forbidden). (xiv)* $\exists A \mathbb{O} A \wedge \exists A \mathbb{F} A$ *is a theorem in every system, and* $\exists A \mathbb{O} A \wedge \exists A \mathbb{P} A \wedge \exists A \mathbb{F} A$ *is a theorem in every system that includes dD.*

Proof. Straightforward. ■

Theorem 16 seems to be philosophically quite interesting since it can be used as an argument against at least some forms of (normative) nihilism. Nihilism is often taken to imply that nothing is obligatory, that everything is permitted and that nothing is forbidden, that is, that the following proposition holds:

Proposition 17 ***((Normative) Nihilism)*** *(i) If nihilism is true, then everything is permitted. (ii) If nihilism is true, then nothing is obligatory. (iii) If nihilism is true, then nothing is forbidden.*[11]

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[11] The expression ‘is true’ in proposition 17 is an abbreviation of ‘is true in a possible world at a moment in time (in a model)’, and ‘is false’ is interpreted similarly. Not every form of nihilism entails these claims. Some nihilists might, for example, assert that all normative sentences lack truth-

If nihilism has these implications, then we can use Theorem 16 to show that nihilism is false. The arguments are very easy. Here are three examples. (i) If nihilism is true, then everything is permitted. It is not the case that everything is permitted (by Theorem 16). Hence nihilism is not true. (ii) If nihilism is true then nothing is obligatory. Some things are obligatory (by Theorem 16). Hence, nihilism is not true. (iii) If nihilism is true, then nothing is forbidden. Some things are forbidden (by Theorem 16). Hence, nihilism is not true.

These arguments refute every kind of nihilism that entails that nothing is obligatory, it refutes every kind of nihilism that entails that everything is permitted, and it refutes every kind of nihilism that entails that nothing is forbidden. Furthermore, if nihilism necessarily entails that nothing is obligatory and/or that everything is permitted and/or that nothing is forbidden, then nihilism (in this sense) is necessarily false, it cannot be true. For it is necessarily the case that some things are obligatory, that not everything is permitted and that not nothing is forbidden (these propositions follow immediately from Theorem 16), and if it is necessary that *A* implies not-*B* and it is necessary that *B* then it is impossible that *A*.

A nihilist might respond to these arguments in at least two ways. She can (1) define 'nihilism' in such a way that it does not entail the problematic theses, or (2) use 'obligatory', 'permitted' and 'forbidden' in some non-standard way and reject every normal deontic logic of the kind used in this paper. (2) seems unreasonable. At the least one would like to know in what sense the 'nihilist' uses the terms if she rejects every kind of normal deontic logic. If (1) is the case 'nihilism' might not have any problematic normative consequences. The arguments above thus suggest that nihilism is either (1) rather harmless, or (2) implausible or needs to be combined with some alternative theory of our normative concepts. Some things are obligatory. Hence, nihilism is true only if it is not the case that nothing is obligatory if nihilism is true. It is not the case that everything is permitted. Hence, nihilism is true only if it is not the case that everything is permitted if nihilism is true. Some things are forbidden. Hence, nihilism is true only if it is not the case that nothing is forbidden if nihilism is true.

Of course, one could argue that this 'refutation' of nihilism is not particularly interesting, since we have not shown that something contingent is obligatory [$\exists X(\triangledown X \wedge \mathbb{O}X) = \exists X((\Diamond X \wedge \Diamond\neg X) \wedge \mathbb{O}X)$], nor that it is not the case that everything that is contingent is permitted [$\neg\forall X(\triangledown X \rightarrow \mathbb{P}X) = \neg\forall X((\Diamond X \wedge \Diamond\neg X) \rightarrow \mathbb{P}X)$], and we have not shown that something contingent is forbidden [$\exists X(\triangledown X \wedge \mathbb{F}X) = \exists X((\Diamond X \wedge \Diamond\neg X) \wedge \mathbb{F}X)$]. Furthermore, we have not established the following propositions: $\exists X(\mathbb{O}X \wedge \neg\Box X)$ (There is something that is obligatory but not neces-

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values. So, they are neither true nor false. However, the main purpose of this paper is not to discuss nihilism. Accordingly, I will not consider every version of this thesis.

sary) and $\exists X(\mathbb{F}X \wedge \neg \diamondminus X)$ (There is something that is forbidden but not impossible). All we have shown, so far, is that $\exists X \mathbb{O} X$ and $\exists X \mathbb{F} X$ are theorems in every system. But this result is compatible with the assertion that boxes 2 and 4 (and 3) in Table 24 are empty.

The following sentences are theorems in every system: $\exists X(\mathbb{O}X \wedge \Box X)$ (There is something that is obligatory and necessary) and $\exists X(\mathbb{F}X \wedge \diamondminus X)$ (There is something that is forbidden and impossible). $\exists X(\mathbb{O}X \wedge \Box X)$ entails $\exists X \mathbb{O} X$, and $\exists X(\mathbb{F}X \wedge \diamondminus X)$ entails $\exists X \mathbb{F} X$. So, as far as we know, $\exists X \mathbb{O} X$ might be true 'because' $\exists X(\mathbb{O}X \wedge \Box X)$ is true and $\exists X \mathbb{F} X$ might be true 'because' $\exists X(\mathbb{F}X \wedge \diamondminus X)$ is true. Hence, we know that box 1 and box 5 in Table 24 are non-empty, but we do not know if the other boxes are non-empty.

A nihilist could define a new concept of obligation in terms of $\mathbb{O}$ and $\Box$ as follows: $\mathbb{O}'A =_{df} (\mathbb{O}A \wedge \neg \Box A)$, and a new concept of prohibition in the following way: $\mathbb{F}'A =_{df} (\mathbb{F}A \wedge \neg \diamondminus A)$ (the idea of defining obligation and prohibition in this way goes back at least to [4]). She could then point out that we have not shown that $\exists A \mathbb{O}' A$, nor that $\exists A \mathbb{F}' A$. Obligations and prohibitions that are necessarily fulfilled are not particularly interesting. In our practical moral lives, we are primarily interested in obligations and prohibitions with a contingent content. Can we show that there is something optional, that there is something that is obligatory but not necessary and that there is something that is forbidden but not impossible? Before we turn to this question, let us introduce a couple of definitions.

Definition 18 ***(Rigorism and Optionalism)*** *(i) (Normative) Rigorism is true iff* $\forall X \mathbb{N} X$ *[=* $\forall X(\mathbb{O}X \vee \mathbb{F}X)$*] (Everything is nonoptional, that is, obligatory or forbidden). (ii) (Normative) Optionalism is true iff* $\exists X \mathbb{K} X$ *[=* $\exists X(\mathbb{P}X \wedge \mathbb{P}\neg X)$*] (There is something that is optional).*[12]

Theorem 19 ***(Relationships between Rigorism and Optionalism)*** *All the following sentences are theorems in every system in this paper:* $\exists X \mathbb{K} X \to \neg \forall X \mathbb{N} X$ *[=* $\exists X(\mathbb{P}X \wedge \mathbb{P}\neg X) \to \neg \forall X(\mathbb{O}X \vee \mathbb{F}X)$*] (If something is optional, then not everything is obligatory or forbidden),* $\forall X \mathbb{N} X \to \neg \exists X \mathbb{K} X$ *[=* $\forall X(\mathbb{O}X \vee \mathbb{F}X) \to \neg \exists X(\mathbb{P}X \wedge \mathbb{P}\neg X)$*] (If everything is obligatory or forbidden, then nothing is optional),* $\exists X \mathbb{K} X \leftrightarrow \neg \forall X \mathbb{N} X$ *[=* $\exists X(\mathbb{P}X \wedge \mathbb{P}\neg X) \leftrightarrow \neg \forall X(\mathbb{O}X \vee \mathbb{F}X)$*] (Something is optional iff not everything is obligatory or forbidden),* $\forall X \mathbb{N} X \leftrightarrow \neg \exists X \mathbb{K} X$ *[=* $\forall X(\mathbb{O}X \vee \mathbb{F}X) \leftrightarrow \neg \exists X(\mathbb{P}X \wedge \mathbb{P}\neg X)$*] (Everything is obligatory or forbidden iff nothing is optional).*

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[12] Again, the expression 'is true' in definition 18 is an abbreviation of 'is true in a possible world at a moment in time (in a model)', and 'is false' is interpreted similarly. Rigorism can be true at some world-moment pair (in some model) and false at some other world-moment pair (in this model).

Proof. Straightforward. ■

Theorem 19 does not tell us whether or not anything is optional. Most systems in this paper are compatible both with the proposition that there is something that is optional and with the proposition that nothing is optional. However, in some systems, we can show that Rigorism is true and that Optionalism is false. The following theorem establishes this:

Theorem 20 ***(Rigorism and the falsity of Optionalism)*** *The sentences* $\forall X(\mathbb{O}X \vee \mathbb{F}X)$ *and* $\neg\exists X(\mathbb{P}X \wedge \mathbb{P}\neg X)$ *are theorems in every system that includes* $T - dF$*. This entails that Rigorism is true and that Optionalism is false in every possible world at every moment in time in every class of models that satisfies C − dF (by the soundness results in Section 5).*

Proof. Straightforward. ■

Intuitively, Rigorism is not a plausible thesis. Since we can prove that everything is either obligatory or forbidden in every system that includes $T - dF$, we must reject this rule if there is something that is optional. $T - dF$ is a derivable rule in every system that includes $T - aF$ and $T - \Box\mathbb{O}$. So, if we want to reject $T - dF$, we must also reject either $T - aF$ or $T - \Box\mathbb{O}$. $T - aF$ is formally similar to $T - dF$. In every system that includes this rule we can prove that everything is either necessary or impossible. Let us consider some theorems that can be established with the help of this rule. However, we will first introduce a couple of technical terms.

Definition 21 ***(Noncontingentism and Contingentism)*** *(i) Noncontingentism is true iff* $\forall X \triangle X$ *[=* $\forall X(\Box X \vee ⟠X)$*] (Everything is necessary or impossible). (ii) Contingentism is true iff* $\exists X \triangledown X$ *[=* $\exists X(\Diamond X \wedge \Diamond\neg X)$*] (There is something that is contingent).*[13]

Theorem 22 ***(T − aF, Contingentism and Noncontingentism)*** *(i)* $\forall X(\Box X \vee ⟠X)$ *and* $\neg\exists X(\Diamond X \wedge \Diamond\neg X)$ *are theorems in every system that includes* $T - aF$*. This entails that Noncontingentism is true and Contingentism is false in every possible world at every moment in time in every class of models that satisfies C − aF (by the soundness results in Section 5). (ii) In every system that includes* $T - aF$ *and* $T - \Box\mathbb{O}$*,* $\forall X(\mathbb{O}X \vee \mathbb{F}X)$ *and* $\neg\exists X(\mathbb{P}X \wedge \mathbb{P}\neg X)$ *are theorems. (iii)* $\exists X \triangledown X \leftrightarrow \neg\forall X \triangle X$ *and* $\forall X \triangle X \leftrightarrow \neg\exists X \triangledown X$ *are theorems in every system in this paper.*

Proof. (i) Left to the reader. (ii) Since we can prove that $\forall X(\Box X \vee ⟠X)$ is a theorem in every system that includes $T - aF$, and $\Box A$ entails $\mathbb{O}A$ and $⟠A$ entails

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[13] Again, the expression ‘is true’ in definition 21 is an abbreviation of ‘is true in a possible world at a moment in time (in a model)’, etc.

$\mathbb{F}A$ in every system that includes $T - \Box\mathbb{O}$, this result is obvious. (iii) Left to the reader. ■

The following theorem tells us something about the relationships between obligations with a non-necessary content and prohibitions with a non-impossible (possible) content:

Theorem 23 ***(Obligations with a non-necessary content and prohibitions with a non-impossible content)*** *In every system in this paper, we can prove the following:* $\exists X(\mathbb{O}X \wedge \neg \Box X) \to \exists X(\mathbb{F}X \wedge \neg \Diamond X)$, $\exists X(\mathbb{F}X \wedge \neg \Diamond X) \to \exists X(\mathbb{O}X \wedge \neg \Box X)$ *and* $\exists X(\mathbb{O}X \wedge \neg \Box X) \leftrightarrow \exists X(\mathbb{F}X \wedge \neg \Diamond X)$. *So, if there is something that is obligatory but not necessary, then there is something that is forbidden but not impossible, and vice versa.*

Proof. Straightforward. ■

Obviously, from this result it immediately follows that $\neg\exists X(\mathbb{O}X \wedge \neg \Box X) \leftrightarrow \neg\exists X(\mathbb{F}X \wedge \neg \Diamond X)$ is a theorem in every system in this paper. Hence, if box 2 in Table 24 is empty, box 4 in this table is empty, and vice versa. As far as I can see $\exists X(\mathbb{O}X \wedge \neg \Box X)$ and $\exists X(\mathbb{P}X \wedge \mathbb{P}\neg X)$ are independent of each other in most systems in this paper, and so are $\exists X(\mathbb{F}X \wedge \neg \Diamond X)$ and $\exists X(\mathbb{P}X \wedge \mathbb{P}\neg X)$. However, in some systems they are equivalent, as is shown by the following theorem:

Theorem 24 ***(Obligations with a non-necessary content, prohibitions with a non-impossible content and Optionalism)*** *(i)* $\Diamond\exists X(\mathbb{O}X \wedge \neg\Box X)$, $\Diamond\exists X(\mathbb{F}X \wedge \neg\Diamond X)$ *and* $\Diamond\exists X(\mathbb{P}X \wedge \mathbb{P}\neg X)$ *are theorems in every system in this paper that includes* $T - \Box\mathbb{O}$ *and* $T - aF$. *Hence, (ii) in those systems all the following sentences are equivalent:* $\exists X(\mathbb{O}X \wedge \neg \Box X)$, $\exists X(\mathbb{F}X \wedge \neg \Diamond X)$ *and* $\exists X(\mathbb{P}X \wedge \mathbb{P}\neg X)$.

Proof. (i). Left to the reader. (ii) follows immediately from (i), since all impossible propositions are necessarily equivalent. ■

We have now seen that there are some systems that exclude the existence of optional states of affairs. Hence, there are systems according to which box 3 in Table 24 is empty. Since $\exists X(\mathbb{P}X \wedge \mathbb{P}\neg X)$ is equivalent with $\exists X(\mathbb{O}X \wedge \neg \Box X)$ and with $\exists X(\mathbb{F}X \wedge \neg \Diamond X)$ in some systems, it follows that there are systems according to which the boxes 2, 3 and 4 in Table 24 are empty. It is left as an open question whether there are any systems according to which 1, 3 and 5 are non-empty and 2 and 4 are empty.

Most systems are compatible with Optionalism and with the existence of obligations with a non-necessary content and prohibitions with a possible content. However, usually we cannot prove that something is optional (for example). We will now consider some consequences that follow if we assume that there is something optional, something obligatory that is non-necessary or something forbidden that is possible.

Theorem 25 ***(Optionalism, Contingentism, etc.)*** *In every system in this paper that includes* $T - \Box\mathbb{O}$, $\exists X(\mathbb{P}X \wedge \mathbb{P}\neg X) \rightarrow \exists X(\Diamond A \wedge \Diamond\neg A)$ *(If something is optional, then something is contingent) is a theorem [Optionalism entails Contingentism]. In every system in this paper that includes* $T - \mathbb{O}\Diamond$, $\exists X(\mathbb{O}X \wedge \neg \Box X) \rightarrow \exists X(\Diamond A \wedge \Diamond\neg A)$. *(If something that is not necessary is obligatory, then there is something contingent) and* $\exists X(\mathbb{F}X \wedge \neg \diamondminus X) \rightarrow \exists X(\Diamond A \wedge \Diamond\neg A)$ *(If something that is not impossible is forbidden, then there is something contingent) are theorems.*

Proof. Straightforward. ■

The following theorem tells us something about obligations and prohibitions with a contingent content.

Theorem 26 ***(Obligations and prohibitions with contingent content, etc.)*** *In every system in this paper, we can prove that something contingent is obligatory iff something contingent is forbidden:* $\exists X((\Diamond X \wedge \Diamond\neg X) \wedge \mathbb{O}X)$ *iff* $\exists X((\Diamond X \wedge \Diamond\neg X) \wedge \mathbb{F}X)$.

In every system in this paper that includes $T - \mathbb{O}\Diamond$, *the following sentences are theorems:* $\exists X(\mathbb{O}X \wedge \neg \Box X) \rightarrow \exists X((\Diamond X \wedge \Diamond\neg X) \wedge \mathbb{O}X)$ *(If something is obligatory but not necessary, then something contingent is obligatory) and* $\exists X(\mathbb{F}X \wedge \neg \diamondminus X) \rightarrow \exists X((\Diamond X \wedge \Diamond\neg X) \wedge \mathbb{F}X)$ *(If something is forbidden but not impossible, then something contingent is forbidden).*

Proof. Straightforward. ■

Theorem 27 ***(Barcan-like formulas)*** *All the following sentences are theorems in every system in this paper:* $\forall X\mathbb{O}X \rightarrow \mathbb{O}\forall XX$ *(If everything ought to be true, then it ought to be that everything is true).* $\mathbb{O}\forall XX \rightarrow \forall X\mathbb{O}X$ *(If it ought to be that everything is true, then everything ought to be true).* $\mathbb{O}\forall XX \leftrightarrow \forall X\mathbb{O}X$ *(It ought to be that everything is true iff everything ought to be true).* $\exists X\mathbb{P}X \rightarrow \mathbb{P}\exists XX$ *(If something is permitted, then it is permitted that something is true).* $\mathbb{P}\exists XX \rightarrow \exists X\mathbb{P}X$ *(If it is permitted that something is true, then something is permitted).* $\mathbb{P}\exists XX \leftrightarrow \exists X\mathbb{P}X$ *(It is permitted that something is true iff something is permitted).* $\exists X\mathbb{O}X \rightarrow \mathbb{O}\exists XX$ *(If something ought to be true, then it ought to be that something is true).* $\mathbb{P}\forall XX \rightarrow \forall X\mathbb{P}X$ *(If it is permitted that everything is true, then everything is permitted).*

Proof. Straightforward. Note that $\mathbb{P}\forall XX \rightarrow \forall X\mathbb{P}X$ is vacuously valid in every system in this paper, since $\neg\mathbb{P}\forall XX$ is a theorem in every system, and that $\exists X\mathbb{O}X \rightarrow \mathbb{O}\exists XX$ is vacuously valid in every system since $\mathbb{O}\exists XX$ is a theorem in every system. Furthermore, $\forall X\mathbb{O}X \rightarrow \mathbb{O}\forall XX$, $\mathbb{O}\forall XX \rightarrow \forall X\mathbb{O}X$ and $\mathbb{O}\forall XX \leftrightarrow \forall X\mathbb{O}X$ are vacuously valid in every system in this paper that includes *dD*; for in every system of this kind both $\neg\forall X\mathbb{O}X$ and $\neg\mathbb{O}\forall XX$ are theorems; and $\exists X\mathbb{P}X \rightarrow \mathbb{P}\exists XX$,

$\mathbb{P}\exists XX \to \exists X\mathbb{P}X$ and $\mathbb{P}\exists XX \leftrightarrow \exists X\mathbb{P}X$ are vacuously valid in every system in this paper that includes dD; for in every system of this kind both $\exists X\mathbb{P}X$ and $\mathbb{P}\exists XX$ are theorems. ■

We have now established some Barcan-like formulas. In fact, we can prove several other similar sentences.

Theorem 28 ***(More Barcan-like formulas)*** *Let* $\blacksquare$ *be* $\mathbb{U}$, $\Box$, $\mathbb{A}$, $\mathbb{G}$, $\mathbb{H}$, $[\mathbb{G}]$ *or* $[\mathbb{H}]$, *and let* $\blacklozenge$ *be the dual of* $\blacksquare$*. Then, all the following sentences (schemas) are theorems in every system in this paper:* $\forall X \blacksquare X \to \blacksquare\forall XX$, $\blacksquare\forall XX \to \forall X \blacksquare X$, $\blacksquare\forall XX \leftrightarrow \forall X \blacksquare X$, $\exists X \blacklozenge X \to \blacklozenge\exists XX$, $\blacklozenge\exists XX \to \exists X \blacklozenge X$, $\blacklozenge\exists XX \leftrightarrow \exists X \blacklozenge X$, $\exists X \blacksquare X \to \blacksquare\exists XX$, $\blacklozenge\forall XX \to \forall X \blacklozenge X$.

Proof. Straightforward. ■

Theorem 29 ***(Some fundamental theorems about The Moral Law and The Good)*** *(i)* $\mathbb{O}\star$ *(It ought to be that The Moral Law is fulfilled) and* $\mathbb{O}\bullet$ *(It ought to be that The Good is realised) are theorems in every (extended) system in this paper that includes* $T - \mathbb{O}dT$*. (ii)* $\mathbb{P}\star$ *(It is permitted that The Moral Law is fulfilled) and* $\mathbb{P}\bullet$ *(It is permitted that The Good is realised) are theorems in every (extended) system in this paper that includes* $T - dD$ *and* $T - \mathbb{O}dT$*. (iii)* $\Diamond\star$ *(It is possible that The Moral Law is fulfilled) and* $\Diamond\bullet$ *(It is possible that The Good is realised) are theorems in every (extended) system in this paper that includes* $T - dD$, $T - \Box\mathbb{O}$ *and* $T - \mathbb{O}dT$ *(and in every (extended) system that includes* $T - \mathbb{O}\Diamond$ *and* $T - \mathbb{O}dT$*). (iv)* $\mathbb{O}\mathbb{G}\star$ *(It ought to be that it is always going to be the case that The Moral Law is fulfilled) and* $\mathbb{O}\mathbb{G}\bullet$ *(It ought to be that it is always going to be the case that The Good is realised) are theorems in every (extended) system in this paper that includes* $T - \mathbb{O}\mathbb{G}dT$*. (v)* $\mathbb{O}[\mathbb{G}]\star$ *(It ought to be that it is and that it is always going to be the case that The Moral Law is fulfilled) and* $\mathbb{O}[\mathbb{G}]\bullet$ *(It ought to be that it is and that it is always going to be the case that The Good is realised) are theorems in every (extended) system in this paper that includes* $T - \mathbb{O}dT$ *and* $T - \mathbb{O}\mathbb{G}dT$*.*

Proof. Straightforward. ■

Theorem 30 ***(Necessary conditions for The Good)*** *All the following sentences are theorems in every (extended) system in this paper: (a)* $\bullet \to \forall A(\mathbb{O}A \to A)$ *(The Good is realised only if everything that ought to be the case is the case; or, in other words, The Good is realised only if every obligation is fulfilled), (b)* $\bullet \to \forall A(A \to \mathbb{P}A)$ *(The Good is realised only if everything is right), (c)* $\bullet \to \neg\exists A(A \wedge \mathbb{F}A)$ *(The Good is realised only if nothing is wrong), (d)* $\bullet \to \neg\exists A(\mathbb{O}A \wedge \neg A)$ *(The Good is realised only if no obligation is violated), (e)* $\bullet \to \neg\exists A(\mathbb{F}A \wedge A)$ *(The Good is realised only if no prohibition is violated), (f)* $\bullet \to \forall A(\mathbb{F}A \to \neg A)$ *(The Good is realised only if every prohibition is respected).*

Proof. We prove (a) and leave the rest to the reader.

$$\begin{array}{c}
(1)\ \neg(\bullet \rightarrow \forall A(\mathbb{O}A \rightarrow A)), w_0t_0 \\
(2)\ \bullet, w_0t_0\ [1, \neg \rightarrow] \\
(3)\ \neg\forall A(\mathbb{O}A \rightarrow A), w_0t_0\ [1, \neg \rightarrow] \\
(4)\ \exists A\neg(\mathbb{O}A \rightarrow A), w_0t_0\ [3, \neg\forall] \\
(5)\ \neg(\mathbb{O}X \rightarrow X), w_0t_0\ [4, \exists] \\
(6)\ \mathbb{O}X, w_0t_0\ [5, \neg \rightarrow] \\
(7)\ \neg X, w_0t_0\ [5, \neg \rightarrow] \\
(8)\ Sw_0w_0t_0\ [2, \bullet E] \\
(9)\ X, w_0t_0\ [6, 8, \mathbb{O}] \\
(10)\ *\ [7, 9]
\end{array}$$

■

Theorem 31 ***(Sufficient conditions for The Good)*** *All the following sentences are theorems in every (extended) system in this paper that includes* $T - \mathbb{O}dT$*: (a)* $\forall A(\mathbb{O}A \rightarrow A) \rightarrow \bullet$ *(If everything that ought to be the case is the case, then The Good is realised; or, in other words, if every obligation is fulfilled, then The Good is realised), (b)* $\forall A(A \rightarrow \mathbb{P}A) \rightarrow \bullet$ *(If everything is right, then The Good is realised), (c)* $\neg\exists A(A \wedge \mathbb{F}A) \rightarrow \bullet$ *(If nothing is wrong, then The Good is realised), (d)* $\neg\exists A(\mathbb{O}A \wedge \neg A) \rightarrow \bullet$ *(If no obligation is violated, then The Good is realised), (e)* $\neg\exists A(\mathbb{F}A \wedge A) \rightarrow \bullet$ *(If no prohibition is violated, then The Good is realised), (f)* $\forall A(\mathbb{F}A \rightarrow \neg A) \rightarrow \bullet$ *(If every prohibition is respected, then The Good is realised).*

Proof. We prove (a) and leave the rest to the reader.

$$\begin{array}{c}
(1)\ \neg(\forall A(\mathbb{O}A \rightarrow A) \rightarrow \bullet), w_0t_0 \\
(2)\ \forall A(\mathbb{O}A \rightarrow A), w_0t_0\ [1, \neg \rightarrow] \\
(3)\ \neg\bullet, w_0t_0\ [1, \neg \rightarrow] \\
(4)\ \mathbb{O}\bullet \rightarrow \bullet, w_0t_0\ [2, \forall] \\
\swarrow \qquad\qquad \searrow \\
\begin{array}{cc}
(5)\ \neg\mathbb{O}\bullet, w_0t_0\ [4, \rightarrow] & (6)\ \bullet, w_0t_0\ [4, \rightarrow] \\
(7)\ \mathbb{P}\neg\bullet, w_0t_0\ [5, \neg\mathbb{O}] & (8)\ *\ [3, 6] \\
(9)\ Sw_0w_1t_0\ [7, \mathbb{P}] & \\
(10)\ \neg\bullet, w_1t_0\ [7, \mathbb{P}] & \\
(11)\ Sw_1w_1t_0\ [9, T - \mathbb{O}dT] & \\
(12)\ \bullet, w_1t_0\ [11, \bullet I] & \\
(13)\ *\ [10, 12] &
\end{array}
\end{array}$$

■

Theorem 32 ***(Necessary and sufficient conditions for The Good)*** *All the following sentences are theorems in every (extended) system in this paper that includes* $T - \mathbb{O}dT$*: (a)* $\bullet \leftrightarrow \forall A(\mathbb{O}A \to A)$ *(The Good is realised iff everything that ought to be the case is the case; or, in other words, iff every obligation is fulfilled), (b)* $\bullet \leftrightarrow \forall A(A \to \mathbb{P}A)$ *(The Good is realised iff everything is right), (c)* $\bullet \leftrightarrow \neg\exists A(A \wedge \mathbb{F}A)$ *(The Good is realised iff nothing is wrong), (d)* $\bullet \leftrightarrow \neg\exists A(\mathbb{O}A \wedge \neg A)$ *(The Good is realised iff no obligation is violated), (e)* $\bullet \leftrightarrow \neg\exists A(\mathbb{F}A \wedge A)$ *(The Good is realised iff no prohibition is violated), (f)* $\bullet \leftrightarrow \forall A(\mathbb{F}A \to \neg A)$ *(The Good is realised iff every prohibition is respected).*

Proof. I will prove part (ii) (directly) and leave the rest to the reader. Obviously, Theorem 32 can also be derived (indirectly) from Theorems 30 and 31.

$$
\begin{array}{ccc}
\multicolumn{3}{c}{(1)\ \neg(\bullet \leftrightarrow \forall A(A \to \mathbb{P}A)), w_0t_0} \\
\multicolumn{3}{c}{\swarrow \qquad\qquad \searrow} \\
(2)\ \bullet, w_0t_0\ [1, \neg \leftrightarrow] & \multicolumn{2}{c}{(3)\ \neg\bullet, w_0t_0\ [1, \neg \leftrightarrow]} \\
(4)\ \neg\forall A(A \to \mathbb{P}A), w_0t_0\ [1, \neg \leftrightarrow] & \multicolumn{2}{c}{(5)\ \forall A(A \to \mathbb{P}A), w_0t_0\ [1, \neg \leftrightarrow]} \\
(6)\ \exists A\neg(A \to \mathbb{P}A), w_0t_0\ [4, \neg\forall] & \multicolumn{2}{c}{(7)\ \neg\bullet \to \mathbb{P}\neg\bullet, w_0t_0\ [5, \forall]} \\
(8)\ \neg(X \to \mathbb{P}X), w_0t_0\ [6, \exists] & \swarrow & \searrow \\
(9)\ X, w_0t_0\ [8, \neg \to] & (10)\ \neg\neg\bullet, w_0t_0\ [7, \to] & (11)\ \mathbb{P}\neg\bullet, w_0t_0\ [7, \to] \\
(12)\ \neg\mathbb{P}X, w_0t_0\ [8, \neg \to] & (13)\ \ast\ [3, 10] & (14)\ Sw_0w_1t_0\ [11, \mathbb{P}] \\
(15)\ \mathbb{O}\neg X, w_0t_0\ [12, \neg\mathbb{P}] & & (16)\ \neg\bullet, w_1t_0\ [11, \mathbb{P}] \\
(17)\ Sw_0w_0t_0\ [2, \bullet E] & & (18)\ Sw_1w_1t_0\ [14, \mathbb{O}dT] \\
(19)\ \neg X, w_0t_0\ [15, 17, \mathbb{O}] & & (20)\ \bullet, w_1t_0\ [18, \bullet I] \\
(21)\ \ast\ [9, 19] & & (22)\ \ast\ [16, 20]
\end{array}
$$

■

Theorem 33 ***(Necessary and sufficient conditions for (the fulfillment of) The Moral Law)*** *The following equivalences hold in every (extended) system in this paper: (i)* $\star \leftrightarrow \forall A(\mathbb{O}A \to A)$*. The Moral Law is fulfilled iff everything that ought to be the case is the case. In other words, The Moral Law is fulfilled iff every obligation is fulfilled. (ii)* $\star \leftrightarrow \neg\exists A(A \wedge \mathbb{F}A)$*. The Moral Law is fulfilled iff nothing is wrong. (iii)* $\star \leftrightarrow \forall A(A \to \mathbb{P}A)$*. The Moral Law is fulfilled iff everything is right. (iv)* $\star \leftrightarrow \neg\exists A(\mathbb{O}A \wedge \neg A)$*. The Moral Law is fulfilled iff no obligation is violated. (v)* $\star \leftrightarrow \neg\exists A(\mathbb{F}A \wedge A)$*. The Moral Law is fulfilled iff no prohibition is violated. (vi)* $\star \leftrightarrow \forall A(\mathbb{F}A \to \neg A)$*. The Moral Law is fulfilled iff every prohibition is respected. (vii) All the following formulas are logically equivalent in every (extended) system in this paper:* $\star$*,* $\forall A(\mathbb{O}A \to A)$*,* $\neg\exists A(A \wedge \mathbb{F}A)$*,* $\forall A(A \to \mathbb{P}A)$*,* $\neg\exists A(\mathbb{O}A \wedge \neg A)$*,* $\neg\exists A(\mathbb{F}A \wedge A)$*,* $\forall A(\mathbb{F}A \to \neg A)$*. This means, for example, that everything that ought* *to be the case is the case iff nothing is wrong, nothing is wrong iff everything is right, everything is right iff everything that ought to be the case is the case, etc.*

Proof. (i) is true by definition. I will prove (ii) and leave the rest to the reader. $\star \leftrightarrow \neg\exists A(A \wedge \mathbb{F}A)$ is by definition equivalent with $\forall A(\mathbb{O}A \rightarrow A) \leftrightarrow \neg\exists A(A \wedge \mathbb{F}A)$. So, to prove the former sentence it is sufficient to establish the latter. Here is the tableau ('*MP*' stands for the derived rule *Modus Ponens*):

$$(1)\ \neg(\forall A(\mathbb{O}A \rightarrow A) \leftrightarrow \neg\exists A(A \wedge \mathbb{F}A)), w_1t_1$$

$$\swarrow \qquad\qquad \searrow$$

(2) $\forall A(\mathbb{O}A \rightarrow A), w_1t_1$ [1, $\neg \leftrightarrow$]
(4) $\neg\neg\exists A(A \wedge \mathbb{F}A), w_1t_1$ [1, $\neg \leftrightarrow$]
(6) $\exists A(A \wedge \mathbb{F}A), w_1t_1$ [4, $\neg\neg$]
(8) $X \wedge \mathbb{F}X, w_1t_1$ [6, $\exists$]
(10) X, w_1t_1 [8, $\wedge$]
(12) $\mathbb{F}X, w_1t_1$ [8, $\wedge$]
(14) $\mathbb{O}\neg X, w_1t_1$ [12, $\mathbb{F}$]
(16) $\mathbb{O}\neg X \rightarrow \neg X, w_1t_1$ [2, $\forall$]
(18) $\neg X, w_1t_1$ [14, 16, *MP*]
(19) $*$ [10, 18]

(3) $\neg\forall A(\mathbb{O}A \rightarrow A), w_1t_1$ [1, $\neg \leftrightarrow$]
(5) $\neg\exists A(A \wedge \mathbb{F}A), w_1t_1$ [1, $\neg \leftrightarrow$]
(7) $\exists A\neg(\mathbb{O}A \rightarrow A), w_1t_1$ [3, $\neg\forall$]
(9) $\forall A\neg(A \wedge \mathbb{F}A), w_1t_1$ [5, $\neg\exists$]
(11) $\neg(\mathbb{O}X \rightarrow X), w_1t_1$ [7, $\exists$]
(13) $\mathbb{O}X, w_1t_1$ [11, $\neg \rightarrow$]
(15) $\neg X, w_1t_1$ [11, $\neg \rightarrow$]
(17) $\neg(\neg X \wedge \mathbb{F}\neg X), w_1t_1$ [9, $\forall$]

$$\swarrow \qquad\qquad \searrow$$

(20) $\neg\neg X, w_1t_1$ [17, $\neg\wedge$]
(22) $*$ [15, 20]

(21) $\neg\mathbb{F}\neg X, w_1t_1$ [17, $\neg\wedge$]
(23) $\mathbb{P}\neg X, w_1t_1$ [21, $\neg\mathbb{F}$]
(24) $Sw_1w_2t_1$ [23, $\mathbb{P}$]
(25) $\neg X, w_2t_1$ [23, $\mathbb{P}$]
(26) X, w_2t_1 [13, 24, $\mathbb{O}$]
(27) $*$ [25, 26]

■

Theorem 34 ***(The Good and The Moral Law)*** *(i) In every system in this paper, The Good is realised only if The Moral Law is fulfilled, that is, the following sentence is a theorem:* $\bullet \rightarrow \star$. *(ii) In every system in this paper that includes* $\mathbb{O}dT$*, The Good is realised if The Moral Law is fulfilled, that is, the following sentence is a theorem:* $\star \rightarrow \bullet$. *(iii) In every system in this paper that includes* $\mathbb{O}dT$*, The Good is realised iff The Moral Law is fulfilled, that is, the following sentence is a theorem:* $\bullet \leftrightarrow \star$. *(iv) In every system in this paper that includes* $\mathbb{O}dT$*, all the following formulas are logically equivalent:* $\bullet$, $\star$, $\forall A(\mathbb{O}A \rightarrow A)$, $\neg\exists A(A \wedge \mathbb{F}A)$, $\forall A(A \rightarrow \mathbb{P}A)$, $\neg\exists A(\mathbb{O}A \wedge \neg A)$, $\neg\exists A(\mathbb{F}A \wedge A)$, $\forall A(\mathbb{F}A \rightarrow \neg A)$. *(System = extended system in this theorem.)*

Proof. Straightforward. ■

Theorem 35 ***(Normative propositions and The Good)*** *In every (extended) system in this paper that includes* $\mathbb{O}dT$, $\Box\mathbb{O}$ *and* $ad4$, *all the following sentences are theorems: (i)* $\mathbb{O}A \leftrightarrow \Box(\bullet \rightarrow A)$ *(It ought to be the case that A iff it is necessary that The Good is realised only if A is the case), (ii)* $\mathbb{P}A \leftrightarrow \Diamond(\bullet \wedge A)$ *(It is permitted that A iff it is possible that The Good is realised and that A is the case), (iii)* $\mathbb{F}A \leftrightarrow \neg\Diamond(\bullet \wedge A)$ *(It is forbidden that A iff it is impossible that The Good is realised and that A is the case), (iv)* $\mathbb{O}A \leftrightarrow (\bullet \Rightarrow A)$ *(It ought to be the case that A iff (the fact that) The Good (is realised) strictly implies A), (v)* $\mathbb{P}A \leftrightarrow (A \bigcirc \bullet)$ *(It is permitted that A iff A is consistent with (the fact that) The Good (is realised)), (vi)* $\mathbb{F}A \leftrightarrow (A \ominus \bullet)$ *(It is forbidden that A iff A is inconsistent with (the fact that) The Good (is realised)).*

Proof. I will prove (i) and leave the rest to the reader. Note that all of the rules $\mathbb{O}dT$, $\Box\mathbb{O}$ and $ad4$ are essential in the proofs of all the parts of this theorem.

$(1)\ \neg(\mathbb{O}A \leftrightarrow \Box(\bullet \rightarrow A)), w_0t_0$

$\swarrow \qquad \searrow$

$(2)\ \mathbb{O}A, w_0t_0\ [1, \neg \leftrightarrow]$

$(4)\ \neg \Box (\bullet \rightarrow A), w_0t_0\ [1, \neg \leftrightarrow]$

$(6)\ \Diamond\neg(\bullet \rightarrow A), w_0t_0\ [4, \neg\Box]$

$(8)\ Rw_0w_1t_0\ [6, \Diamond]$

$(10)\ \neg(\bullet \rightarrow A), w_1t_0\ [6, \Diamond]$

$(12)\ \bullet, w_1t_0\ [10, \neg \rightarrow]$

$(14)\ \neg A, w_1t_0\ [10, \neg \rightarrow]$

$(16)\ Sw_1w_1t_0\ [12, \bullet E]$

$(17)\ Sw_0w_1t_0\ [8, 16, ad4]$

$(20)\ A, w_1t_0\ [2, 17, \mathbb{O}]$

$(23)\ *\ [14, 20]$

$(3)\ \neg\mathbb{O}A, w_0t_0\ [1, \neg \leftrightarrow]$

$(5)\ \Box(\bullet \rightarrow A), w_0t_0\ [1, \neg \leftrightarrow]$

$(7)\ \mathbb{P}\neg A, w_0t_0\ [3, \neg\mathbb{O}]$

$(9)\ Sw_0w_1t_0\ [7, \mathbb{P}]$

$(11)\ \neg A, w_1t_0\ [7, \mathbb{P}]$

$(13)\ Rw_0w_1t_0\ [9, \Box\mathbb{O}]$

$(15)\ \bullet \rightarrow A, w_1t_0\ [5, 13, \Box]$

$\swarrow \qquad \searrow$

$(18)\ \neg\bullet, w_1t_0\ [15, \rightarrow]$

$(21)\ Sw_1w_1t_0\ [9, \mathbb{O}dT]$

$(24)\ \bullet, w_1t_0\ [21, \bullet I]$

$(25)\ *\ [18, 24]$

$(19)\ A, w_1t_0\ [15, \rightarrow]$

$(22)\ *\ [11, 19]$

■

Theorem 36 ***(Normative propositions and The Moral Law)*** *In every (extended) system in this paper that includes* $\mathbb{O}dT$, $\Box\mathbb{O}$ *and* $ad4$, *all the following sentences are theorems: (i)* $\mathbb{O}A \leftrightarrow \Box(\star \rightarrow A)$ *(It ought to be the case that A iff it is necessary that The Moral Law is fulfilled only if A is the case), (ii)* $\mathbb{P}A \leftrightarrow \Diamond(\star \wedge A)$ *(It is permitted that A iff it is possible that The Moral Law is fulfilled and that A is the case), (iii)* $\mathbb{F}A \leftrightarrow \neg \Diamond (\star \wedge A)$ *(It is forbidden that A iff it is impossible that The Moral Law is fulfilled and that A is the case), (iv)* $\mathbb{O}A \leftrightarrow (\star \Rightarrow A)$ *(It ought to be the case that A iff (the fact that) The Moral Law (is fulfilled) strictly implies A), (v)* $\mathbb{P}A \leftrightarrow (A \bigcirc \star)$ *(It is permitted that A iff A is consistent with (the fact that)*

The Moral Law (is fulfilled)), (vi) $\mathbb{F}A \leftrightarrow (A \ominus \star)$ *(It is forbidden that A iff A is inconsistent with (the fact that) The Moral Law (is fulfilled)).*

Proof. I will prove (ii) and leave the rest to the reader. Once again, note that all of the rules $\mathbb{O}dT$, $\Box\mathbb{O}$ and $ad4$ are essential in the proofs of all the parts of this theorem. $\mathbb{P}A \leftrightarrow \Diamond(\star \wedge A)$ is by definition equivalent with $\mathbb{P}A \leftrightarrow \Diamond(\forall A(\mathbb{O}A \rightarrow A) \wedge A)$. So, to prove (ii), we prove this sentence.

$$\begin{array}{cccc}
\multicolumn{4}{c}{(1)\ \neg(\mathbb{P}A \leftrightarrow \Diamond(\forall A(\mathbb{O}A \rightarrow A) \wedge A)), w_0t_0} \\
\multicolumn{4}{c}{\swarrow \qquad\qquad \searrow} \\
\multicolumn{2}{c}{(2)\ \mathbb{P}A, w_0t_0} & \multicolumn{2}{c}{(3)\ \neg\mathbb{P}A, w_0t_0} \\
\multicolumn{2}{c}{(4)\ \neg\Diamond(\forall A(\mathbb{O}A \rightarrow A) \wedge A), w_0t_0} & \multicolumn{2}{c}{(5)\ \Diamond(\forall A(\mathbb{O}A \rightarrow A) \wedge A), w_0t_0} \\
\multicolumn{2}{c}{(6)\ \Box\neg(\forall A(\mathbb{O}A \rightarrow A) \wedge A), w_0t_0} & \multicolumn{2}{c}{(7)\ \mathbb{O}\neg A, w_0t_0} \\
\multicolumn{2}{c}{(8)\ Sw_0w_1t_0} & \multicolumn{2}{c}{(9)\ Rw_0w_1t_0} \\
\multicolumn{2}{c}{(10)\ A, w_1t_0} & \multicolumn{2}{c}{(11)\ \forall A(\mathbb{O}A \rightarrow A) \wedge A, w_1t_0} \\
\multicolumn{2}{c}{(12)\ Rw_0w_1t_0} & \multicolumn{2}{c}{(13)\ \forall A(\mathbb{O}A \rightarrow A), w_1t_0} \\
\multicolumn{2}{c}{(14)\ \neg(\forall A(\mathbb{O}A \rightarrow A) \wedge A), w_1t_0} & \multicolumn{2}{c}{(15)\ A, w_1t_0} \\
\swarrow & \searrow & \multicolumn{2}{c}{(16)\ \mathbb{O}\neg A \rightarrow \neg A, w_1t_0} \\
(17)\ \neg\forall A(\mathbb{O}A \rightarrow A), w_1t_0 & (18)\ \neg A, w_1t_0 & \swarrow & \searrow \\
(19)\ \exists A\neg(\mathbb{O}A \rightarrow A), w_1t_0 & (20)\ * & (21)\ \neg\mathbb{O}\neg A, w_1t_0 & (22)\ \neg A, w_1t_0 \\
(23)\ \neg(\mathbb{O}X \rightarrow X), w_1t_0 & & (24)\ \mathbb{P}\neg\neg A, w_1t_0 & (25)\ * \\
(26)\ \mathbb{O}X, w_1t_0 & & (27)\ Sw_1w_2t_0 & \\
(28)\ \neg X, w_1t_0 & & (29)\ \neg\neg A, w_2t_0 & \\
(30)\ Sw_1w_1t_0 & & (31)\ Sw_0w_2t_0 & \\
(32)\ X, w_1t_0 & & (33)\ \neg A, w_2t_0 & \\
(34)\ * & & (35)\ * &
\end{array}$$

■

Theorem 35 proves that all deontic operators in principle are definable in all systems in this paper that include $\mathbb{O}dT$, $\Box\mathbb{O}$ and $ad4$. Let S be a system that contains these rules. Then, $\mathbb{O}A$ can be defined as $\Box(\bullet \rightarrow A)$ in S since $\mathbb{O}A \leftrightarrow \Box(\bullet \rightarrow A)$ is a theorem in S; $\mathbb{P}A$ can be defined as $\Diamond(\bullet \wedge A)$ in S since $\mathbb{P}A \leftrightarrow \Diamond(\bullet \wedge A)$ is provable in S, etc. So, in those systems, it is not necessary to treat the deontic operators as primitive. Theorem 36 is similar to Theorem 35. However, since we have defined $\star$ in terms of $\mathbb{O}$, we cannot use $\star$ to 'eliminate' all deontic operators by defining $\mathbb{O}$ in terms of $\star$ and the alethic operators, etc. Since the equivalences in Theorem 35 do not hold in every system, the deontic operators are not definable in terms of $\bullet$ and the alethic operators in every system. Hence, it is reasonable to treat (at least some of) of them as primitive. Nevertheless, $\mathbb{O}dT$, $\Box\mathbb{O}$ and $ad4$ seem reasonable to me (even though I will not try to defend them in this paper).

5 Soundness and completeness theorems

In this section, I will show that every tableau systems (without propositional quantifiers) in this paper is sound and complete with respect to its semantics and that every augmented or extended system is sound with respect to its semantics.

Let us begin by defining these concepts.

Definition 37 ***(Soundness and completeness)*** *Let $S = T-A_1, ..., T-A_n$ be a temporal alethic deontic tableau system as defined in Section 4.2 (where $T-A_1, ..., T-A_n$ are the non-basic tableau rules in S). Then we shall say that the class of models, $\mathfrak{M}$, corresponds to S iff $\mathfrak{M} = \mathfrak{M}(C - A_1, ..., C - A_n)$.*

Now, let S be any system in this paper. Then, S is sound with respect to $\mathfrak{M}$ iff $\Gamma \vdash_S A$ entails $\mathfrak{M}, \Gamma \Vdash A$, and S is complete with respect to $\mathfrak{M}$ iff $\mathfrak{M}, \Gamma \Vdash A$ entails $\Gamma \vdash_S A$ (where $\mathfrak{M}$ corresponds to S).

5.1 Soundness theorem

Let $\mathcal{M} = \langle \mathfrak{W}, \mathfrak{T}, <, \mathfrak{R}, \mathfrak{S}, \mathfrak{v} \rangle$ be any model and $\mathcal{B}$ any branch of a tableau. Then $\mathcal{B}$ is satisfiable in $\mathcal{M}$ iff there is a function f from $w_0, w_1, w_2, \ldots$ to $\mathfrak{W}$, and a function g from $t_0, t_1, t_2, \ldots$ to $\mathfrak{T}$ such that

(i) A is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$, for every node $A, w_i t_j$ on $\mathcal{B}$;

(ii) if $Rw_i w_j t_k$ is on $\mathcal{B}$, then $\mathfrak{R} f(w_i) f(w_j) g(t_k)$ in $\mathcal{M}$;

(iii) if $S w_i w_j t_k$ is on $\mathcal{B}$, then $\mathfrak{S} f(w_i) f(w_j) g(t_k)$ in $\mathcal{M}$;

(iv) if $t_i < t_j$ is on $\mathcal{B}$, then $g(t_i) < g(t_j)$ in $\mathcal{M}$;

(v) if $t_i = t_j$ is on $\mathcal{B}$, then $g(t_i) = g(t_j)$ in $\mathcal{M}$;

(vi) if $w_i = w_j$ is on $\mathcal{B}$, then $f(w_i) = f(w_j)$ in $\mathcal{M}$.

If these conditions are fulfilled, we say that f and g show that $\mathcal{B}$ is satisfiable in $\mathcal{M}$.

Lemma 38 ***(Soundness Lemma I)*** *Let $\mathcal{B}$ be any branch of a tableau and $\mathcal{M}$ be any model. If $\mathcal{B}$ is satisfiable in $\mathcal{M}$ and a tableau rule is applied to it, then there is an extension of $\mathcal{B}$, $\mathcal{B}'$, such that $\mathcal{B}'$ is satisfiable in $\mathcal{M}$.*

Proof. The proof is by induction on the height of the derivation. I will only consider some of the steps to illustrate the method.

Let f and g be functions that show that the branch $\mathcal{B}$ is satisfiable in $\mathcal{M}$.

($\mathbb{O}$). Suppose that $\mathbb{O}D, w_i t_k$, and $S w_i w_j t_k$ are on $\mathcal{B}$, and that we apply the $\mathbb{O}$-rule. Then we get an extension of $\mathcal{B}$ that includes $D, w_j t_k$. Since $\mathcal{B}$ is satisfiable in $\mathcal{M}$, $\mathbb{O}D$ is true in $f(w_i)$ at $g(t_k)$. Moreover, for any w_i and w_j such that $S w_i w_j t_k$ is on $\mathcal{B}$, $\mathfrak{S} f(w_i) f(w_j) g(t_k)$. Consequently, by the truth conditions for $\mathbb{O}D$, D is true in $f(w_j)$ at $g(t_k)$.

($\mathbb{F}$). Suppose that $\mathbb{F}D, w_it_j$ is on $\mathcal{B}$ and that we apply the $\mathbb{F}$-rule to get an extension of $\mathcal{B}$ that includes nodes of the form $t_j < t_k$ and D, w_it_k. Since $\mathcal{B}$ is satisfiable in $\mathcal{M}$, $\mathbb{F}D$ is true in $f(w_i)$ at $g(t_j)$. Hence, for some τ in $\mathfrak{T}$, $g(t_j) < \tau$ and D is true in $f(w_i)$ at τ [by the truth conditions for $\mathbb{F}D$]. Let g' be the same as g except that $g'(t_k) = \tau$. Since g and g' differ only at t_k, f and g' show that $\mathcal{B}$ is satisfiable in $\mathcal{M}$. Furthermore, by definition $g'(t_j) < g'(t_k)$, and D is true in $f(w_i)$ at $g'(t_k)$.

($T - \mathbb{O}\mathbb{G}dT$). Suppose that $Sw_iw_jt_l$ and $t_l < t_m$ are on $\mathcal{B}$, and that we apply ($T - \mathbb{O}\mathbb{G}dT$) to give an extended branch containing $Sw_jw_jt_m$. Since $\mathcal{B}$ is satisfiable in $\mathcal{M}$, $\mathfrak{S}f(w_i)f(w_j)g(t_l)$ and $g(t_l) < g(t_m)$. Hence, $\mathfrak{S}f(w_j)f(w_j)g(t_m)$, for $\mathcal{M}$ satisfies condition $C - \mathbb{O}\mathbb{G}dT$. Consequently, the extension of $\mathcal{B}$ is satisfiable in $\mathcal{M}$.

($T - \mathbb{O}\mathbb{G}\mathbb{O}\Diamond$). Suppose that $Sw_iw_jt_l$ and $t_l < t_m$ are on $\mathcal{B}$, and that we apply ($T - \mathbb{O}\mathbb{G}\mathbb{O}\Diamond$) to give an extended branch containing $Rw_jw_kt_m$ and $Sw_jw_kt_m$, where w_k is new. Since $\mathcal{B}$ is satisfiable in $\mathcal{M}$, $\mathfrak{S}f(w_i)f(w_j)g(t_l)$ and $g(t_l) < g(t_m)$. Accordingly, for some ω in $\mathfrak{W}$, $\mathfrak{R}f(w_j)\omega g(t_m)$ and $\mathfrak{S}f(w_j)\omega g(t_m)$, since $\mathcal{M}$ satisfies condition $C - \mathbb{O}\mathbb{G}\mathbb{O}\Diamond$. Let f' be the same as f except that $f'(w_k) = \omega$. Since w_k does not occur on $\mathcal{B}$, f' and g show that $\mathcal{B}$ is satisfiable in $\mathcal{M}$. Furthermore, $\mathfrak{R}f'(w_j)f'(w_k)g(t_m)$ and $\mathfrak{S}f'(w_j)f'(w_k)g(t_m)$ by construction. Hence, f' and g show that the extension of $\mathcal{B}$ is satisfiable in $\mathcal{M}$. ■

Theorem 39 ***(Soundness Theorem I)*** *Every (non-extended) system S in this paper is sound with respect to its semantics.*

Proof. Assume that B does not follow from Γ in $\mathfrak{M}$, where $\mathfrak{M}$ is the class of models that corresponds to S. Then every premise in Γ is true and the conclusion B false in some world ω at some time τ in some model $\mathcal{M}$ in $\mathfrak{M}$. Consider an S-tableau whose first nodes consists of A, w_0t_0 for every $A \in \Gamma$ and $\neg B, w_0t_0$, where 'w_0' refers to ω and 't_0' refers to τ. The initial list in this tableau is satisfiable in $\mathcal{M}$. Every time we apply a rule to our tree it produces at least one extension that is also satisfiable in $\mathcal{M}$ (by the Soundness Lemma). Consequently, we can find a whole branch such that every initial section of this branch is satisfiable in $\mathcal{M}$. Suppose that this branch is closed. Then some sentence is both true and false in some possible world at some time in $\mathcal{M}$. But this is impossible. So, the whole tableau is open. It follows that B is not derivable from Γ in S. In conclusion, if B is derivable from Γ in S, then B follows from Γ in $\mathfrak{M}$. ■

5.2 Completeness theorem

In this section, I will show that every (non-extended) system in this paper is complete with respect to its semantics. First, however, I will define the concept of an induced model (other important concepts are defined elsewhere in this paper or are used in a standard way).

Definition 40 *(**Induced Model**) Suppose that $\mathcal{B}$ is an open and complete branch of a tableau, that I is the set of numbers on $\mathcal{B}$ immediately preceded by a 't' and that C is the set of numbers on $\mathcal{B}$ immediately preceded by a 'w'. Let $i \rightleftharpoons j$ iff $i = j$, or '$t_i = t_j$' or '$t_j = t_i$' is on $\mathcal{B}$. $\rightleftharpoons$ is an equivalence relation and $[i]$ is the equivalence class of i. Let $i \sim j$ iff $i = j$, or '$w_i = w_j$' or '$w_j = w_i$' is on $\mathcal{B}$. Again, $\sim$ is an equivalence relation and $[i]$ is the equivalence class of i.*

The model $\mathcal{M} = \langle \mathfrak{W}, \mathfrak{T}, <, \mathfrak{R}, \mathfrak{S}, \mathfrak{v} \rangle$ induced by $\mathcal{B}$ is defined as follows. $\mathfrak{W} = \{\omega_{[i]} : i \in C\}$, $\mathfrak{T} = \{\tau_{[i]} : i \in I\}$, $\tau_{[i]} < \tau_{[j]}$ iff $t_i < t_j$ occurs on $\mathcal{B}$, $\mathfrak{R}\omega_{[i]}\omega_{[j]}\tau_{[k]}$ iff $Rw_iw_jt_k$ occurs on $\mathcal{B}$ and $\mathfrak{S}\omega_{[i]}\omega_{[j]}\tau_{[k]}$ iff $Sw_iw_jt_k$ occurs on $\mathcal{B}$. Suppose A is a propositional variable. Then if A, w_it_j occurs on $\mathcal{B}$, then A is true in $\omega_{[i]}$ at $\tau_{[j]}$ ($\langle \omega_{[i]}, \tau_{[j]} \rangle \in \mathfrak{v}(A)$). $\top$ is true in every possible world at every moment in time and $\bot$ is false in every possible world at every moment in time.

If our tableau system includes neither $T - FC$, $T - PC$ nor $T - C$, $\rightleftharpoons$ is reduced to identity and $[i] = \{i\}$. Hence, in such systems, we may take $\mathfrak{T}$ to be $\{\tau_i : t_i \textit{ occurs on } \mathcal{B}\}$ and dispense with the equivalence classes. Likewise, if our tableau system includes neither $T - aF$ nor $T - dF$, $\sim$ is reduced to identity and $[i] = \{i\}$. Consequently, in such systems, we may take $\mathfrak{W}$ to be $\{\omega_i : w_i \textit{ occurs on } \mathcal{B}\}$ and dispense with the equivalence classes.

Lemma 41 *(**Completeness Lemma**) Let $\mathcal{B}$ be an open branch in a complete tableau and let $\mathcal{M}$ be a model induced by $\mathcal{B}$. Then, for every formula A:*

(i) if A, w_it_j is on $\mathcal{B}$, then $\mathcal{M}, \omega_{[i]}, \tau_{[j]} \Vdash A$, and
(ii) if $\neg A, w_it_j$ is on $\mathcal{B}$, then $\mathcal{M}, \omega_{[i]}, \tau_{[j]} \nVdash A$.

Proof. The proof is by induction on the complexity of A.

(i) Atomic formulas. Propositional variables (and $\top$). The result is true by definition.

($\bullet$). Suppose that $\bullet, w_it_j$ is on $\mathcal{B}$. Since the tableau is complete the $\bullet E$-rule has been applied and $Sw_iw_it_j$ is on $\mathcal{B}$. By the induction hypothesis, $\mathfrak{S}\omega_{[i]}\omega_{[i]}\tau_{[j]}$. Hence, $\bullet$ is true in $\omega_{[i]}$ at $\tau_{[j]}$, as required.

Truth-functional connectives. Straightforward.

Temporal, modal and deontic operators. Let us consider three examples.

($\mathbb{U}$). Suppose $\mathbb{U}D, w_it_k$ is on $\mathcal{B}$. Then, since the branch is complete, the $\mathbb{U}$-rule has been applied to $\mathbb{U}D, w_it_k$ and for every w_j and t_l, D, w_jt_l is on $\mathcal{B}$. By the induction hypothesis, D is true in every $\omega_{[j]}$ at every $\tau_{[l]}$. It follows that $\mathbb{U}D$ is true in $\omega_{[i]}$ at $\tau_{[k]}$, as required.

($\mathbb{F}$). Suppose $\mathbb{F}D, w_i t_j$ is on $\mathcal{B}$. Since the branch is complete $\mathbb{F}$ has been applied to $\mathbb{F}D, w_i t_j$. So, for some new t_k, $t_j < t_k$ and $D, w_i t_k$ occur on $\mathcal{B}$. By the induction hypothesis, $\tau_{[j]} < \tau_{[k]}$, and D is true in $\omega_{[i]}$ at $\tau_{[k]}$. Hence, $\mathbb{F}D$ is true in $\omega_{[i]}$ at $\tau_{[j]}$, as required.

($\mathbb{O}$). Suppose $\mathbb{O}D, w_i t_k$ is on $\mathcal{B}$. Since the branch is complete $\mathbb{O}$ has been applied to $\mathbb{O}D, w_i t_k$. Hence, for all w_j such that $S w_i w_j t_k$ is on the branch, $D, w_j t_k$ occurs on $\mathcal{B}$. By the induction hypothesis, for all $\omega_{[j]}$ such that $\mathfrak{S}\omega_{[i]}\omega_{[j]}\tau_{[k]}$, D is true in $\omega_{[j]}$ at $\tau_{[k]}$. It follows that $\mathbb{O}D$ is true in $\omega_{[i]}$ at $\tau_{[k]}$, as required.

(ii) Let us consider two examples. The other steps are similar.

($\neg\mathbb{F}$). Suppose $\neg\mathbb{F}D, w_i t_j$ is on $\mathcal{B}$. Since the branch is complete $\neg\mathbb{F}$ has been applied to $\neg\mathbb{F}D, w_i t_j$ and we have $\mathbb{G}\neg D, w_i t_j$. Again, since the branch is complete, the $\mathbb{G}$-rule has been applied to $\mathbb{G}\neg D, w_i t_j$ and for every t_k such that $t_j < t_k$ is on $\mathcal{B}$, $\neg D, w_i t_k$ is on $\mathcal{B}$. By the induction hypothesis, D is false in $\omega_{[i]}$ at any time $\tau_{[k]}$ such that $\tau_{[k]}$ is later than $\tau_{[j]}$. It follows that $\mathbb{F}D$ is false in $\omega_{[i]}$ at $\tau_{[j]}$, as required.

($\neg\mathbb{O}$). Suppose $\neg\mathbb{O}D, w_i t_k$ is on $\mathcal{B}$. Since the branch is complete $\neg\mathbb{O}$ has been applied to $\neg\mathbb{O}D, w_i t_k$. Hence, $\mathbb{P}\neg D, w_i t_k$ is on $\mathcal{B}$. Furthermore, since $\mathcal{B}$ is complete $\mathbb{P}$ has been applied to $\mathbb{P}\neg D, w_i t_k$. Consequently, for some w_j, $S w_i w_j t_k$ and $\neg D, w_j t_k$ occur on $\mathcal{B}$. By the induction hypothesis, $\mathfrak{S}\omega_{[i]}\omega_{[j]}\tau_{[k]}$ and D is false in $\omega_{[j]}$ at $\tau_{[k]}$. It follows that $\mathbb{O}D$ is false in $\omega_{[i]}$ at $\tau_{[k]}$, as required. ■

Theorem 42 ***(Completeness Theorem)*** *Every (non-extended) system in this paper is complete with respect to its semantics.*

Proof. First, I will prove that the theorem holds for our weakest temporal modal deontic tableau system $\mathcal{S}$. Then, I will extend the theorem to all stronger (non-extended) systems. Let $\mathfrak{M}$ be the class of models that corresponds to $\mathcal{S}$.

Suppose that D is not derivable from Γ in $\mathcal{S}$. Then it is not the case that there is a closed $\mathcal{S}$-tableau that begins with $A, w_0 t_0$ for every A in Γ and $\neg D, w_0 t_0$. Let $\mathcal{T}$ be a complete $\mathcal{S}$-tableau whose first nodes comprises $A, w_0 t_0$ for every A in Γ and $\neg D, w_0 t_0$. Obviously, $\mathcal{T}$ is open. It follows that there is at least one open branch, $\mathcal{B}$, in $\mathcal{T}$. According to the model induced by $\mathcal{B}$, all the premises in Γ are true and D false in $\omega_{[0]}$ at $\tau_{[0]}$. Hence, it is not the case that D follows from Γ in $\mathfrak{M}$. Consequently, if D follows from Γ in $\mathfrak{M}$, then D is derivable from Γ in $\mathcal{S}$.

I will now prove that all extensions of $\mathcal{S}$ are complete with respect to their semantics. To show this we have to verify that the model induced by the open branch $\mathcal{B}$ is of the right kind in every case. First, we must go through every single semantic condition and prove that the induced model is of the right kind. Then we combine our proofs. Let us consider some steps to illustrate the method.

$C - FC$. Suppose that $\tau_{[i]} < \tau_{[j]}$ and $\tau_{[i]} < \tau_{[k]}$. Then $t_i < t_j$ and $t_i < t_k$ occur on $\mathcal{B}$ [by the definition of an induced model]. Since $\mathcal{B}$ is complete ($T - FC$) has been applied. Hence, $t_j < t_k$ or $t_j = t_k$ or $t_k < t_j$ occurs on $\mathcal{B}$. If $t_j = t_k$ is on $\mathcal{B}$, $j \rightleftharpoons k$, and if $j \rightleftharpoons k$, then $[j] = [k]$. It follows that $\tau_{[i]} < \tau_{[k]}$, $\tau_{[j]} = \tau_{[k]}$ or $\tau_{[k]} < \tau_{[j]}$, as required [by the definition of an induced model].

$C - a4$. Suppose that $\mathfrak{R}\omega_{[i]}\omega_{[j]}\tau_{[l]}$ and $\mathfrak{R}\omega_{[j]}\omega_{[k]}\tau_{[l]}$. Then, both $Rw_iw_jt_l$ and $Rw_jw_kt_l$ occur on $\mathcal{B}$ [by the definition of an induced model]. Since $\mathcal{B}$ is complete, $(T - a4)$ has been applied and $Rw_iw_kt_l$ occurs on $\mathcal{B}$. It follows that $\mathfrak{R}\omega_{[i]}\omega_{[k]}\tau_{[l]}$, as required [by the definition of an induced model].

$C - dD$. Suppose that $\omega_{[i]}$ is in $\mathfrak{W}$ and that $\tau_{[k]}$ is in $\mathfrak{T}$. Then w_i and t_k occur on $\mathcal{B}$ [by the definition of an induced model]. Since $\mathcal{B}$ is complete $(T - dD)$ has been applied. Hence, for some w_j, $Sw_iw_jt_k$ is on $\mathcal{B}$. Accordingly, for some $\omega_{[j]}$, $\mathfrak{S}\omega_{[i]}\omega_{[j]}\tau_{[k]}$, as required [by the definition of an induced model].

$C - ad4$. Assume that $\mathfrak{R}\omega_{[i]}\omega_{[j]}\tau_{[l]}$ and $\mathfrak{S}\omega_{[j]}\omega_{[k]}\tau_{[l]}$. Then, both $Rw_iw_jt_l$ and $Sw_jw_kt_l$ occur on $\mathcal{B}$ [by the definition of an induced model]. Since $\mathcal{B}$ is complete, $(T - ad4)$ has been applied and $Sw_iw_kt_l$ occurs on $\mathcal{B}$. Hence, $\mathfrak{S}\omega_{[i]}\omega_{[k]}\tau_{[l]}$, as required [by the definition of an induced model].

$C - \mathbb{O}\mathbb{G}dT$. Suppose that $\mathfrak{S}\omega_{[i]}\omega_{[j]}\tau_{[l]}$ and $\tau_{[l]} < \tau_{[m]}$. Then $Sw_iw_jt_l$ and $t_l < t_m$ occur on $\mathcal{B}$ [by the definition of an induced model]. Since $\mathcal{B}$ is complete $(T - \mathbb{O}\mathbb{G}dT)$ has been applied. Hence, $Sw_jw_jt_m$ is on $\mathcal{B}$. It follows that $\mathfrak{S}\omega_{[j]}\omega_{[j]}\tau_{[m]}$, as required [by the definition of an induced model].

$C - DR$. Suppose that $\mathfrak{S}\omega_{[i]}\omega_{[j]}\tau_{[l]}$, $\tau_{[l]} < \tau_{[m]}$ and $\mathfrak{S}\omega_{[j]}\omega_{[k]}\tau_{[m]}$. Then $Sw_iw_jt_l$, $t_l < t_m$ and $Sw_jw_kt_m$ is on $\mathcal{B}$ [by the definition of an induced model]. Since $\mathcal{B}$ is complete $(T - DR)$ has been applied. Consequently, $Sw_iw_kt_l$ is on $\mathcal{B}$. In conclusion, $\mathfrak{S}\omega_{[i]}\omega_{[k]}\tau_{[l]}$, as required [by the definition of an induced model].

$C - BT$. Let A be a propositional variable. Suppose A is true in $\omega_{[j]}$ at $\tau_{[l]}$ and that $\mathfrak{R}\omega_{[i]}\omega_{[j]}\tau_{[l]}$. Then, A, w_jt_l and $Rw_iw_jt_l$ occur on $\mathcal{B}$ [by the definition of an induced model]. Since $\mathcal{B}$ is complete, $(T - BT)$ has been applied and A, w_it_l is on $\mathcal{B}$. Consequently, A is true in $\omega_{[i]}$ at $\tau_{[l]}$, as required [by the definition of an induced model]. ■

5.3 Soundness for systems with propositional quantifiers

In this section, I will show that all extended systems in this paper are sound with respect to their semantics. In those systems, a tableau branch can include propositional quantifiers. Accordingly, we must modify the previous soundness proof slightly. First, we modify the soundness lemma.

Lemma 43 ***(Soundness Lemma II).*** *Suppose that the branch $\mathfrak{B}$ is satisfiable in the model $\mathcal{M}$ and that a tableau rule is applied to it. Then there is a model $\mathcal{M}'$ and an extension of $\mathfrak{B}$, $\mathfrak{B}'$, such that $\mathfrak{B}'$ is satisfiable in $\mathcal{M}'$.*

Proof. Most steps are trivial modifications of the steps in the proof for Soundness Lemma I above (just let $\mathcal{M}'$ be $\mathcal{M}$). The only new interesting cases are the steps for the quantifiers.

($\forall$). Straightforward.

($\exists$). Suppose that $\exists XA, w_i t_j$ is on $\mathfrak{B}$ and that we apply ($\exists$) to this node. Then we obtain an extension, $\mathfrak{B}'$, of $\mathfrak{B}$ that includes $A[Y/X], w_i t_j$, where Y is a propositional variable new to the branch. $\exists XA$ is true in $f(w_i)$ at $g(t_j)$, for $\mathfrak{B}$ is satisfiable in $\mathcal{M}$. Therefore, there is some sentence D in $\mathcal{L}$ such that D is substitutable for X in A and $A[D/X]$ is true in $f(w_i)$ at $g(t_j)$. Let $\mathcal{M}' = \langle \mathfrak{W}, \mathfrak{T}, <, \mathfrak{R}, \mathfrak{S}, \mathfrak{v}' \rangle$ be like $\mathcal{M}$, except that $\mathfrak{v}'(Y) = \{\langle f(w_k), g(t_l) \rangle : \mathcal{M}', f(w_k), g(t_l) \Vdash D\}$. Then $A[Y/X]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}'$. ■

Theorem 44 ***(Soundness Theorem II).*** *All extended systems in this paper are sound with respect to their semantics.*

Proof. Once we have shown Soundness Lemma II, the rest of the proof is straightforward. ■

References

[1] Anderson, A. R. (1956). The formal analysis of normative systems. In Rescher (ed.) (1967), pp. 147–213.

[2] Anderson, A. R. (1958). A reduction of deontic logic to alethic modal logic. *Mind*, Vol. 67, No. 265, pp. 100–103.

[3] Anderson, A. R. (1959). On the logic of commitment. *Philosophical Studies* 10, pp. 23–27.

[4] Anderson, A. R. (1967). Some Nasty Problems in the Formal Logic of Ethics. *Noûs*, Vol. 1, No. 4, pp. 345–360.

[5] Åqvist, L. (1999). The Logic of Historical Necessity as Founded on Two-Dimensional Modal Tense Logic. *Journal of Philosophical Logic* 28, pp. 329–369.

[6] Åqvist, L. (2002). Deontic Logic. In Gabbay and Guenthner (eds.) *Handbook of Philosophical Logic*, 2nd Edition, Vol. 8, Dordrecht/Boston/London: Kluwer Academic Publishers, pp. 147–264.

[7] Åqvist, L. (2003). Conditionality and Branching Time in Deontic Logic: Further Remarks on the Alchourrón and Bulygin (1983) Example. In Segerberg and Sliwinski (2003), pp. 13–37.

[8] Åqvist, L. (2005). Combinations of tense and deontic modality: On the Rt approach to temporal logic with historical necessity and conditional obligation. *Journal of Applied Logic* 3, pp. 421–460.

[9] Åqvist, L. and Hoepelman, J. (1981). Some theorems about a "tree" system of deontic tense logic. In Hilpinen (ed.) (1981), pp. 187–221.

[10] Bailhache, P. (1986). *Les normes dans le temps et sur l'action, Essai de logique déontique*. Université de Nantes.

[11] Bailhache, P. (1991). *Essai de logique déontique*. Paris, Librarie Philosophique, Vrin, Collection Mathesis.

[12] Bailhache, P. (1993). The Deontic Branching Time: Two Related Conceptions. *Logique et Analyse*, 36, pp. 159–175.

[13] Bailhache, P. (1997). Canonical Models for Temporal Deontic Logic. *Logique et Analyse*, 149, pp. 3–21.

[14] Bartha, P. (1993). Conditional obligation, deontic paradoxes, and the logic of agency. *Annals of Mathematics and Artificial Intelligence* 9, pp. 1–23.

[15] Bartha, P. (1999). Moral Preference, Contrary-to-Duty obligation and Defeasible Oughts. In McNamara and Prakken (eds.) (1999), pp. 93–108.

[16] Barringer, H., Fisher, M., Gabbay, D. and Gough, G. (eds.) (2000). *Advances in Temporal Logic*. Springer.

[17] Belnap, N., Perloff, M. and Xu, M. (2001). *Facing the Future: Agents and Choices in Our Indeterminist World*. Oxford, Oxford University Press.

[18] Blackburn, P., de Rijke, M. and Venema, Y. (2001). *Modal Logic*. Cambridge University Press.

[19] Blackburn, P., van Benthem, J. and Wolter, F. (eds.) (2007). *Handbook of Modal Logic*. Elsevier.

[20] Broersen, J. M. (2006). Strategic Deontic Temporal Logic as a Reduction to ATL, with an Application to Chisholm's Scenario. In Goble and Meyer (eds.). (2006), pp. 53–68.

[21] Brown, M. A. (1999). Agents with Changing and Conflicting Commitments: A Preliminary Study. In McNamara and Prakken (eds.) (1999), pp. 109–125.

[22] Brown, M. A. (2000). Conditional Obligation and Positive Permission for Agents in Time. *Nordic Journal of Philosophical Logic*. Vol. 5, No. 2, pp. 83–112.

[23] Brown, M. A. (2004). Rich deontic logic: a preliminary study. *Journal of Applied Logic* 2, pp. 19–37.

[24] Brunel, J., Bodeveix, J.-P., Filali, M. (2006). A Stat/Event Temporal Deontic Logic. In Goble and Meyer (eds.) (2006), pp. 85–100.

[25] Brunel, J. (2007). *Combinaison des logiques temporelle et déontique pour la specification de politiques de sécurité*. Université Toulouse III.

[26] Bull, R. (1969). On Modal Logic with Propositional Quantifiers. *Journal of Symbolic Logic*, Vol. 34, No. 2, pp. 257–263.

[27] Burgess, J. P. (1984). Basic Tense Logic. In D. Gabbay and F. Guenthner (eds.) (1984) *Handbook of Philosophical Logic*, vol. 2, Dordrecht: Reidel, pp. 89–133.

[28] Castañeda, H.-N. (1975). Ought, Time, and the Deontic Paradoxes. *The Journal of Philosophy*, Vol. 74, No. 12 (Dec.), pp. 775–791.

[29] Chellas, B. F. (1969). *The Logical Form of Imperatives*. Stanford: Perry Lane Press.

[30] Chellas, B. F. (1980). *Modal Logic: An Introduction*. Cambridge: Cambridge University Press.

[31] Ciuni, R. and Zanardo, A. (2010). Completeness of a Branching-Time Logic with Possible Choices. *Studia Logica* 96, pp. 393–420.

[32] D'Agostino, M., Gabbay, D. M., Hähnle, R. and Posegga, J. (eds.) (1999). *Handbook of Tableau Methods*. Dordrecht, Kluwer Academic Publishers.

[33] DiMaio, M. C. and Zanardo, A. (1998). A Gabbay-Rule Free Axiomatization of T × W Validity. *Journal of Philosophical Logic* 27, pp. 435–487.

[34] Feldman, F. (1986). *Doing the Best We Can: An Essay in Informal Deontic Logic*. Dordrecht: D. Reidel Publishing Company.

[35] Fine, K. (1970). Propositional quantifiers in modal logic. *Theoria* 36, pp. 336–346.

[36] Finger, M. Gabbay, D. and Reynolds, M. (2002). Advanced Tense Logic. In D. Gabbay and F. Guenthner (eds.) (2002) *Handbook of Philosophical Logic*, Vol. 7, Kluwer Academic Publishers, pp. 43–203.

[37] Fisher, M. (1962). A System of Deontic-Alethic Modal Logic. *Mind, New Series*, Vol. 71, No. 282, pp. 231–236.

[38] Fitting, M. and Mendelsohn, R. L. (1998). *First-Order Modal Logic*. Kluwer Academic Publishers.

[39] Gabbay, D. M. (1971). Montague Type Semantics for Modal Logics with Propositional Quantifiers. *Mathematical Logic Quarterly*, 17, pp. 245–249.

[40] Gabbay, D. M., Kurucz, A., Wolter, F. and Zakharyaschev, M. (2003). *Many-Dimensional Modal Logics: Theory and Applications*. Amsterdam, Elsevier.

[41] Gabbay, D., Horty, J., Parent, X., van der Meyden, E. and van der Torre, L. (eds.) (2013). *Handbook of Deontic Logic and Normative Systems*. College Publications.

[42] Gallin, D. (1975). *Intensional and Higher-Order Modal Logic: With Applications to Montague Semantics*. Amsterdam: North-Holland.

[43] Garson, J. W. (2006). *Modal Logic for Philosophers*. New York, Cambridge University Press.

[44] Goble, L. and Meyer, J.-J. Ch. (eds.) (2006). *Deontic Logic and Artificial Normative Systems*. Springer.

[45] Goldblatt, R. (1992). *Logics of Time and Computation*. CSLI.

[46] Hansen, J. (1999). On Relations between Åqvist's Deontic System G and Van Eck's Deontic Temporal Logic. In McNamara and Prakken (eds) (1999), pp. 127–144.

[47] Hilpinen, R. (ed.) (1971). *Deontic Logic: Introductory and Systematic Readings*. Dordrecht: D. Reidel Publishing Company.

[48] Hilpinen, R. (ed.) (1981). *New Studies in Deontic Logic: Norms, Actions, and the Foundation of Ethics*. Dordrecht, D. Reidel Publishing Company.

[49] Hilpinen, R. and McNamara, P. (2013). Deontic Logic: A Historical Survey and Introduction. In Gabbay et.al. (eds.) (2013). *Handbook of Deontic Logic and Normative Systems*, pp. 3–136.

[50] Horty, J. F. (2001). *Agency and Deontic Logic*. Oxford, Oxford University Press.

[51] Horty, J. F. and Belnap, N. (1995). The deliberative stit: a study of action, omission, ability, and obligation. *Journal of Philosophical Logic*, pp. 583–644.

[52] Hughes, G. E. and Cresswell, M. J. (1968). *An Introduction to Modal Logic*. London, Routledge.

[53] Hughes, G. E. and Cresswell, M. J. (1996). *A New Introduction to Modal Logic*. London: Routledge.

[54] Jeffrey, R. C. (1967). *Formal Logic: Its Scope and Limits*. McGraw-Hill, New York.

[55] Kanger, S. (1957). New Foundations for Ethical Theory. In Hilpinen (ed.) (1971), pp. 36–58.

[56] Kaplan, D. (1970). S5 with quantifiable propositional variables. *Journal of Symbolic Logic* 35, p. 355.

[57] Kracht, M. (1999). *Tools and Techniques in Modal Logic*. Number 142 in Studies in Logic. Amsterdam, Elsevier.

[58] Kripke, S. (1959). A Completeness Theorem in Modal Logic. *Journal of Symbolic Logic*, 24(1), pp. 1–14.

[59] Kröger, F. and Merz, S. (2008). *Temporal Logic and State Systems*. Springer.

[60] Lewis, C. I. and Langford, C. H. (1932). *Symbolic Logic*. New York: Dover Publications. Second edition 1959.

[61] McNamara, P. (2010). Deontic Logic. *Stanford Encyclopedia of Philosophy*, http://plato. stanford.edu/entries/logic-deontic/.

[62] McNamara, P. and Prakken, H. (eds.) (1999). *Norms, Logics and Information Systems: New Studies in Deontic Logic and Computer Science*. Amsterdam: IOS Press.

[63] Montague, R. (1960). Logical necessity, physical necessity, ethics and quantifiers. *Inquiry* 4, pp. 259–269.

[64] Øhrstrøm P. and Hasle, P. F. V. (1995). *Temporal Logic: From Ancient Ideas to Artificial Intelligence*. Dordrecht/Boston/London: Kluwer Academic Publishers.

[65] Priest, G. (2008). *An Introduction to Non-Classical Logic*. Cambridge, Cambridge University Press.

[66] Prior, A. (1967). *Past, Present and Future*. Oxford, Clarendon.

[67] Rescher, N. (ed.) (1967). *The Logic of Decision and Action*. Pittsburgh: University of Pittsburgh Press.

[68] Rescher, N. and Urquhart, A. (1971). *Temporal logic*. Wien, Springer-Verlag.

[69] Rönnedal, D. (2009). Dyadic Deontic Logic and Semantic Tableaux. *Logic and Logical Philosophy*, Vol. 18, No. 3–4, pp. 221–252.

[70] Rönnedal, D. (2012). Temporal Alethic-Deontic Logic and Semantic Tableaux. *Journal of Applied Logic* 10, pp. 219–237.

[71] Rönnedal, D. (2012). Bimodal Logic. *Polish Journal of Philosophy*. Vol. VI, No. 2, pp. 71–93.

[72] Rönnedal, D. (2015). Quantified Temporal Alethic Deontic Logic. *Logic and Logical Philosophy*, Vol 24, No 1, pp. 19–59.

[73] Rönnedal, D. (2019). Semantic Tableau Versions of Some Normal Modal Systems with Propositional Quantifiers. *Organon F* 26 (3), pp. 505–536.

[74] Segerberg, K. (1971). *An Essay in Classical Modal Logic*. 3 vols. Uppsala: University of Uppsala.

[75] Segerberg, K. and Sliwinski, R. (eds.) (2003). *Logic, law, morality: thirteen essays in practical philosophy in honour of Lennart Åqvist*. Uppsala philosophical studies 51. Uppsala: Uppsala University.

[76] Smullyan, R. M. (1966). Trees and Nest Structures. *Journal of Symbolic Logic* 31, pp. 303–321.

[77] Smullyan, R. M. (1968). *First-Order Logic*. Heidelberg, Springer-Verlag.

[78] Thomason, R. (1981). Deontic Logic as Founded on Tense Logic. In Hilpinen (ed.) (1981), pp. 165–176.

[79] Thomason, R. (1981). Deontic logic and the role of freedom in moral deliberation. In Hilpinen (ed.) (1981), pp. 177–186.

[80] Thomason, R. (2002). Combinations of Tense and Modality. In D. M. Gabbay and F. Guenthner, (eds.), *Handbook of Philosophical Logic* 2, (1984), pp. 135–165, (2nd edition 7, 2002, pp. 205–234).

[81] van Benthem, J. (2010). *Modal Logic for Open Minds*. CSLI Publications, Stanford.

[82] van Eck, J. E. (1981). *A System of Temporally Relative Modal and Deontic Predicate Logic and its Philosophical Applications*. Department of Philosophy, University of Groningen, The Netherlands.

[83] von Kutschera, F. (1997). T × W Completeness. *Journal of Philosophical Logic* 26, pp. 241–250.

[84] von Wright, G. H. (1951). Deontic Logic. *Mind* 60, pp. 1–15.

[85] Weber, M. E. (2013). Satisficing. In *The International Encyclopedia of Ethics*. Edited by Hugh LaFollette, pp. 4705–4710.

[86] Wölfl, S. (1999). Combinations of Tense and Modality for Predicate Logic. *Journal of Philosophical Logic* 28, pp. 371–398.

[87] Zakharyaschev, M., Segerberg, K., de Rijke and Wansing, H. (eds.) (2001). *Advances in Modal Logic, Vol 2*. CSLI Publications.

[88] Zanardo, A. (1996). Branching-Time Logic with Quantification over Branches: the Point of View of Modal Logic. *The Journal of Symbolic Logic*, vol. 61 number 1, pp. 1–39.