S (for Syllogism) Revisited
"The Revolution Devours its Children"

Robert K. Meyer and Errol P. Martin

Introduction

In 1978, the authors began a paper, "S (for Syllogism)," henceforth [S4S], intended as a *philosophical* companion piece to the technical solution [SPW] of the Anderson-Belnap P–W problem. [S4S] has gone through a number of drafts, which have been circulated among close friends. Meanwhile other authors have failed to see the point of the *semantics* which we introduced in [SPW]. It will accordingly be our purpose here to *revisit* that semantics, while giving our present views on syllogistic matters past, present and future, especially as they relate to *not* begging the question via such *dubious* theses as $A \to A$. We shall investigate in particular a *paraconsistent* attitude toward such theses.

Which arguments are valid? This has been the central question of logic. "Reasoning is an argument in which, certain things being laid down, something *other than these* necessarily comes about through them," said Aristotle.[1] The emphasis is ours. "He who repeats himself does not reason," as Strawson [STR] correctly notes. The fallacy of concluding what one has assumed is almost universally condemned. Some of the rubrics under which it is condemned are the following: *circular reasoning, begging the question, petito principii.*

What, we ask, is the *formal counterpart* of this well-known fallacy? Is it not just $A \to A$ itself?[2] Why then is $A \to A$ equally universally approved? One reason is that the principle is (mis)identified as the Law of Identity.

[1] *Topics* 100a 25–27, as translated by W. A. Pickard-Cambridge in [RWA]. Essentially the same remark may be found in the *Prior Analytics* at 24b 18–20.

[2] We use *upper-case* letters like 'A' as syntactical variables for *formulas*; and *lower-case* ones like 'p' for *atoms*. Some of our readers, alas, will have been *Quinized*; they may think it important where the quotation marks go. We follow in principle the conventions of Curry's [FML]. Perhaps because he was once dogged by Quine on use-mention issues, Curry is the most careful logician we've known on such matters.

Laws are solemn things. This holds doubly for Laws of Logic. And it is not for us, we mostly agree, to tinker with solemn things.

Wait a minute. Surely the law of identity is that A *is* A, whatever the type or category of A. With this law, rightly so called, we have no quarrel. Rather the point of the traditional complaint is that, *just because* A is A, it cannot *properly* be said to *follow from* itself. It occurs to us to allow A to follow *improperly* from itself, *paraconsistently.* Meanwhile, let us become *more* Aristotelian.

1 Vindicating Aristotle

Logic was invented by Aristotle, noted Kant. He added that it had not afterwards been able to advance a single step. In the two centuries since Kant, logic has been symbolized and made mathematical; by common consent, it has advanced many steps. It will be our purpose here to undermine this consent; and insofar as it lies within us, to restore what Kant perceived.

Aristotle was the father of many sciences. Again and again in the modern era further progress has been possible only when the dead hand of Aristotelian dogma has been lifted from the throat of, say, astronomy. (Who any more would wish to hold *immutable* what is in and above the sphere of the moon?) True, Logic came *late* to the arena in which Aristotle was (or anyway was accused of being) a blithering idiot.

Accusation came nonetheless and (eventually) in force. We shall not recall or add to it. Rather it will be for us to dwell on what Aristotle *saw*, by way of founding logic. What he saw was that premisses *combine* to produce a conclusion validly. The Granddaddy of these valid arguments is the so-called syllogism in BARBARA. Since we agree with Anderson and Belnap's [ENT1] that implication is the Heart of Logic, we write down the pure implicational basis of this syllogism. (We use mnemonics from Curry's Combinatory Logic, henceforth CL.)

$$\text{Ax } B. \quad (C \to D) \to ((A \to C) \to (A \to D))$$

B respects the convention that, in stating syllogisms, the major premiss comes first. There is another implicational principle that looks very much like it, namely

$$\text{Ax } B'. \quad (A \to C) \to ((C \to D) \to (A \to D))$$

So easy is it to confuse these principles that they are known indifferently in some texts as "syllogism".[3] This is a sound intuition. But except in the presence of some *further* assumptions that will *not* be made here, B and B' are *different* (and *non-equivalent*) principles.[4]

It is crashingly clear how the premisses of B combine to produce its conclusion. The rubric is

(a) The 2nd implies the 3rd;

(b) The 1st implies the 2nd. So

(c) The 1st implies the 3rd.

B' is just the same, save that premisses (a) and (b) are switched. And both testify to the same root insight, which we identify as the traditional

Dictum de omni. Implication is *transitive*.[5]

Readers familiar with the P–W problem may at this point complain that we have *left out* an axiom. You still *need*, they may insist

Ax I. $A \to A$

Maybe they are *right*, for reasons into which we shall delve below. For now, we view Ax I as just too *crude*. It is not reasoning to argue in a circle, concluding what one has assumed. It is not reasoning to beg the question. It is not reasoning to commit *petitio principii*.

Petitio principii is represented, in every system of logic that it has been the custom to take seriously, by the theorem scheme Ax I. [ENT] waxes almost rhapsodic on the point (in which, it must be wryly confessed, we have sometimes joined it; to oneself, however, all is forgiven). According to [ENT, 8], $A \to A$ is the "archetypal form of inference," the "trivial foundation of all reasoning". More than that, [ENT] confers upon A the honorific status of being (logically) necessary iff A follows from $A \to A$; for the *necessitive* $\Box A$ is *defined as* $(A \to A) \to A$.

[3] We adopt the contemporary λ-CL (Lambda calculus-Combinatory Logic) convention that $\to$'s associate to the *right*. Thus we may henceforth write B' as $(A \to C) \to (C \to D) \to A \to D$, etc. This conflicts, alas, with the *left*ward style of our mentors.

[4] If there is a choice, an Aristotelian preference should go to B over B'. Alas, this trivializes the [SPW] proof of Martin's theorem. It *cannot* be right thus to make a hard and suggestive problem trivial.

[ENT] claimed to present the logic of relevance and necessity. In large measure the claim is *true*. Yet for 2000 years logic has been only peripherally involved with relevance or necessity as such. What the tradition that dates from Aristotle has been concerned with centrally is syllogistic reasoning; and, especially, the role of the *middle term* in separating the valid from the invalid arguments. What assures relevance, traditionally, is that a good argument *goes through* a middle term. So viewed, it is not so hard to find where *irrelevance* attacks, producing the most common fallacies.

Fallacies of relevance creep in when one has got confused about the role and function of the middle term. For example, there might be no way of decently providing a middle term at all—fallacies *ad hominem* or *ad baculum* have this character. Or, by being vague or ambiguous, one might convince oneself that one has a middle term when one doesn't. If one *equivocates* about B, then a purported argument from A to C through B might not *really go through* B at all. (In the traditional lingo, this is the *fallacy of four terms.*) Finally one might have a middle term, and an unambiguous one, while appealing nonetheless to an argument form that is simply invalid (e.g., IAI in the first figure).[6]

The purported ubiquity of $B \to B$, then, as *the* grounding principle of logic, would seem to hark back to the *shared content* which alone undergirds the validity of a *good* argument. Most conspicuously, taking our Ax B as paradigmatic, the "middle term" B of $B \to C$ must be the *same* B as that of $A \to B$. (This is the Law of Identity in its wholesome "B *is* B" form.) Of course $B \to B$ also harks back, in these days of triumph for Gentzen-style consecution calculi, to the ubiquity of $B \vdash B$ as *the axiom* for such systems. So far has (apparent) fallacy *emerged victorious!*

2 Implicational logics

We concentrate for the moment on pure implicational logics, built up freely from a countable stock of *atoms* (for which we use 'p', etc.) under the single binary connective $\to$. The class of all *formulas* (for which we use 'A', etc.) will be $F[\to]$, which when clear in context we abbreviate simply as 'F'. As stated above, for ease in reading formulas we may omit parentheses by associating $\to$'s *to the right.*

In order to turn the *Dictum de omni* into a *System*, we take B and B' as

[6]IAI Example: I: Some Mammals are Cats. A: All Dogs are Mammals. SO, I: Some Dogs are Cats. Given that the premises are TRUE but the conclusion is FALSE, something is WRONG here!

Axiom Schemes for a pure $\to$ logic, which we call S (for syllogism).[7] To get some theorems we need a Rule. As we have said before (though from drafts of [S4S] with increasing hesitation), these days the one rule you can trust is the *modus ponens* principle $\to$E. Using $\Rightarrow$ *metalogically*, we write this as

$$\text{Ru} \to E. \quad A \to C \Rightarrow (A \Rightarrow C)$$

to mean that if $A \to C$ is a theorem then if A is a theorem then C is a theorem.

Ah, but maybe we should *not* be so trusting. $\to$E as just stated *looks like* an instance of the question-begging $D \to D$. Thanks to a *metavaluations* argument of Dwyer, we may replace $\to$E with prefixing, suffixing and transitivity rules directly derived from the S axioms, namely

$$\text{Ru } B. \qquad (C \to D) \Rightarrow (A \to C) \to (A \to D)$$

$$\text{Ru } B'. \qquad (A \to C) \Rightarrow (C \to D) \to (A \to D)$$

$$\text{Ru } BB. \quad (C \to D) \Rightarrow ((A \to C) \Rightarrow (A \to D))$$

When convenient, we shall think of BB' as formulated with these *Dwyer rules*. (See [SPW].)

3 Implicational antitheorems and Powers' problem

$S_{\to}$ may be formulated with axiom schemes B and B' just above, with $\to \mathsf{E}$ or the Dwyer rules.

We turn now to a problem posed by Larry Powers in [ALP]. Consider again $A \to A$. [PCM. Powers' Conjecture, and Martin's 1^{st} theorem.] *No* instances of Ax I are theorems of S. In [SPW], [JSL], [SENT] and elsewhere.

Martin's theorem is quite beyond the familiar sorts of *independence results* in propositional logic. To show an axiom scheme Y *independent* of the remaining axioms $X_1, \ldots, X_n$ for a system, it suffices in common parlance to show that *some* instance of Y is not derivable from the X's. A simple

[7] S here is *not* a CL mnemonic. When careful we call it S[$\to$], adopting in principle Restall's [ISL] convention on primitive particles. We may also call S[$\to$] simply BB', for its *axioms*.

3-valued matrix will do this trick for S, validating BB' and $\to E$ but rejecting the instance $p \to p$ of I, where p is a propositional *atom.* But there is an elementary argument, due in principle to Tarski, that *no finite matrix* similarly rejects *all* instances of I.

We are at the heart of the Heart of Logic, in the pure theory of $\to$. We are often told that valid argument is just a matter of *form.* Symbolized, this means that the *good guys* are not only good themselves, but so also are all their (substitution) *instances.* The *bad guys* differ. This has tricky consequences for the *translation exercises* in elementary logic classes. The job there is to separate the (logical) sheep from the (bad) goats. To identify a (good) sheep, it suffices to find *some* theorem of which the candidate argument can be construed as an instance. But this doesn't work for goats. If we wish to be recalcitrant, every argument with one premiss may be taken to boil down to $p \to q$, letting p stand in for the premiss and a distinct q for the conclusion. And logic would be futile if we were allowed to take *all* arguments as invalid (since $p \to q$ is *bad*).

Are there not some arguments that are *always bad*, just as much to be condemned for their form as the good arguments are to be commended? Outside of *pure* $\to$ logic, even truth-functionalists will agree that this *sometimes* happens, for example in

SC. Snow is white, or it's not. Therefore, coal is black, and it's not.

When more *fully formalized*, [8] SC finds the counterpart

SC$'$. $(p \vee \sim p) \to (q \wedge \sim q)$.

SC$'$ is reprehensible *even to classical logicians*, as the *negation* of a truth-functional tautology.

Some terminology is here in order. A formula A of logic L will be *universally valid* just in case A and *all* its substitution instances are theorems of L. We will say that A is *potentially valid* iff *some* instance of A is a theorem; and *potentially invalid* iff some instance of A is a non-theorem. Finally, and most significantly for now, we'll call A *universally invalid* iff *no* instances of A are theorems. We moreover call the universally invalid formulas *antitheorems* of L.

Since logics are in general closed under substitution, the universal validity of a formula A comes ordinarily to the same thing as the theoremhood

[8] For spotting the goats, *full formalization* is practiced in the texts!

of A. For the same reason, A is potentially *invalid* (normally) iff A is a non-theorem of L. But it is a *different kettle of fish* to call A *potentially valid.* It is well-known that the intuitionist logic H of [HEY] rejects

XM. $p \vee {\sim}p$

Similarly, relevant logics like R reject

DS. $p \wedge ({\sim}p \vee q) \rightarrow q$

These formulas, although non-theorems (and rightly so, say those philosophers who despise them) are only *potentially* invalid in their respective systems. To see that XM is *also potentially valid* for H, just substitute $q \wedge {\sim}q$ for p. And in DS for R, put in p for q.

And now we will see that potential validity is the *usual* situation for *all formulas* in the *positive parts* of many famous logics. Recall our discussion above. Let a *positive* formula be one built up from atoms by $\rightarrow$ (and perhaps other connectives from among $\wedge$, $\vee$, $\leftrightarrow$, etc.). Then [Anti-antitheorem theorem for $\mathsf{2}_+$] All positive classical formulas A are potentially valid.

Positive sentential formulas are satisfied classically just by assigning T to all their variables. Since a substitution instance is wanted, pick a theorem (say $p \rightarrow p$) to substitute uniformly for *all* the atoms of A. The result will be a theorem, equivalent to $p \rightarrow p$ itself. So, classically, there are no *ineluctably bad* positive arguments. There are only bad *instances* of potentially *good* arguments. Nor is classical logic the only offender. The real point of the little argument just concluded is that there is a theorem t of $\mathsf{2}$ which is an *idempotent* under all positive operations. We took t, for our little argument, as $p \rightarrow p$. And the idempotence of the t so chosen just means that it is logically equivalent, in $\mathsf{2}$, to each of $\mathsf{t} \rightarrow \mathsf{t}$, $\mathsf{t} \wedge \mathsf{t}$, $\mathsf{t} \vee \mathsf{t}$, $\mathsf{t} \leftrightarrow \mathsf{t}$. We can make the same choice, with the same result, in the positive part $\mathsf{H}_{\vdash}$ of the intuitionist logic H of [HEY]; or of the relevant positive logics R_+ and E_+ of [ENT].

4 The quest for a Minimal logic

We pause for another definition. A system L is *lax* iff every formula of L is potentially valid. We have just observed (and bewailed) that laxity is endemic among positive logics—even the best known relevant ones like $\mathsf{R}_{\rightarrow}$ and E_+ . But relevant logics are themselves the outcome of the search for a *minimal* logic—a search that perhaps reached its zenith in the 1950's. It was then that Moh Shaw-Kwei in [TNV] and Church in [WTI] published their versions of what came to be known as $\mathsf{R}_{\to}$. While in 1956 Ackermann introduced in [BSI] what Anderson and Belnap reformulated (e. g., in [FOE]) as E (of entailment).

Is there a minimal implicational logic? Curry may be taken to have suggested that there is *not*. But Anderson and Belnap were (apparently) on the other side of the question.[9] They proposed the following criterion for *minimality*.

(ABC). If $A \to C$ and $C \to A$ are both theorems, then A and C are the *same formula*.

The BB'IW system $\mathsf{T}_{\to}$ of [ENT1] was an initial candidate to satisfy (ABC). But, letting t again be $p \to p$ as above, Belnap quickly dashed this hope by proving in $\mathsf{T}_{\to}$ both of

1. $(\mathsf{t} \to \mathsf{t}) \to (\mathsf{t} \to (\mathsf{t} \to \mathsf{t}))$ and
2. $(\mathsf{t} \to (\mathsf{t} \to \mathsf{t})) \to (\mathsf{t} \to \mathsf{t})$,

whence $\mathsf{t} \to \mathsf{t}$ and $\mathsf{t} \to (\mathsf{t} \to \mathsf{t})$ are a pair A, C that refute (ABC).[10]

Nothing daunted, Anderson and Belnap proposed the $BB'I$ system of pure $\to$ calculus as an (ABC) candidate.[11] They were *correct*.

[ABCM. Anderson-Belnap Conjecture, and Martin's 2$^{\text{nd}}$ theorem.] For *no* two *distinct* formulas A and B of $\mathrm{F}[\to]$ are both $A \to B$ and $B \to A$ theorems of $BB'I$.

In [SPW], [JSL], [SENT] and elsewhere.

5 Implicational theories

Let L (officially, $\mathsf{L}[\to]$) be an implicational logic. We define

D $<_{\mathsf{L}}$. $A <_{\mathsf{L}} C =_{\mathrm{df}} \mathsf{L} \vdash A \to C$

[9] For Curry, cf. his [CGT]. Anderson introduced $\mathsf{T}_{\to}$in [ESM]. Cf. [ENT1], 94f.

[10] Could it be, the reader might hope, that the positive fragment T_{+} of the [ENT] system of *ticket entailment* isn't lax? Exercise: Although t above is *not* T_{+} idempotent, refute that hope anyway!

[11] Following Belnap (though he long *denied* it), this system was called P–W in [SPW]. [ENT1] calls it $\mathsf{T}_{\to}$-W, in introducing the (ABC) conjecture. We *stick* to P–W as the name of the pure $\to$ fragment $BB'I$.

That is, $<_{\mathsf{L}}$ is the *entailment relation* according to L. Clearly $<_{BB'}$ is *transitive*, while $<_{BB'I}$ is moreover *reflexive*. Let x be any set of formulas. We call x an (*Implicational*) L-*Theory* (*ILT*) iff it is closed under $<_{\mathsf{L}}$. I. e.,

DILT. $x \in ILT$ iff $\forall A \forall C(A <_{\mathsf{L}} C \Rightarrow A \in x \Rightarrow C \in x)$

Let x, y be sets of formulas. We define

D $\circ$. $x \circ y = \{C : \exists A(A \to C \in x \text{ and } A \in y)\}$

That is, as an operation on sets of formulas $\circ$ is what Powers in [ALP] called *modus ponens product* (and which we sometimes call *fusion*), taking *major premises* from its left argument and *minor premises* from its right argument. More than that, Powers also saw that closing the *axioms* of $BB'I$ under $\circ$ both produces and exhausts the $\mathsf{BB'I}$ theorems! Put otherwise and in general, any implicational logic L formulated with $\to E$ as sole rule has as its set of theorems $\mathsf{L\,T}$ the *smallest* subset of F satisfying

(i) A is an axiom of $\mathsf{L} \Rightarrow A \in \mathsf{L\,T}$

(ii) $x \subseteq \mathsf{L\,T}$ & $y \subseteq \mathsf{L\,T} \Rightarrow x \circ y \subseteq \mathsf{L\,T}$

[PO. (*Prefixing Observation*).] Suppose $\mathsf{L}[\to]$ is a logic closed under the prefixing rule Ru B. Then if $x \in ILT$ and $y \subseteq F$, we have $x \circ y \in ILT$. Let x be an implicational L-theory. We must show $x \circ y \in ILT$, where y is any set of formulas. Suppose $C <_{\mathsf{L}} D$ and $C \in x \circ y$. By D $\circ$ there is an $A \in y$ such that $A \to C \in x$. Because $C <_{\mathsf{L}} D$, we have by Ru B that $A \to C <_{\mathsf{L}} A \to D$. Accordingly $A \to D \in x$, whence $D \in x \circ y$. Done! The prefixing observation PO is more than enough to show that very many implicational logics L, certainly including S, are such that the class ILT of their theories is closed under the fusion operation $\circ$.

6 Combinators and formula sets

We continue to develop Powers' observations, building on his [ALP]. Let us simply *identify* the "combinator" B with the set of all instances of Ax B. Similarly identify B' with the instances of Ax B' and I with all $A \to A$. Show as in [IFM] that the 1-step reduction rules of Combinatory Logic then *hold*, as full set-theoretic equalities, for *all* sets x, y, and z of formulas from $F[\to]$. That is, dropping '$\circ$' for simple juxtaposition and associating to the left, we quickly verify the following facts:

Fact B. $$Bxyz = x(yz)$$

Fact B'. $$B'xyz = y(xz)$$

Fact I. $$Ix = x$$

Monotonicity. $x \supseteq y \Rightarrow xz \supseteq yz$, and $y \supseteq z \Rightarrow xy \supseteq xz$[12]

In a nutshell, on the BB' and $BB'I$ fragments of CL, the [IFM] Fools Model is *perfect*. [13]

7 Are S and P–W the SAME system?

We became interested in S on account of Powers' Conjecture. Yet this conjecture arose out of his desire to solve the P–W problem. Powers saw that there was *minimal interaction* between the I axioms and the syllogistic B and B' in actually deriving *theorems* in $BB'I$. (It may be that Belnap already grasped this lack of interaction in formulating and working on (ABC) for the system.) It accordingly occurs to us to ask whether, at root, the systems are *conceptually distinct*.

We shall see that, conceptually, provable S implications express a kind of proper $<$ relation, which is perhaps sensibly mated with a $\leqslant$ from P–W. It perhaps behooves us to take a thorough-going *relational* view of both systems, distinguishing them *not* by their $\rightarrow$'s but by their $\vdash$'s. If we say P–W $\vdash A \rightarrow$ C, we mean $A \leqslant C$; if $S \vdash A \rightarrow C$, then $A < C$. And we now reserve $<$ *for the balance of this paper* for $<_S$; and $\leqslant$ for $<_{P--W}$.[14].

[12] It is even more obvious that $x = y \Rightarrow xz = yz$, and $y = z \Rightarrow xy = xz$.

[13] This extends to all of BCI, which is the fragment LL[$\rightarrow$] of the Linear Logic of Girard [GLL].

[14] There is a bonus in removing further parentheses, on the convention that $\rightarrow$ *binds more strongly* than $<$ or $\leqslant$. Note that we only use the relational symbols to indicate that some $\rightarrow$ statement *is a theorem*

8 Semantical ingredients for Martin's theorems

We saw above that S-theories are closed under $\circ$. We view them henceforth as a *structure*

$$\mathsf{IST} = \langle IST, \supseteq, \circ \rangle$$

where IST is the set of implicational S-theories, $\supseteq$ is superset, and $\circ$ is defined by D $\circ$.

All this sets up an *operational semantics* for $BB'I$. The intuition is that the *points* of this semantics are just $BB'I$ *theories.* A $BB'I$ model structure (henceforth, Sms) will be a triple $K = \langle K, \supseteq, \circ \rangle$, where K is a non-empty set; $\supseteq$ is a binary relation on K, and $\circ$ is a binary operation on K (which we indicate by simple juxtaposition, associating to the *left*). We impose the following postulates, for all $x, y, z \in K$:

p B. $\quad xyz \supseteq x(yz)$

p B'. $\quad xyz \supseteq y(xz)$

p I. $\quad x \supseteq x$

p $\supseteq$. $\quad y \supseteq z \Rightarrow x \supseteq y \Rightarrow x \supseteq z$

p $\circ$. $\quad y \supseteq z \Rightarrow xy \supseteq xz$, and $x \supseteq y \Rightarrow xz \supseteq yz$

The B, B' and I postulates reflect the (Curry-style) *reduction rules* for the *combinators* that have the corresponding axioms as their *types* (in the vocabulary of [CL1]). Intuitively, $\supseteq$ is the *superset* relation on theories, whence $p \supseteq$ records that superset is transitive. A final intuitive hook is that $\circ$ is the *modus ponens* product operation on theories defined by D $\circ$.

We seem to have taken a giant step *back* from the smooth facts of section 6. And why have we called our frames Sms, when we have set them up for the supersystem $BB'I$ of $\mathsf{S}[\rightarrow]$? Patience, gentle reader, patience! We have already suggested in section 7 that, deep down, S and $\mathsf{P\text{–}W}$ are the *same* system. For Powers first noted and Dwyer more smoothly proved,[15] [PDF. Powers-Dwyer Fact.] Let A and C be *distinct* formulas of F. Then $A < C$ iff $A \leqslant C$. Left to right is obvious. For the converse, define with Dwyer

[15] A referee finds Dwyer's smooth argument the nicest thing in the paper. We like it too. It was communicated by Dwyer to Meyer back in 1974, when we were both at Pittsburgh. Thanks, Robin! And, if ANYONE knows where you are, PLEASE GET IN TOUCH!

a *metavaluation* $v : F \to 2$ by

$v(p) = 0$ if p is an atom

$v(D \to E) = 1$ iff (i) $D < E$ or $D = E$, and (ii) $v(D) \leqslant v(E)$.

The '=' in (i) applies iff D and E are *exactly the same formula.* Show then by deductive induction, for all D and E in $F[\to]$, that if $BB'I \vdash D \to E$ then $v(D \to E) = 1$.[16]. Since the fact assumes $A \leqslant C$ for *distinct* A and C, (i) then enforces $A < C$. Done! We have the immediate

[PDF Corollary.] A set x of $\to$ formulas is a BB'-theory iff x is a $BB'I$-theory. So IST is *indifferently* the set of all BB'-theories and of all $BB'I$-theories. Note too

[STM Theorem.] IST is an Sms. All the postulates but p B and p B' are clearly OK by D $\circ$ and properties of $\supseteq$. As for p B', recall that our postulates are being asserted of S-theories x. So $x \supseteq$B'$\circ x$ in this case, since closure under $\leqslant$ is imposed on theories.[17]. Two applications of Monotonicity then produce $xyz \supseteq$B'xyz, while $B'xyz = y(xz)$ by Fact B' for *any* formula sets. This shows that $xyz \supseteq y(xz)$ for S-theories x, y, z. Note that the inclusion can *not* in general be reversed. A similar argument verifies p B as well, ending the proof. We now *complete* our operational semantical story for $BB'I$ by adding some interpretative machinery. Let K be an Sms. Any function $I : F \times K \to 2$ is a *possible interpretation* in K, assigning either 1 (*true*) or 0 (*false*) to each formula A at each point x in K. Where I is a possible interpretation fixed in context, let us write simply 'Aa' for $I(A, a) = 1$, and '$\sim A\ a$' for $I(A, a) = 0$. Then a possible interpretation I is moreover an *interpretation* if it satisfies two further conditions, a *hereditary* condition H and a *truth* condition T$\to$, for all $a, b \in K$ and $B, C \in F$:

H. $\quad a \supseteq b \Rightarrow Cb \Rightarrow Ca$

T$\to$. $\quad [B \to C]a = \forall b \in K(Bb \Rightarrow C(a \circ b))$

The heredity condition H may be restricted to propositional variables p, since an easy induction using T$\to$ and Monotonicity then establishes it for all the $B \to C$ as well. Check out the intuition behind H. If theory a is a

[16]PDF is independent of Martin's theorems. But the '=' clause under (i) takes up the slack in verifying AxI. Check the cases, involving delicate interplay between syntactic (i) and semantic (ii) in verifying B and B'

[17]B' here is simply the *set* of all suffixing axioms, which is *not* itself an S-theory

supertheory of theory b, then any C in b is most certainly in a also.

T$\rightarrow$ is the nub of the matter. Suppose $B \rightarrow C \in a$ and $B \in b$. Then, by D $\circ$, we expect $C \in a \circ b$. This is the idea from left to right. From right to left suppose that *whenever* $B \in b$ then $C \in a \circ b$. Surely $B \in [B, \infty)$, the *principal theory* containing B and all that B entails. So, the thought goes, $C \in a \circ [B, \infty)$, which if luck is with us is the set of all D such that $B \rightarrow D \in a$. Luck is with us; so in particular $B \rightarrow C \in a$, as desired.

We are merely sketching ideas here, to make them *plausible.* (Again as Belnap once noted, in Philosophy that *counts as a proof.*) But we are doing Logic, which is *less forgiving.* We complete the interpretative story thus, on interpretation I in K:

Ent I. $\quad B$ *I-entails* $C = \forall a \in K(Ba \Rightarrow Ca)$

Ent K. $\quad B$ *K-entails* $C = B$ I-entails C on every interpretation I in K

Ent S. $\quad B$ *S-entails* $C = B$ K-entails C in every Sms K

[Soundness theorem for $BB'I$.] $A \leqslant C \Rightarrow A$ S-entails C. Formulate $BB'I$ with Ax B, Ax B', Ax I and the Dwyer rules Ru B, Ru B', Ru BB. Pick an arbitrary interpretation I in any Sms K. Check that each antecedent of an axiom I-entails its consequent. Show that this property is preserved under the Dwyer rules. Since I and K are arbitrary, the axioms yield S-entailments and the rules preserve this property. Done! For completeness we appeal to the STM theorem, recalling that IST is an Sms. We define a *canonical interpretation* CI by setting, for each S-theory x and formula A,

$$CI(A, x) = 1 \text{ iff } A \in x$$

Evidently the canonical interpretation respects the heredity condition H (since in IST the $\supseteq$ relation really *is* superset). It also respects $\mathsf{T}\rightarrow$, since the intuitions we were courting just above may now be established as formal facts. We conclude [Completeness theorem for BB'I.] A S-entails $C \Rightarrow A \leqslant C$. Assume that A S-entails C. In particular A CI-entails C on the canonical interpretation in IST. I. e., $\forall x \in IST(A \in x \Rightarrow C \in x)$. Let $[A, \infty) = \{D : A \leqslant D\}$. Instantiating, $A \in [A, \infty) \Rightarrow C \in [A, \infty)$. But $A \in$ the S-theory $[A, \infty)$ by Ax I. So $C \in [A, \infty)$. I. e., $A \leqslant C$, ending the proof.

9 The Semantics turns Paraconsistent

Readers who skimmed through the proofs just offered did *not* miss much. While perhaps *caviar to the general*, they are direct and straightforward, and entirely in the spirit of similar developments in [RLR]. We may assume moreover that we have been working in a wholly classical metalogic, with '$\Rightarrow$' in definitions and proofs simply taken as classical material $\supset$.

In this section, which we beg you *not* to skim, we set out the semantical thinking that was a direct ingredient into Martin's theorems. We appealed in the last section to a theory

$$[A, \infty) = \{C : \; A \; \leqslant \; C\}.$$

$[A, \infty)$ is the *principal* $BB'I$*-theory determined by* A. It contains A, by Ax I. But there is *another* principal theory not far off. (A, ∞). We define

$$(A, \infty) = \{C : A < C\}.$$

Where A is any formula, (A, ∞) is the *principal* BB'*-theory determined by* A. Question: Is A in *its own* principal BB'-theory (A, ∞)? Answer: Never! (Apply Martin's 1^{st} theorem.)

Our notation takes off from that for intervals on the real line. We distinguish the *open* interval (m, ∞), which does not contain m, from the *half-closed* $[m, \infty)$, of which m is the initial member.) While Powers' conjecture was open, it was possible that, for some A,

$\diamond$. $\quad (A, \infty) = [A, \infty)$.

The input of the semantics into Martin's [SPW] proof lies in showing that $\diamond$ *never* happens. Equivalent to Section 3's PCM is the generalized *negated* equality

PCM'. $\quad$ For all A we have $(A, \infty) \neq [A, \infty)$.

It is PCM' that may be most conveniently viewed as *the lemma* for the semantic proof of PCM.[18] Our purpose here is to lay bare some of the philosophical background for that lemma. The idea is to *separate* the formulas *properly entailed* by a formula A from those that are *entailed by or identical*

[18] Since the essence of the [SPW] proof lies in adroit symbol-pushing, Martin was long aware that a more directly syntactical treatment also delivers PCM. See his [FM] with Fine, which extended the PCM result to $\mathsf{S}[\rightarrow\wedge]$ a decade ago. But bringing $\vee$ also under the tent has proved recalcitrant.

with A. Since this distinction can *always* be made, the key conjectures were *true*.

The first thought was to turn $\mathsf{T}\!\!\rightarrow$ above into *two* truth-conditions, one for *weak* truth (of the $\leqslant$ variety) and the other for *strong* truth (the same for $<$). It turned out that these conditions can be combined if we switch the metalogic from $\mathsf{2}$ to the semi-relevant logic $\mathsf{RM3}$, which has a 3-valued characteristic matrix.[19] Add $1/2$ to 0 and 1 above, and make $1/2$ its own negation. A nice thing about $\mathsf{RM3}$ is that it has a characteristic 3 point matrix, whose $\wedge$ and $\vee$ tables may be read off the following Hasse diagram.[20]

The characteristic matrix TNF for RM3:

$$\begin{array}{c} 1 \\ | \\ 1/2 \\ | \\ 0 \end{array}$$

The $\rightarrow$ and $\sim$ tables for $\mathsf{RM3}$ are just the following:

$\rightarrow$	1	$1/2$	0
$*1$	1	0	0
$*1/2$	1	$1/2$	0
0	1	1	1

	$\sim$
$*1$	0
$*1/2$	$1/2$
0	1

Both T and N count as designated elements in TNF (originally due to Sobociński in [Sob52]). Note also that the value of $\sim p$ is just that of $p \rightarrow 1/2$
.

An interpretation I in an Sms $K = \langle K, \supseteq, \circ \rangle$ now assigns one of these 3 values to every formula A at every point x in K. A formula B is *strongly true* at x on I just in case $I(B, x) = 1$; *weakly true*, if $I(B, x) \neq 0$. Particularly delicate is the case where $I(B, x) = 1/2$, a sort of *crossover point* from falsity to truth. For the hereditary condition H, when reinterpreted in this $\mathsf{RM3}$ way, refuses to *linger* at $1/2$. Suppose $x \neq y$, where $I(B, x) = 1/2$. Then if $y \supseteq x$, then $I(B, y) = 1$; while if $x \supseteq y$, then $I(B, y) = 0$.

[19] $\mathsf{RM3}$ is a *wonderful logic*, good for everything from showing relevant arithmetic *non-trivial* (take *that*, Gödel!) to solving the P–W problem (for years the *Fermat's Last Theorem* of relevant logics).

[20] "Make the $\mathsf{RM3}$ matrix *explicit*," said the referee, "to keep things self-contained." Recall that adding $A \rightarrow (A \rightarrow A)$ and $A \vee (A \rightarrow B)$ as new axiom schemes produces $\mathsf{RM3}$ from the relevant logic R of [ENT]. Some readers may be more familiar with the $\mathsf{RM3}$ truth values 1, $1/2$ and 0 as 'T' 'N' and 'F' respectively.

Similarly, the truth-conditions $\mathsf{T}\!\rightarrow$ and the S-entailment were reinterpreted in the 3-valued way. It sufficed to create a *new* canonical interpretation, and to show that every formula A was assigned the *boundary value* $1/2$ on this interpretation, at *its own* principal theory (A, ∞). Since $1/2 \rightarrow 1/2 = 1/2$ in the characteristic matrix for $\mathsf{RM3}$, the result is that *all* the $A \rightarrow A$ turn out weakly but not strongly true. And $\mathsf{PCM'}$ was proved.

10 Adding Inferential Negation

We now extend $F[\rightarrow]$ to a language $F[\rightarrow, \mathsf{f}]$, by adding the constant f to the formation apparatus. This enables an *inferential* definition of negation (as Bull once suggested that an idea going back to C. S. Peirce be applied here).

D $\sim$. $\quad \sim\! A =_{\mathrm{df}} A \rightarrow \mathsf{f}$

We are not certain that the *minimal* syllogistic negation defined by D $\sim$ is the one that we shall eventually want for S. But we expect that we shall *at least* wish the transposition and related properties that this definition engenders. We get immediately from Ax B and Ax B'

Th $B\sim$. $\quad \sim\! C \rightarrow (A \rightarrow C) \rightarrow \sim\! A$

Th $B'\sim$. $\quad (A \rightarrow C) \rightarrow \sim\! C \rightarrow \sim\! A$

In view of Martin's 1^{st} theorem, it makes sense to extend S with the following new axioms

Ax $\sim\! I$. $\quad \sim\!(A \rightarrow A)$

We call the resulting system $BB' \sim I$, and note the following pleasant result.

[Conservative extension theorem for BB'$\sim$I.] Let A be a formula in our original vocabulary $F[\rightarrow]$. Then A is theorem of $BB'\!\sim\! I$ iff A is already a theorem of BB'. Right to left is trivial. For the converse, suppose for *reductio* that A is not a theorem of BB', but that it has nonetheless a proof $A_1, \ldots, A_n$ in $BB' \sim I$. Some of the A_i in this proof must be of the form (in primitive notation) $(C \rightarrow C) \rightarrow \mathsf{f}$, since otherwise A would already be provable in BB'. Since A is a non-theorem of BB', there is a strong logical matrix $\mathsf{M} = \langle M, \rightarrow, D \rangle$ and an interpretation I in M that refutes A—i.e., $I(A) \notin$ the designated set D of matrix elements. *Enlarge* M to a new matrix $\mathsf{M'} = \langle M', \rightarrow', D' \rangle$. M' is M with additional elements T and F.

For all $a, b \in M$, we require $a \to' b = a \to b$. Moreover $\to'$ is subject to the *rigorous compactness* condition that, for all a in M', $\mathsf{F} \to' a = a \to' \mathsf{T} = \mathsf{T}$; while otherwise $a \to' \mathsf{F} = \mathsf{T} \to' a = \mathsf{F}$. (The effect is to make T a matrix Top element and F a Bottom element, *isolated from* the original matrix elements in M.) It is readily observed as in [CPL] that, since M is a strong matrix for BB', so also is M' , setting $D' = D \cup \{\mathsf{T}\}$. Extending I to an interpretation I' with $I'(\mathsf{f}) = \mathsf{T}$, all of $A_1, \ldots, A_n$ are designated on I', by straightforward deductive induction. Since $A_n = A$, this is a contradiction. Even our minimal negation produces some welcome metatheorems. The *antitheorem* $A \to A$ of $\mathsf{S}[\to]$ has been *transmuted* into the *axiom* $\sim(A \to A)$ of $BB' \sim I$. Other antitheorems are quick to follow. For example, by substitution in the theorem scheme B'B'(B'B'), we have

(1) $(A \to (A \to B) \to B) \to (A \to (A \to B)) \to A \to A \to B$

As an instance of Ax$\sim$I, we get

(2) $\sim((A \to (A \to B)) \to A \to A \to B)$.

Applying *modus tollens* to (1) and (2), justified by your choice of Th $B\sim$ or Th $B'\sim$,

(3) $\sim(A \to (A \to B) \to B)$,

signalling the *antitheoremhood* of the CI principle in S.

Life gets interesting when we throw in Ax I as another primitive principle. Now we become *explicitly contradictory*, since Ax $\sim I$ remains. Can we live with this? We believe that we can. For we have seen the I axiom as *on the border* between *valid inference* and *fallacy*. $A \to A$, in a certain sense, is *both*. It is what happens when one takes the *directed* $<$ of an honest entailment as an *equality*.

Save for $A \to A$, the antitheorems of BB' *remain antitheorems* in $BB'I$. So in what we call $BB'I \sim I$, the $\sim CI$ theorem (3) *continues to signal* that *no instances* of $A \to (A \to B) \to B$ are syllogistically OK. That is, *explicit inconsistency* is restricted to the troublesome $A \to A$. As it *should* be!

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