A Note on the Relevance of Semilattice Relevance Logic

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Abstract

A propositional logic has the *variable sharing property* if $\phi \to \psi$ is a theorem only if ϕ and ψ share some propositional variable(s). In this note, I prove that positive semilattice relevance logic ($\mathbf{R}_u^+$) and its extension with an involution negation ($\mathbf{R}_u^-$) have the variable sharing property (as these systems are not subsystems of $\mathbf{R}$, these results are not automatically entailed by the fact that $\mathbf{R}$ satisfies the variable sharing property). Typical proofs of the variable sharing property rely on ad hoc, if clever, matrices. However, in this note, I exploit the properties of rather more intuitive arithmetical structures to establish the variable sharing property for the systems discussed.

1 Introduction

One of the characteristic features of relevance logics such as **R** and **E** is that they have the variable sharing property:

DEFINITION 1. Let $\Pi(\phi)$ be the set of propositional variables occurring ϕ. A logic **L** has the *variable sharing property* if $\vdash_{\mathbf{L}} \phi \to \psi$ only if $\Pi(\phi) \cap \Pi(\psi) \neq \emptyset$.

That logics such as **R** have the variable sharing property has typically been proved using complicated and somewhat ad hoc, if clever, matrices.[1] However, many relevance logics have fairly natural models in elegant mathematical structures and at least some interesting logical conclusions have

[1] The classic example is Belnap's use of an eight-valued logic to show that **E** has the variable sharing property in [3]. Similar matrix methods remain the standard tool used for proving this property (see, for example, [8], [9], and [12]).

been drawn by considering such models. A case in point is Meyer's proof that there are infinitely many non-equivalent formulae in two propositional variables in $\mathbf{R}_{\to}$, the implicational fragment of $\mathbf{R}$ [10] (see also [1, §29.2]).

Meyer's model, which is based on an ordering of the integers by divisibility, is suggestive. In this note, I use related structures to prove that positive semilattice relevance logic ($\mathbf{R}_u^+$) and its extension with an involution negation ($\mathbf{R}_u^\neg$) have the variable sharing property. Note that, since positive semilattice relevance logic is not a subsystem of the positive fragment of $\mathbf{R}$ (see [13, p. 163] and [2, §47.4]), that the latter enjoys the variable sharing property implies nothing for the former. Rather surprisingly, no proof seems to have been given in the literature that the semilattice relevance logics in fact have the variable sharing property, despite the fact that satisfying this condition is generally taken to be a necessary condition on being a relevance logic.[2] This note shall properly establish the claims of these systems (and their subsystems) to being relevance logics.

As already advertised, my focus in this paper will be on systems which have a semilattice semantics in the sense of Urquhart's seminal [13]. I review the rudiments of the semilattice semantics in section 2. In section 3, I present a simple arithmetical semilattice model which establishes that positive semilattice relevance logic has the variable sharing property. I show how to extend the result to semilattice relevance logic with an involution negation in section 4. A couple concluding remarks are given in section 5.

2 Review of Semilattice Semantics

In this section, I review the basic features of the semilattice semantics for relevance logic due to Urquhart [13]. The language is understood to consist of a countable set of propositional variables $\Pi = \{p_0, p_1, \ldots\}$ and, as determined by context, some subset of the connectives $\{\to, \wedge, \vee, \neg\}$ (the formation rules are all standard). I use $p, q, \ldots$ for arbitrary propositional variables and $\phi, \psi, \ldots$ for arbitrary formulae.

DEFINITION 2. A *positive semilattice frame* is a structure $\mathfrak{F} = \langle S, 0, \cup \rangle$ where $\langle S, \cup \rangle$ is a join-semilattice and $0 \in S$ is lattice bottom.

DEFINITION 3. An *involution semilattice frame* is a structure $\mathfrak{I} = \langle \mathfrak{F}, * \rangle$ where $\mathfrak{F}$ is a positive semilattice frame and $* : S \to S$ is an involution, i.e. $x^{**} = x$.

[2] Thus Anderson and Belnap, writing of $\mathbf{E}_{\to}$ in particular: "we propose as a *necessary*, but by no means sufficient, condition for the relevance of A to B in the pure calculus of entailment, that A and B must share a variable" [1, §5.1.2].

Note that the condition specified in definition 3 is weaker than that examined by Urquhart [13, p. 164], in that it is not required that $0^* = 0$.[3] Consequently, this will characterize a weaker (though not uninteresting) negation.[4]

DEFINITION 4. A *semilattice model* is a structure $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ where $\mathfrak{F}$ is a semilattice frame and $V : \Pi \rightarrow \mathcal{P}(S)$. Given $\mathfrak{M}$ and $x \in S$, define the relation $\models^{\mathfrak{M}}_x$ as follows (omit the conditions for omitted connectives as needed):

1. $\models^{\mathfrak{M}}_x p$ if and only if $x \in V(p)$
2. $\models^{\mathfrak{M}}_x \phi \wedge \psi$ if and only if $\models^{\mathfrak{M}}_x \phi$ and $\models^{\mathfrak{M}}_x \psi$
3. $\models^{\mathfrak{M}}_x \phi \vee \psi$ if and only if $\models^{\mathfrak{M}}_x \phi$ or $\models^{\mathfrak{M}}_x \psi$
4. $\models^{\mathfrak{M}}_x \neg\phi$ if and only if $\not\models^{\mathfrak{M}}_{x^*} \phi$ (where $\mathfrak{F}$ is involutive)
5. $\models^{\mathfrak{M}}_x \phi \rightarrow \psi$ if and only if for all $y \in S$, $\not\models^{\mathfrak{M}}_y \phi$ or $\models^{\mathfrak{M}}_{x \cup y} \psi$

DEFINITION 5. ϕ is *valid in* $\mathfrak{M}$ ($\models^{\mathfrak{M}} \phi$) if $\models^{\mathfrak{M}}_0 \phi$. ϕ is *valid* ($\models \phi$) if for all $\mathfrak{M}$, $\models^{\mathfrak{M}} \phi$.

Besides the class of all semilattice models, some interest attaches to subclasses in which certain constraints are imposed on V. For $\mathbf{J}_{\rightarrow}$, the implicational fragment of intuitionistic logic, one can impose the condition that $x \in V(p)$ implies $x \cup y \in V(p)$ [2, §47.4]. As in Kripke semantics for intuitionistic logic (see, e.g., [7]), imposing this condition on propositional variables yields a sort of "heredity lemma" result for all formulae in the language. Similarly, a pure implicational relevance logic in which the Mingle axiom $(\phi \rightarrow (\phi \rightarrow \phi))$ is valid, $\mathbf{RM0}_{\rightarrow}$, can be obtained by imposing the (weaker) condition that $x \in V(p)$ and $y \in V(p)$ imply $x \cup y \in V(p)$ [2, §47.4].[5]

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[3] Philosophically, it is problematic to require that $0^* = 0$ anyway, for precisely the reasons Urquhart [13, p. 166] mentions.

[4] I should also note that Urquhart [13, §5] discusses other species of negation which I will not discuss in this note.

[5] Note that the semantics satisfying this condition does *not* characterize the implicational fragment of **RM**, which in fact coincides with the implicational fragment of a system of Sobociński's [11, p. 294] (see also [1, §8.15]).

For all formulae ϕ in the connectives $\{\rightarrow, \wedge\}$, imposing this condition on propositional variables yields the result that $\models^{\mathfrak{M}}_x \phi$ and $\models^{\mathfrak{M}}_y \phi$ imply $\models^{\mathfrak{M}}_{x \cup y} \phi$. However, this is not the case if $\vee$ is added in. To see this, let $\mathfrak{M} = \langle S, 0, \cup, V \rangle$ be defined as follows: $S = \{1, 2, 3, 6\}$, '0' is 1, $x \cup y = \mathrm{lcm}(x, y)$, $V(p) = \{2\}$, $V(q) = \{3\}$, and otherwise V is empty. This model clearly meets the conditions of definition 2 and the constraint on V. Then it's the case that $\models^{\mathfrak{M}}_2 p \vee q$ and $\models^{\mathfrak{M}}_3 p \vee q$ but $\not\models^{\mathfrak{M}}_{\mathrm{lcm}(2,3)} p \vee q$.

3 Positive Semilattice Relevance Logic

Let $\mathbf{R}_u^+$ be the set of formulae (in the connectives $\mathcal{C}^+ = \{\to, \wedge, \vee\}$) valid in all semilattice models (definition 4). $\mathbf{R}_u^+$ was axiomatized by Fine [6] and Charlwood [4]. Since $\mathbf{R}_u^+$ properly extends $\mathbf{R}^+$, the positive fragment of $\mathbf{R}$, that the variable sharing property holds of the latter does not guarantee that it holds of the former. In this section, I prove that the variable sharing property holds of $\mathbf{R}_u^+$.

I will make use of a more elaborate semilattice frame to prove this result than is strictly necessary. Nevertheless, the structure, besides being quite elegant and related to that discussed by Meyer [10] in a different context, bears a somewhat interesting algebraic relation to the frame that will be used in the following section. These factors collectively make it worthy of examination.

DEFINITION 6. The *ω-frame* is the positive semilattice frame $\mathfrak{F}^\omega = \langle \omega, 1, \mathrm{lcm}\}$ where $\omega = \{1, 2, \ldots\}$ and $\mathrm{lcm}(i, j)$ is the least common multiple of i and j.

In order to prove that the variable sharing property holds of $\mathbf{R}_u^+$ (and all subsystems thereof), I first prove some lemmata. As these will make clear, the idea of the proof is to exploit the parity properties of the natural numbers to construct a simple countermodel.

Note that, where X is a set of propositional variables, $\mathcal{LANG}(X)$ is the set of all formulae in the connectives $\mathcal{C}^+$ built up from variables in X. I use $\mathcal{E}$ for $\{2k : k \in \omega\}$ and $\mathcal{O}$ for $\{2k - 1 : k \in \omega\}$.

LEMMA 1. *Let $X = \{p_i, \ldots, p_m\}$ be a set of propositional variables and let $\mathfrak{M} = \langle \mathfrak{F}^\omega, V \rangle$ be such that, for each $p \in X$, $V(p) = \mathcal{E}$. Then for ϕ in $\mathcal{LANG}(X)$ and $i \in \mathcal{E}$, $\models_i^{\mathfrak{M}} \phi$.*

Proof. The proof is by induction on ϕ. For the basis case, if ϕ is a propositional variable p, then if $i \in \mathcal{E}$, by definition, $i \in V(p)$, whence $\models_i^{\mathfrak{M}} p$. Suppose the result holds for ψ and θ. For $\wedge$, consider an arbitrary $i \in \mathcal{E}$. By the induction hypothesis, $\models_i^{\mathfrak{M}} \psi$ and $\models_i^{\mathfrak{M}} \theta$, that is, $\models_i^{\mathfrak{M}} \psi \wedge \theta$. (The case of $\vee$ is equally trivial and is omitted.) For $\to$, consider an arbitrary $i \in \mathcal{E}$ and arbitrary $j \in \omega$. Since $\mathrm{lcm}(i, j) \in \mathcal{E}$, by the induction hypothesis, $\models_{\mathrm{lcm}(i,j)}^{\mathfrak{M}} \theta$. Therefore, $\models_i^{\mathfrak{M}} \psi \to \theta$. □

LEMMA 2. *Let $X = \{p_i, \ldots, p_m\}$ be a set of propositional variables and let $\mathfrak{M} = \langle \mathfrak{F}^\omega, V \rangle$ be such that, for each $p \in X$, $V(p) = \mathcal{O}$. Then for ϕ in $\mathcal{LANG}(X)$: $\models_i^{\mathfrak{M}} \phi$ if and only if $i \in \mathcal{O}$.*

Proof. Again, the proof is by induction on ϕ. The basis case is by definition. Suppose the result holds for ψ and θ. For $\vee$, consider, first, an arbitrary $i \in \mathcal{O}$. By the induction hypothesis, $\models_i^{\mathfrak{M}} \psi$ and $\models_i^{\mathfrak{M}} \theta$, from which the result is immediate. Alternatively, if $i \in \mathcal{E}$ (i.e. $i \notin \mathcal{O}$), by the induction hypothesis, $\not\models_i^{\mathfrak{M}} \psi$ and $\not\models_i^{\mathfrak{M}} \theta$, that is, $\not\models_i^{\mathfrak{M}} \psi \vee \theta$. (The case of $\wedge$ is omitted.) For $\rightarrow$, suppose $i \in \mathcal{O}$ and consider an arbitrary $j \in \omega$. If j is even, by the induction hypothesis, $\not\models_j^{\mathfrak{M}} \psi$. Alternatively, if j is odd, $\mathrm{lcm}(i, j)$ is odd as well. Consequently, by the induction hypothesis, $\models_{\mathrm{lcm}(i,j)}^{\mathfrak{M}} \theta$. Thus, either $\not\models_j^{\mathfrak{M}} \psi$ or $\models_{\mathrm{lcm}(i,j)}^{\mathfrak{M}} \theta$, from which it follows that $\models_i^{\mathfrak{M}} \psi \rightarrow \theta$. For the converse, suppose that $i \in \mathcal{E}$ and pick any $j \in \mathcal{O}$. By the induction hypothesis, $\models_j^{\mathfrak{M}} \psi$. Since $\mathrm{lcm}(i, j) \in \mathcal{E}$, by the induction hypothesis, $\not\models_{\mathrm{lcm}(i,j)}^{\mathfrak{M}} \theta$. Therefore, $\not\models_i^{\mathfrak{M}} \psi \rightarrow \theta$. □

THEOREM 1. $\boldsymbol{R}_u^+$ *has the variable sharing property.*

Proof. Suppose that $\Pi(\phi) \cap \Pi(\psi) = \emptyset$. Let $\mathfrak{M} = \langle \mathfrak{F}^\omega, V \rangle$ be such that:

1. For each $p \in \Pi(\phi)$, $V(p) = \mathcal{E}$.
2. For each $p \in \Pi(\psi)$, $V(p) = \mathcal{O}$.

Pick any even number, say 2. By lemma 1, $\models_2^{\mathfrak{M}} \phi$. By lemma 2, $\not\models_2^{\mathfrak{M}} \psi$. Then $\not\models_1^{\mathfrak{M}} \phi \rightarrow \psi$ since both $\models_2^{\mathfrak{M}} \phi$ and $\not\models_{\mathrm{lcm}(1,2)}^{\mathfrak{M}} \psi$. Since $\not\models^{\mathfrak{M}} \phi \rightarrow \psi$, $\not\models \phi \rightarrow \psi$, which was to be proved. □

COROLLARY 1. $\boldsymbol{RM0}_{\rightarrow}$ *has the variable sharing property.*[6]

Proof. Let $\mathfrak{M} = \langle \mathfrak{F}^\omega, V \rangle$ be exactly as in theorem 1 *except* that if $p \notin \Pi(\phi) \cup \Pi(\psi)$, set $V(p) = \emptyset$. It suffices for the result to show that $\mathfrak{M}$, so defined, satisfies the constraint that $x \in V(p)$ and $y \in V(p)$ imply $x \cup y \in V(p)$. If $p \notin \Pi(\phi) \cup \Pi(\psi)$, then the result holds vacuously. So suppose $p \in \Pi(\phi)$. The result holds vacuously if either x or y is odd. Alternatively, if they are both even, then clearly $\mathrm{lcm}(x, y) \in \mathcal{E} = V(p)$. Finally, suppose $p \in \Pi(\psi)$. The result holds vacuously unless both x and y are odd, in which case $\mathrm{lcm}(x, y) \in \mathcal{O} = V(p)$, as desired.[7] □

[6] A proof of this fact using a matrix is given in [1, §8.15].

[7] The same argument works for $\mathbf{RM0}_{\rightarrow,\wedge}$ (the implication-conjunction fragment of **RM0**) but, for the reason noted in footnote 5, in a language including $\vee$, it will not generally be guaranteed that every instance of the mingle axiom is valid (the proof of the validity of mingle in the class of models satisfying the pertinent constraint relies on a version of the heredity lemma). In any case, that the positive fragment of **RM0** has the variable sharing property has already been proved using matrices by Méndez [8, p. 286].

It may be of some interest to note where this argument breaks down in the case of $\mathbf{J}_{\to}$. Given the model $\mathfrak{M} = \langle \mathfrak{F}^{\omega}, V \rangle$ defined above, let $p \in \Pi(\psi)$. If i is odd, then i is in $V(p)$; but if j is any even number, $\mathrm{lcm}(i, j) \notin V(p)$. Therefore, $\mathfrak{M}$ does not satisfy the constraint required for $\mathbf{J}_{\to}$ semilattice models.

4 Adding Negation

The negation extensions of $\mathbf{R}_u^+$ seem to be much less well understood than $\mathbf{R}_u^+$ itself. Below, I will only examine the extension of $\mathbf{R}_u^+$ by an involution negation $\neg$. In particular, let $\mathbf{R}_u^{\neg}$ be the set of formulae (in the connectives $\mathcal{C}^{\neg} = \{\to, \wedge, \vee, \neg\}$) valid in all semilattice models based on involution semilattice frames. To the best of my knowledge, $\mathbf{R}_u^{\neg}$ has not been given a complete axiomatization. Nevertheless, among others, all of the following schemes and rules are easily shown to be valid:

$$\phi \to \neg\neg\phi \tag{1}$$

$$\neg\neg\phi \to \phi \tag{2}$$

$$\neg(\phi \wedge \psi) \to (\neg\phi \vee \neg\psi) \tag{3}$$

$$(\neg\phi \vee \neg\psi) \to \neg(\phi \wedge \psi) \tag{4}$$

$$\frac{\phi \to \psi}{\neg\psi \to \neg\phi} \tag{5}$$

In this section, I prove that $\mathbf{R}_u^{\neg}$ has the variable sharing property. The frame that I will use for this purpose bears a mildly interesting relationship to $\mathfrak{F}^{\omega}$. Order the elements of ω under the 'divides' relation $|$.[8] For $i \in \omega$, define $\nabla(i) = \{k \in \omega : k \mid i\}$. $\nabla(i)$ is an *ideal*, that is, a downward closed set closed under lcm, which contains 1. Consequently, $\nabla(i)$, like ω itself, furnishes a natural join-semilattice with lattice bottom 1.

The particular frame I will be concerned with hereafter is based on $\nabla(6)$. It is the structure $\mathfrak{F}^{\mathbf{6}} = \langle \nabla(6), 1, \mathrm{lcm}, f \rangle$, where everything is fairly self-explanatory (see the Hasse diagram in figure 1 below) except f, which is defined piecewise as follows:

$$f(i) = \begin{cases} 3 & \text{if } i = 1 \\ 1 & \text{if } i = 3 \\ 6 & \text{if } i = 2 \\ 2 & \text{if } i = 6 \end{cases}$$

[8] Recall that $\mathrm{lcm}(i, j) = j$ if and only if $i \mid j$.

In short, f maps a given odd to the other odd, and maps a given even to the other even. It is obviously an involution, so $\mathfrak{F}^{\mathbf{6}}$ is an involution semilattice frame.

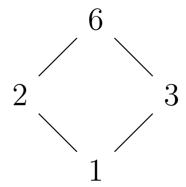

Figure 1: The semilattice $\mathfrak{F}^{\mathbf{6}}$

LEMMA 3. *Let* $X = \{p_i, \ldots, p_m\}$ *be a set of propositional variables and let* $\mathfrak{M} = \langle \mathfrak{F}^{\mathbf{6}}, V \rangle$ *be such that, for each* $p \in X$, $V(p) = \{1, 6\}$. *Then for* ϕ *in* $\mathcal{LANG}(X)$: $\models_i^{\mathfrak{M}} \phi$ *if and only if* $i \in \{1, 6\}$.

Proof. The proof is by induction on the complexity of ϕ. I consider only the cases of $\neg$ and $\rightarrow$. The induction hypothesis is that the result holds for ψ and θ. For $\rightarrow$, let $i \in \{1, 6\}$ and consider j such that $\models_j^{\mathfrak{M}} \psi$; by the induction hypothesis, $j \in \{1, 6\}$, hence $\text{lcm}(i, j) \in \{1, 6\}$. Thus, by the induction hypothesis, $\models_{\text{lcm}(i,j)}^{\mathfrak{M}} \theta$, and consequently, $\models_i^{\mathfrak{M}} \psi \rightarrow \theta$. Conversely, suppose that $i \notin \{1, 6\}$. By the induction hypothesis, $\models_1^{\mathfrak{M}} \psi$ and since $\text{lcm}(i, 1) \in \{2, 3\}$, $\not\models_{\text{lcm}(i,1)}^{\mathfrak{M}} \theta$. Thus, $\not\models_i^{\mathfrak{M}} \psi \rightarrow \theta$. For $\neg$, if $i \in \{1, 6\}$, then $f(i) \notin \{1, 6\}$ from which it follows, by the induction hypothesis, that $\not\models_{f(i)}^{\mathfrak{M}} \psi$. Thus, $\models_i^{\mathfrak{M}} \neg\psi$, as desired. Alternatively, if $i \notin \{1, 6\}$, since $f(i) \in \{1, 6\}$, $\models_{f(i)}^{\mathfrak{M}} \psi$, from which it follows that $\not\models_i^{\mathfrak{M}} \neg\psi$. □

LEMMA 4. *Let* $X = \{p_i, \ldots, p_m\}$ *be a set of propositional variables and let* $\mathfrak{M} = \langle \mathfrak{F}^{\mathbf{6}}, V \rangle$ *be such that, for each* $p \in X$, $V(p) = \{1, 2\}$. *Then for* ϕ *in* $\mathcal{LANG}(X)$: $\models_i^{\mathfrak{M}} \phi$ *if and only if* $i \in \{1, 2\}$.

Proof. Again, the argument is by induction on the complexity of ϕ. Suppose the result holds for ψ and θ. For $\rightarrow$, consider $i \in \{1, 2\}$ and suppose $\models_j^{\mathfrak{M}} \psi$; by the induction hypothesis and fact that $\text{lcm}(i, j) \in \{1, 2\}$, $\models_{\text{lcm}(i,j)}^{\mathfrak{M}} \theta$, from which it follows that $\models_i^{\mathfrak{M}} \psi \rightarrow \theta$. Conversely, if $i \notin \{1, 2\}$, by the induction hypothesis and fact that $\text{lcm}(i, 1) \notin \{1, 2\}$, $\models_1^{\mathfrak{M}} \psi$ and $\not\models_{\text{lcm}(i,1)}^{\mathfrak{M}} \theta$. Thus, $\not\models_i^{\mathfrak{M}} \psi \rightarrow \theta$. For $\neg$, if $i \in \{1, 2\}$, $f(i) \notin \{1, 2\}$, from which $\not\models_{f(i)}^{\mathfrak{M}} \psi$ and $\models_i^{\mathfrak{M}} \neg\psi$ follow by the induction hypothesis. Finally, if $i \notin \{1, 2\}$, $f(i) \in \{1, 2\}$, from which $\models_{f(i)}^{\mathfrak{M}} \psi$ and $\not\models_i^{\mathfrak{M}} \neg\psi$ follow by parallel reasoning. □

THEOREM 2. $\boldsymbol{R}_u^-$ *has the variable sharing property.*

Proof. Suppose that $\Pi(\phi) \cap \Pi(\psi) = \emptyset$. Let $\mathfrak{M} = \langle \mathfrak{F}^{\mathbf{6}}, V \rangle$ be such that:

1. For each $p \in \Pi(\phi)$, $V(p) = \{1, 6\}$.
2. For each $p \in \Pi(\psi)$, $V(p) = \{1, 2\}$.

By lemma 3, $\models_6^{\mathfrak{M}} \phi$. By lemma 4, $\not\models_{\mathrm{lcm}(1,6)}^{\mathfrak{M}} \psi$. Therefore, $\not\models_1^{\mathfrak{M}} \phi \to \psi$, as desired. □

The shape of $\mathfrak{F}^{\mathbf{6}}$ may suggest **FDE** and, in fact, any model in $\mathfrak{F}^{\mathbf{6}}$ is a model of **FDE**. Thus, **FDE** has the variable sharing property (lest you forgot).[9]

5 Concluding Remarks

In this note, I proved that the semilattice relevance logics $\mathbf{R}_u^+$ and $\mathbf{R}_u^-$ have the variable sharing property. To prove this, I used fairly natural arithmetical structures (division semilattices), rather than many-valued matrices. While the main results of this paper are, to the best of my knowledge, novel, I take one of the principal contributions of this paper to lie in showing the value of intuitive models of relevance logics for demonstrating some interesting logical properties.

References

[1] Alan Ross Anderson and Nuel D. Belnap, Jr. *Entailment: The Logic of Relevance and Necessity*, volume I. Princeton University Press, Princeton, 1975.

[2] Alan Ross Anderson, Nuel D. Belnap, Jr., and J. Michael Dunn. *Entailment: The Logic of Relevance and Necessity*, volume II. Princeton University Press, Princeton, 1992.

[3] Nuel D. Belnap, Jr. Entailment and relevance. *Journal of Symbolic Logic*, 25(2):144–146, 1960.

[9]The previous comments notwithstanding, observe that $\mathfrak{F}^{\mathbf{6}}$, taken algebraically as the structure $\langle \nabla(6), \mathrm{lcm}, \mathrm{gcd}, f \rangle$, is *not* a de Morgan lattice (**FDE** is algebraically characterized by de Morgan lattices, for which see [5, §3.3]). To be a de Morgan lattice, $\mathfrak{F}^{\mathbf{6}}$ would have to satisfy the condition that $a \mid f(b)$ imply $b \mid f(a)$, but this is not so: $1 \mid 6$ and $2 \nmid 3$.

[4] G. Charlwood. An axiomatic version of positive semilattice relevance logic. *Journal of Symbolic Logic*, 46(2):233–239, 1981.

[5] J. Michael Dunn and Greg Restall. Relevance logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 6, pages 1–128. Kluwer Academic Publishers, Dordrecht, 2002.

[6] Kit Fine. Completeness for the semilattice semantics with disjunction and conjunction (abstract). *Journal of Symbolic Logic*, 41(2):560, 1976.

[7] Saul A. Kripke. Semantical analysis of intuitionistic logic I. In John N. Crossley and Michael A. E. Dummett, editors, *Formal Systems and Recursive Functions: Proceedings of the Eighth Logic Colloquium, Oxford, July 1963*, Studies in Logic and the Foundations of Mathematics, pages 92–130. North-Holland Publishing Company, Amsterdam, 1965.

[8] José M. Méndez. The compatibility of relevance and mingle. *Journal of Philosophical Logic*, 17(3):279–297, 1988.

[9] José M. Méndez, Gemma Robles, and Francisco Salto. Ticket entailment plus the mingle axiom has the variable-sharing property. *Logic Journal of the IGPL*, 20(1):355–364, 2012.

[10] Robert K. Meyer. R_I—the bounds of finitude. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 16(7):385–387, 1970.

[11] Robert K. Meyer and Zane Parks. Independent axioms for the implicational fragment of Sobociński's three-valued logic. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 18(19-20):291–295, 1972.

[12] Gemma Robles and José M. Méndez. A general characterization of the variable-sharing property by means of logical matrices. *Notre Dame Journal of Formal Logic*, 53(2):223–244, 2012.

[13] Alasdair Urquhart. Semantics for relevant logics. *Journal of Symbolic Logic*, 37(1):159–169, 1972.