A NOTE ON GODDARD AND ROUTLEY'S SIGNIFICANCE LOGICS

DAMIAN SZMUC AND HITOSHI OMORI

Abstract. The present note revisits the joint work of Leonard Goddard and Richard Routley on significance logics (namely, logics able to handle nonsignificant sentences) with the aim of shedding new light on their understanding by studying them under the lens of recent semantic developments, such as the plurivalent semantics developed by Graham Priest. These semantics allow sentences to receive one, more than one, or no truth-value at all from a given carrier set. Since nonsignificant sentences are taken to be neither true nor false, i.e. truth-value gaps, in this essay we show that with the aid of plurivalent semantics it is possible to straightforwardly instantiate Goddard and Routley's understanding of how the connectives should work within significance logics.

1. Introduction

1.1. **Aim.** Richard Routley (later Sylvan) argued, in a series of philosophical writings including the papers "On a significance theory" ([20]), "The need for nonsense" ([21]) and "Don't care was made to care" (with Ross Brady, [4]), and the book "The Logic of Significance and Context, Volume 1" (with Leonard Goddard, [11]), for the need for a logic of *nonsense* or, as he called it, a *significance logic*. That is, an inferential framework that allows to handle sentences which, although grammatical, are deprived of meaning and, thus, have "no truth-value" [11, p. 266]. Among these works, Goddard and Routley's book seems to be less appreciated than it should be. Reassessing what Goddard and Routley achieved in their book concerning significance logics is an enormous project, one which we intend not to complete but merely to start in this note.

The aim of the present note is, thus, to revisit Goddard and Routley's motivations for the development of a logic of significance, putting special emphasis on the need for a suitable *semantic* framework, and to offer a formalization of Goddard and Routley's ideas with the aid of plurivalent semantics, as developed by Graham Priest. Plurivalent semantics are a kind of semantics which allow sentences to receive one, more than one, or no truth-value at all. Goddard and Routley did think of nonsignificant sentences as being neither true nor false and, thus, it is worth exploring whether or not the sort of semantic behavior expected of significance logics can be modeled within a plurivalent setting.

1.2. **Outline.** To achieve our goal, the paper is structured as follows. In the next section, namely §2, we provide an overview of Goddard and Routley's reasons to think that there is an actual need for a logic of nonsense, together with some of the most important objections they considered, and the corresponding replies they gave to them. This will be followed by §3 in which we propose a formalization of Goddard and Routley's account of nonsignificant sentences with the help of the plurivalent semantics developed by Priest. Finally, in §4 we wrap things up with some concluding remarks. An appendix considers functionally complete expansions of significance logics not discussed by Goddard and Routley, but novel to this work.

1.3. **Preliminaries.** Our language $\mathcal{L}$ consists of a finite set $\{\sim, \wedge, \vee\}$ of propositional connectives and a countable set Prop of propositional variables which we denote by p, q, etc. Furthermore, we denote by Form the set of formulas defined as usual in $\mathcal{L}$. We denote a formula of $\mathcal{L}$ by A, B, C, etc. and a set of formulas of $\mathcal{L}$ by Γ, Δ, Σ, etc.

2. Revisiting Goddard and Routley's ideas

2.1. **The need for nonsense.** Goddard and Routley were of the opinion that nonsignificance was a notion quite central to the philosophy of Russell, the logical positivists and Wittgenstein [11, p. 1]. For example, Russell clearly speaks of nonsignificant sentences as those that violate his own criterion of significance devised to deal with the foundations of mathematics, namely the *theory of types*. Thus, according to this position, any sentence expressing the self-membership of a given class, for example "The class of all classes which are not member of themselves is (or is not) a member of itself", is meaningless [11, p. 2]. Alternatively, the logical positivists understood nonsignificant sentences as those that violated their own criterion of significance, namely *empirical verifiability*. Thus, according now to this latter position, any sentence expressing something that is not empirically verifiable, for example "The Absolute is green", is meaningless [11, p. 310]. Now, even if these scholars did in fact have different understandings of what it takes for a sentence to be nonsignificant or meaningless, it seems nonetheless clear that nonsignificance is a notion quite dear to much of contemporary philosophy.

Goddard and Routley always took nonsignificant sentences to be non-truth-valued, i.e. as receiving no truth-value at all. In nowadays terminology, it is fair to say that meaningless sentences should be classified, under this reading, as truth-value *gaps*. In addition to that, it was central to Goddard and Routley's project to develop a *logic* of significance. That is, a formal inferential framework in which nonsignificant sentences are allowed to exist, and to participate in consequence relations [11, p. 5-6].

By the above remarks, it duly follows that a logic of this sort must, therefore, allow for sentences to belong to one of the following three sets: that of the *true* sentences, that of the *false* sentences, and that of the *nonsignificant* and, thus, *neither true nor false* sentences (cf. [21, p. 383], [11, p. 257]).

Goddard and Routley's own approach throughout their work is to model such a tripartite classification with the help of many-valued semantics, i.e. semantics containing the truth-values **t** and **f** (standing for truth and falsity, respectively) and some other additional non-classical "algebraic" values. In the case concerning us here, this third value will be referred to as **u** and will be devised to be paradigmatically assigned to meaningless sentences.[1] We should highlight, then, that Goddard and Routley did not take this third value to be a genuine truth-value at all, but instead as a device to mark the complete lack of truth-values suffered by the sentence in question.

However, all of the above hardly makes for a logical framework. To obtain that we need to settle for a target set of logical *connectives*, decide which are going to be the *truth-functions* for those connectives and, finally, take some stance towards the definition of *logical consequence*. In the next sections we review Goddard and Routley's decisions concerning these issues.

2.2. **Towards a significance logic: the connectives.** Which connectives should be included in a significance logic? It seems reasonable to claim, with Goddard and Routley, that at least those of Classical Logic (**CL**, hereafter), i.e. $\sim, \wedge, \vee$ (to be interpreted as negation, conjunction and disjunction, respectively). The reason for this is, as Goddard and Routley put it, that a significance logic should cover, if possible, the same grounds than **CL** and this seems a fair way to achieve that (cf. [11, p. 258]).

Having said this, we must decide which are going to be the truth-functions for those connectives. This is, needless to say, a non-trivial question. Of course we know how these connectives

[1] Goddard and Routley refer to this value as **n** in [11], but here we opted for referring to it as **u**, in order to avoid confusion with the characteristic non-classical value of Strong Kleene logic $\mathbf{K_3}$, from [13].

work in **CL**, but we have many options to go for when we allow for meaningless sentences, i.e. for truth-value gaps to be thrown into the mix.[2]

It is, then, this question that is more pressing, and which prompts Goddard and Routley to propose that the target connectives should comply with two essential requirements. They must: (i) coincide with **CL** for the classical truth-values **t** and **f**; (ii) obey the so-called principle of *Component Homogeneity*, by means of which *any compound sentence with a nonsignificant component is itself nonsignificant* [11, p. 260].[3] The aforementioned constraints, together, render the following truth-tables, usually called the Weak Kleene truth-tables after the work of Stephen Cole Kleene in [13].

	$\sim$
t	**f**
u	**u**
f	**t**

$\wedge$	**t**	**u**	**f**
t	**t**	**u**	**f**
u	**u**	**u**	**u**
f	**f**	**u**	**f**

$\vee$	**t**	**u**	**f**
t	**t**	**u**	**t**
u	**u**	**u**	**u**
f	**t**	**u**	**f**

It shall be noticed that Goddard and Routley's understanding of the Weak Kleene truth-tables was completely "two-valued", meaning that they did not take the third value to represent a legitimate semantic status alongside truth and falsity, but merely the illegitimacy of these qualifications. This effort, i.e. that of making sense of the Weak Kleene truth-tables from a two-valued perspective, will guide our investigations below, when we revisit Goddard and Routley's ideas with the help of Priest's plurivalent semantics.

In any case, how are we to justify the previous requirements, which render the Weak Kleene truth-tables? Compliance with **CL** when the classical truth-values are taken into account seems to be justified by the idea that significance logics *extend* **CL**. By this we mean that significance logic contain **CL**, i.e. that if no meaningless sentence were to exist, what would remain is plain old two-valued **CL**. It looks like it is the principle of *Component Homogeneity*, perhaps the most distinctive semantic feature of significance logics, that needs to be justified.

To achieve this, Goddard and Routley appeal to a number of different accounts of nonsignificance, showing that each of them explicitly or implicitly entails this principle. First, it is argued in [11, p. 261-262] that Russell himself supported in *Principia Mathematica* the view that a sentence $\sim A$ is nonsignificant iff A is nonsignificant and, additionally, that $A \supset B$ is significant iff A is significant and B is significant. Whence, by the usual definitional equivalences of $A \vee B$ as $\sim A \supset B$ and of $A \wedge B$ as $\sim(\sim A \vee \sim B)$, it is inferred that $A \wedge B$ is nonsignificant iff either A or B is nonsignificant, and similarly for $A \vee B$. Second, in [11, p. 262-264], it is argued that the logical positivist ascribed to two different understandings of nonsignificance which support the principle of Component Homogeneity. On the one hand, as we said above, nonsignificance is often conflated with empirical verifiability. Whence, $\sim A$ is empirically verifiable iff A is empirically verifiable, and also $A \wedge B$ is empirically verifiable iff both A and B are, with the same requirement for disjunction. On the other hand, nonsignificance was also paired by logical positivist with ill-formation and, thus, A is ill-formed iff $\sim A$ is, with the same holding for $A \wedge B$ and $A \vee B$ whenever A or B are ill-formed.

Goddard and Routley consider two main groups of objections to this account. Both consist, in a nutshell, in resisting that the Weak Kleene truth-functions are appropriate for a significance

[2]Presenting this as a matter of deciding which truth-functions should we endow these connectives with, entails something that Goddard and Routley also assume. That is, that in the context of a significance logic these connectives must have *extensional* semantics. Extensionality is a widely accepted requirement for notions so basic as negation, conjunction and disjunction (which does not mean that there cannot, or that it is uninteresting to consider intensional accounts of these logical connectives, but these will be admittedly unusual) and, thus, it needs no special justification in itself.

[3]Univalent semantics featuring values obeying the principle of Component Homogeneity are sometimes referred as inducing *infectious* logics (see [25], [16]), whence these characteristic values are also dubbed *infectious values.*

logic, although they focus their resistance in different places. The first group challenges the adoption of the above peculiar three-valued truth-tables, favoring instead the use of Kleene's *strong* regular tables. The second group questions the adoption of three-valued semantics to deal with nonsignificant sentences, claiming instead that two-valued logics, like **CL**, can deal with them successfully.

The first kind of objection suggests that nonsignificant sentences can be unproblematically conflated with other semantic categories in which sentences might be classified and, moreover, that if that is the case the appropriate truth-tables for the connectives $\sim, \wedge, \vee$ are the strong regular Kleene ones. For instance, Kleene himself motivated the move to a three-valued logic in the idea of partially recursive functions underlying a given language, where the status assigned to those sentences which cannot be algorithmically proven to be true, or false, is *undefined*. Now, as Goddard and Routley notice, Kleene suggested that instead of undefined, this can also be understood as meaningless and thus, nonsignificant. Since Kleene sided for his strong regular tables (which, most importantly, do *not* satisfy the principle of Component Homogeneity), the conclusion is that there can be significance logics which do not adopt the same truth-tables that Goddard and Routley did.

Goddard and Routley's response to this line of argumentation is that the third value of the strong Kleene tables cannot be read as nonsignificant. Nonsignificant sentences are genuinely non-truth-valued, i.e. they are neither true nor false, properly speaking. But undefined sentences, according to Kleene's reading, are not genuinely speaking neither true nor false, but actually either true or false. It is just that the given algorithmic procedure is not able to determine whether they are true or false, and thus the third value stands for an epistemic and not an ontological gap between truth and falsity. Undecided sentences are, still, one of true or false. Therefore, nonsignificance cannot be identified with undefinedness. Goddard and Routley consider, in addition to the standard Kleene interpretation for the strong tables, alternative readings of the third value as *undecidable* and *contingent*, for which the previous critique also applies, i.e. that they represent "a cross-classification of true-or-false sentences and not a genuine classification of three-valued sentences" [11, p. 269].

The second kind of objection suggests that there is no need for nonsense or, more specifically, for a third value representing the case where a sentence is neither true nor false but nonsignificant. The main idea here, is that nonsingifincant sentences are *plainly false*, because their negations must be assigned the truth-value true.[4] This take is, probably, justified by the idea that if a nonsignificant sentence (e.g. "This stone is thinking about Vienna") is queer, then its negation (i.e. "This stone is not thinking about Vienna") is less so.

Goddard and Routley's response to this line of argumentation goes as follows. Such a view will require that "ordinary equivalences derived from synonymy relations could no longer hold" [11, p. 12]. If nonsignificant negative sentences should be considered as true, but nonsignificant positive sentences should be taken to be plainly false, then obvious paraphrasing of positive nonsignificant sentences (e.g. "the square root of -1 is in motion") into negative ones (i.e. "the square root of -1 is not at rest") clash with the different semantic status that these sentences are bound to have. Moreover, if it is admitted that the negation of a nonsignificant sentence is itself nonsignificant, and it is still maintained that nonsignificant sentences should be plainly false, then we face a huge logical problem, for negation ceases to be a contradictory-forming operator and turns out to be a contrary-forming operator.

[4] Goddard and Routley consider, also, the thesis that nonsignificant sentences are such that both them and their negations should be true; and that one of them is true and the other is false. For matters of space and brevity, we will not summarize Goddard and Routley's discussion of these alternatives here. But let us notice that Goddard and Routley highlight that similar critiques apply to these and to the case we consider.

The upshot is, for Goddard and Routley, that there is not only a clear need for a significance logic, but also that such logical framework should handle (at least) logical negations, conjunctions and disjunctions, allowing them to work classically when sentences are either true or false, and behaving according to the principle of Component Homogeneity otherwise. Nevertheless, as Goddard and Routley clearly point out in [11, p. 273], settling for a set of connectives with their corresponding three-valued truth-tables does not imply having a significance logic. What we need, for that purpose, is to provide a corresponding definition of the notion of *logical consequence* at play. We now turn to review Goddard and Routley's discussion of this issue.

2.3. **Towards a significance logic: logical consequence.** What inferences are valid in a significance logic? What sentences are implied by nonsignificant sentences? Do meaningless sentences actually imply any sentence whatsoever? Answering these questions systematically amounts to define the notion of logical consequence in the context of a significance logic.

There are nowadays many —*many*— ways of doing this. However, one of the most popular options when dealing with many-valued logics and indeed the one which Goddard and Routley favored, takes logical consequence to be defined by the *preservation*, from premises to conclusion, of a given set of *designated values*. In a nutshell, the idea is that an inference is valid if and only if when all the premises are assigned designated values, so is the conclusion. In this vein, the *theorems* of a given logic are those sentences which are assigned a designated value in every valuation. Of course, in the case of two-valued **CL** this boils down to the idea that a valid argument is one which preserves truth from premises to conclusion, and a theorem is a sentence which is true in every valuation. But this need not be the case if we allow other values other than **t** to belong to the set of designated values.

Recall that, as per Goddard and Routley's understanding of nonsignificance, we should consider three possible cases: that a sentence is true (and thus assigned the truth-value **t**), that a sentence is false (and thus assigned the truth-value **f**), and that a sentence is nonsignificant, hence neither true nor false (and thus not assigned either of **t** or **f**, but the value **u**). Concerning the definition of logical consequence as preservation of designated values, Goddard and Routley's setting allows us to consider a number of options to configure the corresponding set of designated values.

Let us start first with the classical cases. To begin with, let us consider if **t** should be designated. If we take into account Goddard and Routley's idea that a significance logic should preserve as much of **CL** as possible, then of course inferences preserving truth should be rendered as valid and, therefore, **t** must be considered as a designated value. Secondly, let us consider if **f** should be designated. Regarding this Goddard and Routley argue in [11, p. 273] that, since sentences that receive a designated value in every valuation seem to play the role of the thesis asserted by the logic, then including **f** among the designated values will amount to the absurdity of asserting sentences that are false at every valuation. Therefore, **f** should not be considered as a designated value.

With **t** in and **f** out of the set of designated values for significance logics, we need to discuss the *inclusion or exclusion* of **u**, the value standing for nonsignificance. This leaves us with the two alternatives appearing in [11, p. 273-276]: either only **t** is designated, or both **t** and **u** are. Goddard and Routley call S-logics the significance logics adopting the former path, and C-logics the ones adopting the latter.[5] Building on some fairly standard terminology in the literature, we might refer to these alternatives as *paracomplete* and *paraconsistent*, respectively.

S-logics have a philosophical advantage and a technical disadvantage. Conceptually, they align well with the tradition, for they keep fixed the understanding of validity as truth-preservation,

[5]The name C-logics makes reference to Sören Halldéns logic C, defined in [12], who was the first to adopt the idea of designating both **t** and **u**.

although now incarnated in a three-valued setting. Significance logics working this way have had clear treatments in contemporary philosophy, the most salient being Kleene's discussion of the paracomplete logic induced by his Weak Kleene truth-tables, and the internal fragment of Dmitri Bochvar's logic of nonsense in [1]. However, Goddard and Routley did not consider Bochvar's system.[6] As many have pointed out, this approach has nevertheless an "overwhelming defect" (see e.g. [11, p. 273]): the resulting logic has no theorems, i.e. no tautologies or logically valid sentences. This is easily observed by the fact that the value $\mathbf{u}$ propagates or *infects* every sentence A featuring the connectives $\sim, \wedge, \vee$ in which it appears. If A has a subsentence e.g. p which is assigned the value $\mathbf{u}$, then A itself is assigned the value $\mathbf{u}$. Given that for every sentence A there are valuations which assign the value $\mathbf{u}$ to at least some of its subsentences, no sentence is a theorem for such kind of logics. Goddard and Routley consider a way out of this problem, on which we are not going to comment now, but below.

C-logics, on the other hand, have a technical advantage and a philosophical disadvantage. Logically speaking, these systems align well with the tradition too, for they keep the classical theorems or tautologies fixed; in fact, their logical truths coincide with those of **CL**. However, philosophically speaking, taking $\mathbf{u}$ to be designated implies that we are willing to affirm sentences that can sometimes be nonsensical, and this is, no wonder, a hard pill to swallow (see e.g. [11, p. 274] and [4, p. 219]).[7] Notwithstanding this issue, it should be pointed out that these systems have also some logical drawbacks, the prime being that the *modus ponens* rule is rendered as invalid. This can be observed by noticing that when A is nonsignificant so is $A \supset B$ and in each of these cases, the formulae are designated. Now, if B is false and thus undesignated, the inference from A and $A \supset B$ to B is invalid in C-logics. In addition to this, it can also be highlighted that functionally complete expansions of C-logics are inconsistent, i.e. they will count with a sentence A such that A and $\sim A$ are both theorems.

Interestingly, Goddard and Routley's take on the previous discussion is to propose some modifications that allow to overcome the technical and philosophical difficulties of both S-logics and C-logics. To present this third option we may notice, as Goddard and Routley do in [11, p. 274], that the lack of theorems of S-logics rests on the implicit assumption that all propositional variables range over the set $\{\mathbf{t}, \mathbf{u}, \mathbf{f}\}$. If this is the case, then since for every formula A there is a valuation in which A is assigned the value $\mathbf{u}$, for no A the formula $A \vee \sim A$ is a theorem. Now, if the above assumption is abandoned, then all the theorems of **CL** can be recovered. Goddard and Routley's abandonment of this requirement drive them to introduce *restricted propositional variables*, in addition to the ordinary propositional variables, in the sense that valuation functions for them range over the set $\{\mathbf{t}, \mathbf{f}\}$, while valuation functions for the ordinary unrestricted propositional variables range over the full set $\{\mathbf{t}, \mathbf{u}, \mathbf{f}\}$. In this vein, Goddard and Routley define logical consequence in some of their S-logics (e.g. in [11, p. 302, 307, 317]) by looking at premises and conclusions built up using only restricted propositional variables. The upshot of this syntactic instrumentation is, of course, that the rule of uniform substitution is invalid. Thus, even if $B \vee \sim B$ is a theorem when B is a restricted variable, we cannot infer from that, that $A \vee \sim A$ is a theorem for any formula A whatsoever, for A might as well be an unrestricted variable, which allows for a nonsignificant instance.

[6] We conjecture that this was caused by the fact that even though Bochvar's work was written in the late 30s in Russian, it was not made available in English until the 80s, via the translation due to Merrie Bergmann (see [2]).

[7] Halldén himself considered a response to this kind of accusation, claiming that a formula A is to be taken as asserting something only under significant (i.e. true or false) replacements [12, p. 47]. This is taken by Goddard and Routley as conceding that in such cases (i.e. when it is assigned $\mathbf{u}$) the formula in question is not strictly speaking designated, but only a substitution instance of a formula that is designated when only the values $\mathbf{t}$ and $\mathbf{f}$ are considered.

In what follows we will devote ourselves to make sense of the truth-functions assigned to the connectives $\sim, \wedge, \vee$ in the context of significance logics, within a framework developed by Priest, i.e. that of *plurivalent logics*. In doing so, we will prove how Goddard and Routley's idea that there are only two truth-values, truth and falsity, and that nonsignificant sentences are deprived of any of these, can be made perfect sense plurivalently speaking. Crucial to this instantiation will be to show that truth-value gaps behave, within plurivalent semantics, according to the widely discussed principle of Component Homogeneity. We will be, thus, focusing primarily in analyzing the three-valued connectives that make a significance logic from a two-valued and plurivalent point of view. Discussion of the consequence relation of the various significance logics proposed by Goddard and Routley will be, therefore, left out of the coming analysis, hoping to discuss it in further works.

3. Making sense of Goddard and Routley à la Priest

3.1. **Priest: Preliminaries of plurivalent semantics.** Plurivalent semantics will be presented, in what follows, as a useful tool for understanding some *many-valued* semantics for logical systems in terms of *less* truth-values —ideally, in terms of two truth-values: truth and falsity, i.e. $\mathbf{t}$ and $\mathbf{f}$. For this purpose, we first need to introduce some formalities concerning many-valued semantics, as they are usually understood in terms of univalent semantics. We basically follow the presentation of Priest's [19], but make some changes in some of the notations as well as the order of definitions.

Definition 3.1 (Univalent semantics)**.** A *univalent semantics* for the language $\mathcal{L}$ is a structure $M = \langle \mathcal{V}, \mathcal{D}, \delta \rangle$, where

- $\mathcal{V}$ is a non-empty set of truth values,
- $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, the designated values,
- δ contains, for every n-ary connective $*$ in the language, a truth-function $\delta_* : \mathcal{V}^n \to \mathcal{V}$.

A *univalent interpretation* is a pair $\langle M, \mu \rangle$, where M is such a structure, and μ is an evaluation function from Prop to $\mathcal{V}$. Given an interpretation, μ is extended to a map from Form to $\mathcal{V}$ recursively, by the following clause:

$$\mu(*(A_1, \ldots, A_n)) = \delta_*(\mu(A_1), \ldots, \mu(A_n)).$$

Finally, $\Sigma \models_u^M A$ iff in every univalent interpretation in which all the formulas of Σ are designated, so is A.[8]

As we previously advanced, plurivalent logics and their semantics can be thought as an alternative way to look at logical frameworks where instead of a formula's single truth-value coming from an arbitrary set, it is allowed for formulae to have one, more than one, or no truth-value at all from a given set. Note that the original idea behind the general construction can be found already in [17]. We begin now with the most general case of plurivalent semantics.

Definition 3.2 (General plurivalent semantics)**.** Given a univalent interpretation, the corresponding *general plurivalent interpretation* is the same, except that it replaces the evaluation function, μ, with a one-many evaluation relation, $\mathfrak{R}$, between Prop and $\mathcal{V}$. Given an interpretation, $\mathfrak{R}$ is extended to a map from Form to $\mathcal{V}$ recursively, following clause (‡):

(‡) $\quad *(A_1, \ldots, A_n)\mathfrak{R}v$ iff for some $v_1, \ldots, v_n$: $(A_i\mathfrak{R}v_i$ and $v = \delta_*(v_1, \ldots, v_n))$.

Finally, we can define two sorts of consequence relation out of this given plurivalent semantics. To this end, we will say that:

- $\mathfrak{R}$ *tolerantly* designates a formula A iff $A\mathfrak{R}v$ for some $v \in \mathcal{D}$,

[8] We will sometimes omit the subscript u, when contexts disambiguate.

- $\mathfrak{R}$ *strictly* designates a formula A iff it is not the case that $A\mathfrak{R}v$ for some $v \notin \mathcal{D}$.

Based on these notions of designation, we define the corresponding notions of tolerant and strict general plurivalent consequence as follows:[9]

- $\Sigma \models^M_{g,t} A$ iff in every general plurivalent interpretation, if $\mathfrak{R}$ tolerantly designates all the formulas of Σ then $\mathfrak{R}$ tolerantly designates A.
- $\Sigma \models^M_{g,s} A$ iff in every general plurivalent interpretation, if $\mathfrak{R}$ strictly designates all the formulas of Σ then $\mathfrak{R}$ strictly designates A.

Thus, we can prove a general relation between the univalent semantics and the general plurivalent semantics. To this end, we need the following definitions.

Definition 3.3. Let $M = \langle \mathcal{V}, \mathcal{D}, \delta \rangle$ be a univalent semantics. Then we can define two univalent semantics $\ddot{M}^t = \langle \ddot{\mathcal{V}}, \ddot{\mathcal{D}}^t, \ddot{\delta} \rangle$ and $\ddot{M}^s = \langle \ddot{\mathcal{V}}, \ddot{\mathcal{D}}^s, \ddot{\delta} \rangle$, where

- $\ddot{\mathcal{V}} = 2^{\mathcal{V}}$,
- $\ddot{\mathcal{D}}^t = \{\ddot{v} \in \ddot{\mathcal{V}} : v \in \ddot{v} \text{ for some } v \in \mathcal{D}\}$,
- $\ddot{\mathcal{D}}^s = \{\ddot{v} \in \ddot{\mathcal{V}} : v \notin \ddot{v} \text{ for all } v \notin \mathcal{D}\}$,
- all $\ddot{\delta}_*$ in $\ddot{\delta}$ are defined recursively by the following clause (‡‡):

(‡‡) $$v \in \ddot{\delta}_*(\ddot{v}_1, \ldots, \ddot{v}_n) \text{ iff for some } v_1, \ldots, v_n : (v_i \in \ddot{v}_i \text{ and } v = \delta_*(v_1, \ldots, v_n)).$$

Proposition 3.4. Given any univalent semantics $M = \langle \mathcal{V}, \mathcal{D}, \delta \rangle$, for any $\Sigma \cup \{A\}$:

- $\Sigma \models^M_{g,t} A$ *iff* $\Sigma \models^{\ddot{M}^t}_u A$,
- $\Sigma \models^M_{g,s} A$ *iff* $\Sigma \models^{\ddot{M}^s}_u A$.

Remark 3.5. The above results show that the corresponding general plurivalent semantics can be seen as univalent semantics $\ddot{M}^t = \langle \ddot{\mathcal{V}}, \ddot{\mathcal{D}}^t, \ddot{\delta} \rangle$ and $\ddot{M}^s = \langle \ddot{\mathcal{V}}, \ddot{\mathcal{D}}^s, \ddot{\delta} \rangle$ respectively.

We now turn to the positive plurivalent semantics, which is obtained by adding a constraint to the general plurivalent semantics.

Definition 3.6 (Positive plurivalent semantics)**.** Given a univalent interpretation, the corresponding *positive plurivalent interpretation* is the same, except that it replaces the evaluation function, μ, with a one-many evaluation relation, $\mathfrak{R}$, between Prop and $\mathcal{V}$ with the following *positivity condition*:

$$\text{for every } p \in \mathsf{Prop}\text{: } p\mathfrak{R}v \text{ for at least one } v \in \mathcal{V}.$$

Given an interpretation, $\mathfrak{R}$ is extended to a map from Form to $\mathcal{V}$ recursively, by the clause (‡). Then,

- $\Sigma \models^M_{p,t} A$ iff in every positive plurivalent interpretation, if $\mathfrak{R}$ tolerantly designates all the formulas of Σ then $\mathfrak{R}$ tolerantly designates A.
- $\Sigma \models^M_{p,s} A$ iff in every positive plurivalent interpretation, if $\mathfrak{R}$ strictly designates all the formulas of Σ then $\mathfrak{R}$ strictly designates A.

Then, we can prove a general relation between the two semantics. To state the result, the following definition will be useful.

Definition 3.7. Let $M = \langle \mathcal{V}, \mathcal{D}, \delta \rangle$ be a univalent semantics. Then we can define a two univalent semantics $\dot{M}^t = \langle \dot{\mathcal{V}}, \dot{\mathcal{D}}^t, \dot{\delta} \rangle$, and $\dot{M}^s = \langle \dot{\mathcal{V}}, \dot{\mathcal{D}}^s, \dot{\delta} \rangle$, where

- $\dot{\mathcal{V}} = 2^{\mathcal{V}} \backslash \{\emptyset\}$,
- $\dot{\mathcal{D}}^t = \{\dot{v} \in \dot{\mathcal{V}} : v \in \dot{v} \text{ for some } v \in \mathcal{D}\}$,

[9] This terminology is not original to Priest, but borrowed from [24].

- $\dot{\mathcal{D}}^s = \{\dot{v} \in \dot{\mathcal{V}} : v \notin \dot{v}$ for all $v \notin \mathcal{D}\}$,
- all $\dot{\delta}_*$ in $\dot{\delta}$ are defined recursively by the clause (‡‡).

Proposition 3.8. Given any univalent semantics $M = \langle \mathcal{V}, \mathcal{D}, \delta \rangle$, for any $\Sigma \cup \{A\}$:

- $\Sigma \models^M_{p,t} A$ *iff* $\Sigma \models^{\dot{M}^t}_u A$,
- $\Sigma \models^M_{p,s} A$ *iff* $\Sigma \models^{\dot{M}^s}_u A$.

Remark 3.9. The above results show that the corresponding positive plurivalent semantics can be seen as the univalent semantics $\dot{M}^t = \langle \dot{\mathcal{V}}, \dot{\mathcal{D}}^t, \dot{\delta} \rangle$, and $\dot{M}^s = \langle \dot{\mathcal{V}}, \dot{\mathcal{D}}^s, \dot{\delta} \rangle$ respectively.

3.2. **Plurivalence and the FDE family: some examples.** So far we have been looking at plurivalent semantics from a rather abstract point of view. Here are some examples, obtained by applying plurivalence to the logics of what Priest calls in [19] the **FDE** family (we will let **b**, **n**, **e** and **a** denote the characteristic non-classical truth-values of **LP**, $\mathbf{K_3}$, $\mathbf{K_3^w}$ and **PWK**, respectively).[10]

Definition 3.10. Let $M = \langle \mathcal{V}, \mathcal{D}, \delta \rangle$ be a univalent semantics of the **FDE** family. Then we define a univalent semantics $M^{\mathbf{b}} = \langle \mathcal{V}^{\mathbf{b}}, \mathcal{D}^{\mathbf{b}}, \delta^{\mathbf{b}} \rangle$, where:

- $\mathcal{V}^{\mathbf{b}} = \mathcal{V} \cup \{\mathbf{b}\}$,
- $\mathcal{D}^{\mathbf{b}} = \mathcal{D} \cup \{\mathbf{b}\}$,
- $\delta^{\mathbf{b}}$ contains a function $\delta^{\mathbf{b}}_*$ that extends, if needed, each δ_* by letting **b** interact with the truth-values in $\mathcal{V}$ as it does in the characteristic univalent semantics for the logics in the **FDE** family.

We also define univalent semantics $M^{\mathbf{n}} = \langle \mathcal{V}^{\mathbf{n}}, \mathcal{D}^{\mathbf{n}}, \delta^{\mathbf{n}} \rangle$ in a similar manner, with $\mathcal{D}^{\mathbf{n}} = \mathcal{D}$.

Definition 3.11. Let $M = \langle \mathcal{V}, \mathcal{D}, \delta \rangle$ be a univalent semantics of the **FDE** family. Then we can define a univalent semantics $M^{\mathbf{e}} = \langle \mathcal{V}^{\mathbf{e}}, \mathcal{D}^{\mathbf{e}}, \delta^{\mathbf{e}} \rangle$, where:

- $\mathcal{V}^{\mathbf{e}} = \mathcal{V} \cup \{\mathbf{e}\}$,
- $\mathcal{D}^{\mathbf{e}} = \mathcal{D}$,
- $\delta^{\mathbf{e}}$ contains a function $\delta^{\mathbf{e}}_*$ that extends, if needed, each δ_* by letting $\delta^{\mathbf{e}}_*(v^{\mathbf{e}}_1, \dots, v^{\mathbf{e}}_n) = \mathbf{e}$ iff $v^{\mathbf{e}}_i = \mathbf{e}$ for some $v^{\mathbf{e}}_i \in \mathcal{V}^{\mathbf{e}}$. Otherwise, $\delta^{\mathbf{e}}_* = \delta_*$.

We also define univalent semantics $M^{\mathbf{a}} = \langle \mathcal{V}^{\mathbf{a}}, \mathcal{D}^{\mathbf{a}}, \delta^{\mathbf{a}} \rangle$ in a similar manner, with $\mathcal{D}^{\mathbf{a}} = \mathcal{D} \cup \{\mathbf{a}\}$.

With the help of these definitions, Priest observed the following results.[11]

Theorem 3.12 ([19])**.** *Let M be a univalent semantics of the* **FDE** *family. Then, for any $\Sigma \cup \{A\}$:*

- $\Sigma \models^M_{g,t} A$ *iff* $\Sigma \models^{M^{\mathbf{b,e}}}_u A$,
- $\Sigma \models^M_{g,s} A$ iff $\Sigma \models^{M^{\mathbf{n,a}}}_u A$,
- $\Sigma \models^M_{p,t} A$ iff $\Sigma \models^{M^{\mathbf{b}}}_u A$,
- $\Sigma \models^M_{p,s} A$ iff $\Sigma \models^{M^{\mathbf{n}}}_u A$.

Proposition 3.13 ([19])**.** Let M be a univalent semantics. Then, for any $\Sigma \cup \{A\}$:

- $\Sigma \models^M_{g,t} A$ iff $\Sigma \models^{M^{\mathbf{e}}}_{p,t} A$,
- $\Sigma \models^M_{g,s} A$ iff $\Sigma \models^{M^{\mathbf{a}}}_{p,s} A$.

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[10] For a detailed presentation of **PWK**, see e.g. [15] and [3].

[11] The second results of both Theorem 3.12 and Proposition 3.13 are not originally stated in Priest's work, but in [24].

Example 3.14. First, let M be the univalent semantics for **CL**. Then, $M^{\mathbf{b,e}}$ and $M^{\mathbf{n,a}}$ are the univalent semantics of Deutsch's system $\mathsf{S}_{\mathtt{fde}}$ ([8]) and for the system $\mathsf{dS}_{\mathtt{fde}}$, i.e. the dual of $\mathsf{S}_{\mathtt{fde}}$.[12] respectively. Thus, for any $\Sigma \cup \{A\}$:

- $\Sigma \models^{M}_{g,t} A$ iff $\Sigma \models^{M^{\mathbf{b,e}}}_{u} A$ iff $\Sigma \models^{\mathsf{S}_{\mathtt{fde}}} A$,
- $\Sigma \models^{M}_{g,s} A$ iff $\Sigma \models^{M^{\mathbf{n,a}}}_{u} A$ iff $\Sigma \models^{\mathsf{dS}_{\mathtt{fde}}} A$.

Second, let M be a univalent semantics for Strong Kleene logic $\mathbf{K_3}$. Then, $M^{\mathbf{b}}$ is the univalent semantics of **FDE**. Thus, for any $\Sigma \cup \{A\}$:

- $\Sigma \models^{M}_{p,t} A$ iff $\Sigma \models^{M^{\mathbf{b}}}_{u} A$ iff $\Sigma \models^{\mathbf{FDE}} A$.

Finally, let M be a univalent semantics for $\mathsf{S}_{\mathtt{fde}}$. Then, $M^{\mathbf{n}}$ is the univalent semantics of Daniels' logic $\mathsf{S}^{*}_{\mathtt{fde}}$ ([7]). Thus, for any $\Sigma \cup \{A\}$:

- $\Sigma \models^{M}_{p,s} A$ iff $\Sigma \models^{M^{\mathbf{n}}}_{u} A$ iff $\Sigma \models^{\mathsf{S}^{*}_{\mathtt{fde}}} A$.

3.3. **Instantiating Routley: negative and singular plurivalent semantics.** It is also possible to focus on alternative restrictions of the general plurivalent framework that allow to modularly connect univalent semantics with plurivalent semantics allowing for sentences not to receive a truth-value at all [11, p. 266], i.e. being a truth-value gap that behaves according to the principle of Component Homogeneity.

The first attempt along these lines was the negative plurivalence due to Hitoshi Omori in [15].

Definition 3.15 (Negative plurivalent semantics)**.** Given a univalent interpretation, the corresponding *negative plurivalent interpretation* is the same, except that it replaces the evaluation function, μ, with a one-many evaluation relation, $\mathfrak{R}$, between Prop and $\mathcal{V}$ with the following *negativity condition*:

$$\text{for every } p \in \mathsf{Prop}\text{: it is not the case that } p\mathfrak{R}v \text{ for all } v \in \mathcal{V}$$

Given an interpretation, $\mathfrak{R}$ is extended to a map from Form to $\mathcal{V}$ recursively, by the clause (‡). Finally,

- $\Sigma \models^{M}_{n,t} A$ iff in every negative plurivalent interpretation, if $\mathfrak{R}$ tolerantly designates all the formulas of Σ then $\mathfrak{R}$ tolerantly designates A.
- $\Sigma \models^{M}_{n,s} A$ iff in every negative plurivalent interpretation, if $\mathfrak{R}$ strictly designates all the formulas of Σ then $\mathfrak{R}$ strictly designates A.

Then, we can again prove a relation between the univalent semantics and negative plurivalent semantics. However, as we will see shortly, the *only* starting point allowed for negative plurivalence to define a proper consequence relation is the two-valued univalent semantics for **CL**.

Definition 3.16. Let $M = \langle \mathcal{V}, \mathcal{D}, \delta \rangle$ be the univalent semantics for **CL**. Then we define two univalent semantics $\dddot{M}^t = \langle \dddot{\mathcal{V}}, \dddot{\mathcal{D}}^t, \dddot{\delta} \rangle$ and $\dddot{M}^s = \langle \dddot{\mathcal{V}}, \dddot{\mathcal{D}}^s, \dddot{\delta} \rangle$, where

- $\dddot{\mathcal{V}} = 2^{\mathcal{V}} \setminus \{\mathcal{V}\}$,
- $\dddot{\mathcal{D}}^t = \{\dddot{v} \in \dddot{\mathcal{V}} : v \in \dddot{v} \text{ for some } v \in \mathcal{D}\}$,
- $\dddot{\mathcal{D}}^s = \{\dddot{v} \in \dddot{\mathcal{V}} : v \notin \dddot{v} \text{ for all } v \notin \mathcal{D}\}$,
- all $\dddot{\delta}_*$ in $\dddot{\delta}$ are defined recursively by the clause (‡‡).

Proposition 3.17. Given the two-valued univalent semantics $M = \langle \mathcal{V}, \mathcal{D}, \delta \rangle$ for **CL**, its corresponding negative plurivalent semantics can be seen as univalent semantics $\dddot{M}^t = \langle \dddot{\mathcal{V}}, \dddot{\mathcal{D}}^t, \dddot{\delta} \rangle$ and $\dddot{M}^s = \langle \dddot{\mathcal{V}}, \dddot{\mathcal{D}}^s, \dddot{\delta} \rangle$, i.e. for any $\Sigma \cup \{A\}$:

[12] The logic $\mathsf{dS}_{\mathtt{fde}}$ is referred to as $\mathsf{L}_{\mathbf{nb'}}$ in [25], and has recently been discussed as suitable to address semantic paradoxes in [6].

- $\Sigma \models^{M}_{n,t} A$ *iff* $\Sigma \models^{\ddot{M}^t}_{u} A$,
- $\Sigma \models^{M}_{n,s} A$ *iff* $\Sigma \models^{\ddot{M}^s}_{u} A$.

Proposition 3.18. Let M be the two-valued univalent semantics for **CL**. Then, for any $\Sigma \cup \{A\}$:

- $\Sigma \models^{M}_{n,t} A$ *iff* $\Sigma \models^{M^{\mathbf{e}}} A$,
- $\Sigma \models^{M}_{n,s} A$ *iff* $\Sigma \models^{M^{\mathbf{a}}} A$.

Remark 3.19. As observed by Omori in [15], negative plurivalence is *not* applicable to all the logics in the **FDE** family, but just to **CL**. By this we mean that negative plurivalence does not always define a consequence relation out of a given univalent semantics from the **FDE** family.

Thus, we might want to find suitable plurivalent semantics that are flexible enough to render, for each univalent semantics M in the **FDE** family, a plurivalent surrogate for both the univalent semantics $M^{\mathbf{e}}$ and $M^{\mathbf{a}}$ (where **e** and **a** are the characteristic truth-values of $\mathbf{K_3^w}$ and **PWK**, respectively). There are, in fact, additional philosophical reasons to be interested in this enterprise, besides technical curiosity.

First, plurivalent semantics are well suited tools for making sense of many-valued univalent semantics with the aid of smaller number of truth-values. For example, in the case of Goddard and Routley's S_0, with the help of negative and singular plurivalent semantics (explained below) we can understand it as two-valued logic that allows for truth-value gaps that behave according to the principle of Component Homogeneity.

Second, although Goddard and Routley did not argue for the existence of meaningful cases of truth-value gaps that did not behave infectiously, and neither did they for the case of truth-value gluts behaving like that, it seems plausible and conceptually interesting to entertain such systems. Some contemporary philosophers and logicians are of the opinion that genuine cases of the former and the latter do exist (e.g. [18], [10]) and, thus, it might also be reasonable to have a semantic framework that allows to handle such gaps and gluts, alongside with nonsignificance. We will present a framework that fills this role, called the *singular plurivalence*, explaining the inclusion of nonsignificance in a plurivalent manner, following Goddard and Routley, who took sentences of this meaningless sort to have no truth-value at all.

Therefore, this investigation will shed some light into the question of how to find, given a univalent semantics M in the **FDE** family, a plurivalent surrogate for both the univalent semantics $M^{\mathbf{e}}$ and $M^{\mathbf{a}}$. To this we now turn.

Definition 3.20 (Singular plurivalent semantics)**.** Given a univalent interpretation, the corresponding *singular plurivalent interpretation* is the same, except that it replaces the evaluation function, μ, with a one-many evaluation relation, $\mathfrak{R}$, between Prop and $\mathcal{V}$ with the following *singularity condition*:

$$\text{for every } p \in \mathsf{Prop}\text{: } p\mathfrak{R}v \text{ for at most one } v \in \mathcal{V}.$$

Given an interpretation, $\mathfrak{R}$ is extended to a map from Form to $\mathcal{V}$ recursively, by the clause (‡). Finally,

- $\Sigma \models^{M}_{s,t} A$ iff for every singular plurivalent interpretation, if $\mathfrak{R}$ tolerantly designates all the formulas of Σ then $\mathfrak{R}$ tolerantly designates A,
- $\Sigma \models^{M}_{s,s} A$ iff for every singular plurivalent interpretation, if $\mathfrak{R}$ strictly designates all the formulas of Σ then $\mathfrak{R}$ strictly designates A.

Then, we can prove a general relation between the two semantics. To state the result, the following definition will be useful.

Definition 3.21. Let $M = \langle \mathcal{V}, \mathcal{D}, \delta \rangle$ be a univalent semantics. Then we can define two univalent semantics $\dddot{M}^t = \langle \dddot{\mathcal{V}}, \dddot{\mathcal{D}}^t, \dddot{\delta} \rangle$ and $\dddot{M}^s = \langle \dddot{\mathcal{V}}, \dddot{\mathcal{D}}^s, \dddot{\delta} \rangle$, where

- $\ddddot{\mathcal{V}} = \mathcal{V} \cup \{\emptyset\}$,
- $\ddddot{\mathcal{D}}^t = \{\ddddot{v} \in \ddddot{\mathcal{V}} : v \in \ddddot{v} \text{ for some } v \in \mathcal{D}\}$,
- $\ddddot{\mathcal{D}}^s = \{\ddddot{v} \in \ddddot{\mathcal{V}} : v \notin \ddddot{v} \text{ for all } v \notin \mathcal{D}\}$,
- all $\ddddot{\delta}_*$ in $\ddddot{\delta}$ are defined recursively by the clause (‡‡).

Proposition 3.22. Given any univalent semantics $M = \langle \mathcal{V}, \mathcal{D}, \delta \rangle$, its corresponding singular plurivalent semantics can be seen as the univalent semantics $\ddddot{M}^t = \langle \ddddot{\mathcal{V}}, \ddddot{\mathcal{D}}^t, \ddddot{\delta} \rangle$ and $\ddddot{M}^s = \langle \ddddot{\mathcal{V}}, \ddddot{\mathcal{D}}^s, \ddddot{\delta} \rangle$, i.e. for any $\Sigma \cup \{A\}$:

- $\Sigma \models^M_{s,t} A$ iff $\Sigma \models^{\ddddot{M}^t}_u A$,
- $\Sigma \models^M_{s,s} A$ iff $\Sigma \models^{\ddddot{M}^s}_u A$.

Let us now highlight some relations between singular plurivalent semantics and the **FDE** family.

Theorem 3.23. *Let M be a univalent semantics of the* **FDE** *family. Then, for any $\Sigma \cup \{A\}$:*

$$\Sigma \models^M_{s,t} A \textit{ iff } \Sigma \models^{M^{\mathbf{e}}}_u A.$$

Proof. Consider a homomorphism between $\langle \ddddot{\mathcal{V}}, \ddddot{\mathcal{D}}^t, \ddddot{\delta} \rangle$ and $M^{\mathbf{e}}$. Let $\theta : \ddddot{\mathcal{V}} \longrightarrow \mathcal{V} \cup \{\mathbf{e}\}$ be such that

- if $\mathbf{t} \in X$, then $\theta(X) = \mathbf{t}$
- else: if $\mathbf{b} \in X$, then $\theta(X) = \mathbf{b}$
- else: if $\mathbf{a} \in X$, then $\theta(X) = \mathbf{a}$
- else: if $\mathbf{f} \in X$, then $\theta(X) = \mathbf{f}$
- else: if $\mathbf{n} \in X$, then $\theta(X) = \mathbf{n}$
- else: if $\mathbf{e} \in X$, then $\theta(X) = \mathbf{e}$
- else: $\theta(X) = \mathbf{e}$

It is easily checked that θ is a homomorphism and that it is onto. Tedious verification also allows establishing that it preserves all the logical operations, as well as designatedness. □

Example 3.24. Let M be a univalent semantics for Strong Kleene logic $\mathbf{K_3}$. Hence, $M^{\mathbf{e}}$ is the univalent semantics for a system Priest's highlights in [19, p. 6] counting with values $\mathbf{n}$ and $\mathbf{e}$ and referred to as $\mathsf{L}_{\mathbf{ne}}$ in [25]. Then, for any $\Sigma \cup \{A\}$: $\Sigma \models^M_{s,t} A$ iff $\Sigma \models^{\mathsf{L}_{\mathbf{ne}}} A$.

Theorem 3.25. *Let M be a univalent semantics of the* **FDE** *family. Then, for any $\Sigma \cup \{A\}$:*

$$\Sigma \models^M_{s,s} A \textit{ iff } \Sigma \models^{M^{\mathbf{a}}}_u A.$$

Proof. Consider a homomorphism between $\langle \ddddot{\mathcal{V}}, \ddddot{\mathcal{D}}^s, \ddddot{\delta} \rangle$ and $\mathcal{M}^{\mathbf{a}}$. Let $\theta : \ddddot{\mathcal{V}} \longrightarrow \mathcal{V} \cup \{\mathbf{a}\}$ be such that

- if $\mathbf{f} \in X$, then $\theta(X) = \mathbf{f}$
- else: if $\mathbf{n} \in X$, then $\theta(X) = \mathbf{n}$
- else: if $\mathbf{e} \in X$, then $\theta(X) = \mathbf{e}$
- else: if $\mathbf{t} \in X$, then $\theta(X) = \mathbf{t}$
- else: if $\mathbf{b} \in X$, then $\theta(X) = \mathbf{b}$
- else: if $\mathbf{a} \in X$, then $\theta(X) = \mathbf{a}$
- else: $\theta(X) = \mathbf{a}$

It is easily checked that θ is a homomorphism and that it is onto. Tedious verification also allows establishing that it preserves all the logical operations, as well as designatedness. □

Example 3.26. Let M be the univalent semantics for **FDE**. Hence, $M^{\mathbf{a}}$ is the univalent semantics for the system $\mathsf{dS}^*_{\mathtt{fde}}$, i.e. the dual of $\mathsf{S}^*_{\mathtt{fde}}$.[13] Then, for any $\Sigma \cup \{A\}$: $\Sigma \models^{M}_{s,s} A$ iff $\Sigma \models^{M^{\mathbf{a}}}_{u} A$ iff $\Sigma \models^{\mathsf{dS}^*_{\mathtt{fde}}} A$.

Remark 3.27. In [19] Priest considers, given a univalent interpretation, the corresponding *unique plurivalent interpretation* to be the same, except that it replaces the evaluation function, μ, with a one-many evaluation relation, $\mathfrak{R}$, between Prop and $\mathcal{V}$ with the following *uniqueness condition*:

$$\text{for every } p \in \mathsf{Prop}\text{: } p\mathfrak{R}v \text{ for exactly one } v \in \mathcal{V}.$$

Highlighting, furthermore, that this plurivalent semantics will just amount to a "notational variant" of the corresponding univalent semantics. We might note, additionally, that the uniqueness condition is *equivalent* to the conjunction of the positive and the singularity conditions. But, otherwise, these conditions do not separately entail the uniqueness condition.

Remark 3.28. When the univalent semantics taken as a starting point are those of **CL**, singular plurivalence and negative plurivalence coincide. However, when other univalent semantics from e.g. the **FDE** family are taken as the basis this is not the case.

3.4. **Reflecting upon truth-value gaps in Dunn and plurivalent semantics.** It is worth noticing that in the context of general, negative and singular plurivalent semantics, truth-values behaving according to the principle of Component Homogeneity can *only* be represented by the *empty set*.

As the previous examples show, this is remarkably so, even if we start with univalent semantics other than two-valued **CL**, such as **LP**, or even **FDE**. If we, additionally, think of a plurivalent truth-value x as being *true* if $\mathbf{t} \in x$, and, respectively, as being *false* if $\mathbf{f} \in x$, then it seems that the plurivalent setting follows Goddard and Routley's intuition that nonsignificant sentences are nothing more than truth-value *gaps*. In other words, these sentences are semantically described as not having *any truth-value at all*. And this last description is fully general: for it not only refers to a sentence not receiving the truth-value $\mathbf{t}$ or $\mathbf{f}$, but also not receiving the truth-values $\mathbf{b}$, $\mathbf{n}$, $\mathbf{e}$ and $\mathbf{a}$.

But this raises a question about the peculiarity of the behavior of truth-value gaps in the context of the plurivalent semantics, as opposed to the natural alternative of Dunn semantics (or relational semantics) first developed by J. Michael Dunn (cf. [9]). Interestingly, it shall be noted that Dunn semantics have more often than not received an informational or *epistemic* reading, by means of which the fact that a given sentence has been assigned no truth-value is interpreted as its truth-value being undecided, or as having no available information concerning its semantic status. All of which are compatible with future stages where its truth-value is decided, or where there is certain information concerning its semantic status.

Much to the contrary, in the context of plurivalent semantics, the fact that a given sentence has been assigned no truth-value at all shall be interpreted under an *ontological* reading, by means of which this fact really represents the absolute lack of a truth-value, not merely the lack of knowledge or certainty of the truth-value of the sentence in question. This, of course, does not preclude the Dunn semantics being able to receive an ontological understanding, as they have in fact received in e.g. [14].

In any case, and notwithstanding these philosophical reflections, one might wonder what is it that causes truth-value gaps to act "infectiously" in plurivalent semantics and "non-infectiously" in the regular Dunn semantics? The answer lies in the way these semantics define the truth and falsity conditions for each of the logical connectives. To show this, we will focus on the special

[13] The logic $\mathsf{dS}^*_{\mathtt{fde}}$ is referred to as $\mathsf{L}_{\mathbf{nbb'}}$ in [25].

cases of conjunction and disjunction. Thus, the *classical* understanding of the truth and falsity conditions for conjunction and disjunction has it that:

- $A \wedge B$ is true *iff* A is true and B is true,
- $A \wedge B$ is false *iff* A is false or B is false,
- $A \vee B$ is true *iff* A is true or B is true,
- $A \vee B$ is false *iff* A is false and B is false.

Regarding this, it shall be noticed that Dunn and plurivalent semantics coincide in their preservation of the classical truth condition for conjunction, and the classical falsity condition for disjunction, but they differ in some other respects, as recently pointed out in [16].

To observe their coincidence, note that the usual account of Dunn semantics asks that[14]

- $(A \wedge B)\mathfrak{r}\mathbf{t}$ *iff* $A\mathfrak{r}\mathbf{t}$ & $B\mathfrak{r}\mathbf{t}$,
- $(A \vee B)\mathfrak{r}\mathbf{f}$ *iff* $A\mathfrak{r}\mathbf{f}$ & $B\mathfrak{r}\mathbf{f}$,

while the plurivalent semantics require that

- $(A \wedge B)\mathfrak{R}\mathbf{t}$ *iff* $A\mathfrak{R}\mathbf{t}$ & $B\mathfrak{R}\mathbf{t}$ and for some $\mathbf{v}_1, \mathbf{v}_2 : A\mathfrak{R}\mathbf{v}_1$ & $B\mathfrak{R}\mathbf{v}_2$,
- $(A \vee B)\mathfrak{R}\mathbf{f}$ *iff* $A\mathfrak{R}\mathbf{f}$ & $B\mathfrak{R}\mathbf{f}$ and for some $\mathbf{v}_1, \mathbf{v}_2 : A\mathfrak{R}\mathbf{v}_1$ & $B\mathfrak{R}\mathbf{v}_2$,

which, by simplification, respectively imply the classical conditions.

To observe where these accounts part ways, note that they differ concerning the falsity condition for conjunction and the truth condition for disjunction. Thus, Dunn semantics require that

- $(A \wedge B)\mathfrak{r}\mathbf{f}$ *iff* $A\mathfrak{r}\mathbf{f}$ or $B\mathfrak{r}\mathbf{f}$,
- $(A \vee B)\mathfrak{r}\mathbf{t}$ *iff* $A\mathfrak{r}\mathbf{t}$ or $B\mathfrak{r}\mathbf{t}$,

while the plurivalent semantics require that

- $(A \wedge B)\mathfrak{R}\mathbf{f}$ *iff* $A\mathfrak{R}\mathbf{f}$ or $B\mathfrak{R}\mathbf{f}$ and for some $\mathbf{v}_1, \mathbf{v}_2 : A\mathfrak{R}\mathbf{v}_1$ & $B\mathfrak{R}\mathbf{v}_2$,
- $(A \vee B)\mathfrak{R}\mathbf{t}$ *iff* $A\mathfrak{R}\mathbf{t}$ or $B\mathfrak{R}\mathbf{t}$ and for some $\mathbf{v}_1, \mathbf{v}_2 : A\mathfrak{R}\mathbf{v}_1$ & $B\mathfrak{R}\mathbf{v}_2$,

which do *not* necessarily entail the classical conditions.

To further highlight their differences, in case e.g. A is false and B is a truth-value gap, notice that the Dunn semantics will judge $A \wedge B$ as false, whereas the plurivalent semantics will take it to be a truth-value gap itself. Similarly, in case e.g. A is true and B is a truth-value gap, the Dunn semantics will take $A \vee B$ to be true, while the plurivalent semantics will consider it as a truth-value gap. This in itself shows that plurivalent semantics treat truth-value gaps according to the principle of Component Homogeneity, which is not the case with Dunn semantics.[15]

This is very much in alignment with Goddard and Routley's motivations. In fact, in their book, they explicitly acknowledge that embracing the principle of Component Homogeneity collides with the adoption of the classical truth and falsity conditions for conjunction and disjunction. In [11, p. 260] they highlight that appropriately dealing with nonsignificant sentences requires taking a distinctive stance towards these issues, by means of which

- $A \wedge B$ is true *iff* A is true and B is true, and both A and B are significant,
- $A \vee B$ is false *iff* A is false and B is false, and both A and B are significant,

which, given being non-significant entails being neither true nor false, can be simplified to arrive to the classical conditions. More interestingly, Goddard and Routley's adoption of Component Homogeneity entails that

- $A \wedge B$ is false *iff* A is false or B is false, and both A and B are significant,

[14] We will use $\mathfrak{r}$ and $\mathfrak{R}$ for relations in Dunn and plurivalent semantics respectively.

[15] This is not to say that logics counting with truth-value gaps behaving in accordance with the principle of Component Homogeneity cannot receive possible realizations in terms of Dunn semantics, but that this will require some tweaking of the truth and falsity conditions of disjunction and conjunction, respectively.

- $A \vee B$ is true *iff* A is true or B is true, and both A and B are significant,

both being stricter conditions than the classical ones.

Does this mean that we should always favor plurivalent semantics over Dunn semantics, when dealing with significance logics? We do not believe so. Even though the former constitute a framework that is more likely to faithfully represent the logical operations in contexts where the principle of Component Homogeneity holds (that is, to give a better understanding of the Weak Kleene operations), it also has it problematic corners, outside of significance logics talk.[16] At any rate, here we did not intend to argue for or against any of these semantics as an all-purpose setting, but to provide a better explanation of the different treatment they give of truth-value gaps, and how it affects the way in which these semantics might or might not be in accordance with Goddard and Routley's approach to significance logics.

4. Concluding remarks

In this note we revisited Goddard and Routley's work on significance logics, with the aim of analyzing their semantic account of these systems with the aid of plurivalent semantics, developed by Priest. The result of such an investigation led us to show that significance logics and their subsystems can be made perfect sense of as plurivalent logics built starting from the univalent semantics for two-valued **CL**. Indeed, this allows to look at significance logics as systems having truth-value gaps which behave complying with the principle of Component Homogeneity, according to which if a sentence has a nonsignificant component, then it is nonsignificant itself.

It will come as no surprise that there is still a huge amount of work to be done in revisiting Goddard and Routley's work on significance logics. Prime among the remaining tasks is the formalization of the significance connectives, i.e those connectives which convert nonsense into sense —like the significance operator $\circ$ in the Appendix below— within the setting of plurivalent semantics. It seems as if none of these significance connectives can be defined plurivalently by taking operations of the carrier algebra. Along this line, it is argued in [16] that e.g. the significance operator $\circ$ might well be understood as a connective saying that the set of truth-values assigned to a given sentence is non-empty. Remarks of this sort make it more plausible that the meta-theoretical nature of these notions precludes representing them as normal plurivalent operations. Confirming or disconfirming such a claim would be extremely interesting.

Yet another important issue that was left out of our discussion here were proof systems. It seems that there are some important motivations for considering proof systems for significance logics. First, Goddard and Routley were specially concerned with the problem of conducting reasoning in significance logics, symptom of which is that they devote an entire chapter of their book, i.e. [11, ch. 6] to the discussion of proof systems for these frameworks. Second, that although Goddard and Routley considered in [11, p. 381] a Hilbert-style axiomatic system for Halldén's significance logic, they only proved the *weak* completeness theorem for the calculus in question, whence it is important to explore possible *strongly* complete calculi.[17] Third, that some recent incursions in this regard are confined only to the $\{\sim, \wedge, \vee\}$-fragment of these logics (see e.g. [5]), whereas we want to focus on the full propositional language considered for these

[16] Thus, for instance, while Strong Kleene logic $\mathbf{K_3}$ is usually taken to be a three-valued logic counting with truth-value gaps, its plurivalent interpretation requires to understand its characteristic non-classical truth-value $\mathbf{n}$, plurivalently as the set $\{\mathbf{t}, \mathbf{f}\}$, which is naturally read as a truth-value glut.

[17] The importance of the strong completeness as opposed to the weak completeness lies in the impossibility to differentiate some non-classical logics such as Priest's **LP** or Halldén's logic C, from **CL**, if we just focus on the theorems or logical truths.

systems, including the significance connectives.[18] We hope to deal with these issues in future work.

Appendix: functionally complete significance logics

Goddard and Routley consider in [11, ch. 5] a *sequence* of more than sixteen significance logics, some of which are S-logics (and, thus, take only $\mathbf{t}$ to be designated), and some of which are C-logics (and, thus, take $\mathbf{t}$ and $\mathbf{u}$ to be designated). Here we will not focus on their motivations to seek for further systems along the way, but instead we will touch upon the functionally complete significance logics that they define in e.g. [11, p. 345-355] and [11, p. 355-365], referred to as 5-logics and 6-logics, i.e. the systems S_5 and C_5, and S_6 and C_6, respectively.

These pairs of logics are obtained by extending the algebra formed by the connectives in the set $\{\sim, \wedge, \vee\}$ endowed with the Weak Kleene truth-tables, as follows. In the case of 5-logics, the expansion includes the connectives in the set $\{\circ, \mathsf{Q}\}$ endowed with their truth-functions below, and in the case of 6-logics the expansion includes the connectives in the set $\{\neg, \mathsf{D}, \rightharpoonup\}$ endowed with their corresponding truth-functions below.

	$\circ$
$\mathbf{t}$	$\mathbf{t}$
$\mathbf{u}$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{t}$

	Q
$\mathbf{t}$	$\mathbf{t}$
$\mathbf{u}$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{u}$

	$\neg$
$\mathbf{t}$	$\mathbf{f}$
$\mathbf{u}$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{t}$

	D
$\mathbf{t}$	$\mathbf{u}$
$\mathbf{u}$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{f}$

$\rightharpoonup$	$\mathbf{t}$	$\mathbf{u}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{u}$	$\mathbf{f}$
$\mathbf{u}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$

We shall notice that $\circ$ is the well-known significance or meaningfulness operator appearing always in the literature about significance logics, but that Goddard and Routley offer no intuitive conceptual reading of Q or D (see [11, p. 346]). To the contrary, $\neg$ seems to be a well-motivated three-valued generalization of a *classical* negation (a connective that is true iff the negated sentence is not true, and false iff the negated sentence is true) and $\rightharpoonup$ is taken by Goddard and Routley to be a "three-valued extension of the two-valued '$\rightarrow$' and its interpretation is approximately —but only approximately— given by the ordinary language connective 'provided that... then...' " [11, p. 324].

But why at all are Goddard and Routley concerned with functionally complete expansions of significance logics, beyond functional completeness being a nice feature of logical frameworks? First, because they crave for significance-complete logics. By this we mean logics such that *all* truth-functions that do not turn sense into nonsense are definable in them. However, this is not the only thing Goddard and Routley aim at. They want something more: to be able to define a *constant nonsignificant sentence*, for some sentences *are* nonsignificant —this is, as the authors put it, a main thesis of their work. However, in no merely significance-complete logic is possible to define a constant nonsignificant sentence [11, p. 351]. Thus, for someone with their goals, it is necessary to move to a *functionally complete* significance logic to achieve them.

Here we will propose a different functionally complete expannsion of the significance logics built with the connectives in the set $\{\sim, \wedge, \vee\}$ endowed with the Weak Kleene truth-tables. We will motivate it with the inclusion of the significance or meaningfulness operator $\circ$ mentioned above, and a connexive[19] implication $\rightarrow$, equipped with the following truth-table.

[18] For the purpose of such a task, however, it is important to keep in mind that some significance logics that were not considered by Goddard and Routley, namely Bochvar's logic of nonsense Σ_0, pose some problems. For, as pointed out by Goddard and Routley, its $\{\sim, \wedge, \vee\}$-fragment has no theorems. Thus, appropriate care must be taken when providing e.g. a Hilbert-style axiomatic system for them.

[19] For connexive logic in general, see [27].

$\to$	**t**	**u**	**f**
t	**t**	**u**	**f**
u	**t**	**u**	**f**
f	**u**	**u**	**u**

Then, as pointed out in [14, Remark 20], the resulting system is functionally complete.

Still, it remains the important question concerning the intuitive reading of this connexive implication. In this vein, [14] argues, following Heinrich Wansing's idea in [26], that the key to obtain a connexive implication in a non-classical setting, e.g. in Dunn semantics, is to replace the more familiar falsity condition for the conditional (i.e. $A \to B$ is false iff A is true and B is false) with the alternative falsity condition $A \to B$ is false iff A is not true or B is false. It is interesting to discuss the implications of this move, but this would take us too far afield now.

Finally, let us notice that Goddard and Routley did talk about connexivity in their work, for example, in [11, p. 291, 443].[20] Quite distinctively, they remarked that the connexive implication seems to be a well-suited implication to employ when formalizing the "interpretation of traditional logic according to which sentences in the traditional forms are nonsignificant if terms occurring in them are empty" [11, p. 350]. Whether or not this is enough to take the previous functionally complete expansion of significance logics to be well-motivated, is a broader task that we leave for future discussion.

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IIF-SADAF, CONICET and Department of Philosophy, University of Buenos Aires, Argentina.
E-mail address: szmucdamian@gmail.com
URL: https://uba.academia.edu/DamianSzmuc

Department of Philosophy, Kyoto University, Japan.
E-mail address: hitoshiomori@gmail.com
URL: https://sites.google.com/site/hitoshiomori/home