SEMANTICAL ANALYSIS OF ENTAILMENT AND RELEVANT IMPLICATION: I

Contents:

WORK IN PROGRESS

DRAFT

Semantical analyses are provided for several intensional logics ; in particular for (substantial parts of) the systems R of relevant implication, □R of relevant implication with necessity, P of ticket entailment, and E of entailment, and what is the same theory as E the system Π of rigorous implication. The analyses provided are used to provide semantical completeness results and decidability results for the main systems discussed, and are applied to settle some of the open questions concerning E and R and their fragments. (On these questions see Anderson [ ]).

The analyses extend the set-up analysis of the first-degree theory of entailment provided in [ ]. (The discussion in [ ] is presupposed in the remainder of this introductory section). The rules for set-up membership for conjunctive, disjunctive and negated formulae are essentially the rules already deployed in [ ]; viz.

$(A \& B)$ is in set-up H_α iff A is in H_α and B is in H_α

$(A \vee B)$ is in H_α iff A is in H_α or B is in H_α

$\sim A$ is in H_α iff A is not in H_α^*,

with complementary set-up H_α^* of H_α explained as in [ ]. The chief innovation is a more sophisticated rule for the evaluation of entailmental formulae, of the form $A \to B$, which enables the design of set-ups which falsify entailmental principles, and in particular of set-ups which falsify the law of identity $A \to A$ for any given A. This is done by evaluating higher degree entailments, not over a single (possible) situation as in strict implication, but over a pair of (relevantly-connected) situations. Thus the general form of the implication rule is as follows :-

$A \to B$ is in H_α iff for every pair of set-ups H_β and H_γ which are $\underline{R}$-related to H_α if A is in H_β then, relatedly B is in H_γ ; in short, if $\underline{R}(H_\alpha, H_\beta, H_\gamma)$ and A is in H_β then B is in H_γ . Canonically relation $\underline{R}$ is the following : $\underline{R}(H_\alpha, H_\beta, H_\gamma)$ iff for every wff B and C, if $B \to C$ is in H_α and B is in H_β then C is in H_γ.

But the general implication rule requires special conditions for practically every pure implicational theory ; so while it is a fine tool for independence proofs and for systems with weak pure entailment parts , it considerably complicates first attempts to prove completeness . To take advantage of known results , e.g. on system E , the implication rule is recast as follows :
$A \to B$ is in H_α iff for every set-up H_β which is R-related to H_α if A is in H_β then B is in $H_{\alpha+\beta}$, where $H_{\alpha+\beta}$ is a certain compounded set-up constructed from H_β taking account of H_α . In fact the conditions may now be made using Anderson's rule of entailment elimination : if $A \in H_\alpha$ and $A \to B \in H_\beta$ then $B \in H_{\alpha+\beta}$, where $\alpha+\beta$ is the set or lattice union of α and β.

For analysis of E, P and R (and for typing) it is convenient to transform H_α into the pair $(\alpha, \underline{H})$ and to consider α and H as independent units. Then for all the systems considered the $\underline{R}$-relation of H_α to H_β , now replaced by the relation of $(\alpha, \underline{H_1})$ to $(\beta, \underline{H_2})$, can be analyzed broken down into two independent relations , of $\underline{H_1}\, \underline{R}\, \underline{H_2}$ and of $\alpha \, \underline{Z} \, \beta$. Relation $\underline{R}$ is the now familiar alternativeness relation of modal logic ; and in the case of system E it is required , as for $S4$, that $\underline{R}$ is reflexive and transitive . In the case of systems like E and R , which , unlike P , countenance inflictive suppression or inflictive commutation principles, the ordering relation $\underline{Z}$ does not figure, since $\alpha \, \underline{Z} \, \beta$ for every α and β; accordingly the implication rule can be simplified in these cases to:

$A \to B$ is in $(\alpha, \underline{H})$ iff for every $\underline{H}_1$ and every β, if $\underline{H}_1 \underline{R} \underline{H}_2$ and A is in $(\beta, \underline{H}_1)$ then B is in $(\alpha+\beta, \underline{H}_2)$.

For system R where relation $\underline{R}$ is an equivalence relation and where the hereditariness condition:

NL) if A is in $(\alpha, \underline{H}_1)$ and $\underline{H}_1 \underline{R} \underline{H}_2$ then A is in $(\alpha, \underline{H}_2)$

NL) is satisfied, a further simplification can be made: relation $\underline{R}$ and its field can be omitted altogether. Thus for system R the implication rule reduces to:

NL) $A \to B$ is in α iff for every β if A is in β then B is in $\alpha+\beta$.

NL) The rule for E can be recovered from this rule for R by combining it with the □ necessity rule for '□'; viz.

NL) $\Box A$ is in $(\alpha, \underline{H})$ iff, for every $\underline{H}_1$, if $\underline{H} \underline{R} \underline{H}_1$ then, materially, A is in $(\alpha, \underline{H}_1)$.

NL) It will follow then from the semantics that E is effectively an S4-modalisation of R.

The strict implication rule is a special case of the entailment rule for E: the strict implication rule results upon equating α with β for every α and β. Thus the semantics include semantics for normal modal logics as special cases.

In the presentation a characteristic function h is used to indicate whether or not a given wff is in or holds in a given situation, i.e. $h(A, (\alpha, \underline{H})) = T$, or $= F$, according as A is in, or is not in, $(\alpha, \underline{H})$. Finally $h(A, (\alpha, \underline{H}))$ is shortened to $h(A, \alpha, \underline{H})$.

The paper is <u>heavily indebted</u> to the work of Anderson and Belnap and Meyer and Dunn and coworkers (and I hope the debt will increase). The paper presupposes some of this work, but it also presupposes semantical analyses of modal logic, especially the work of Kripke.

The methods of the paper may be applied to provide semantics for a number of other systems related to those studied.

§1. The axiomatic systems.

The postulates of system E are (in forward Anderson & Belnap form) as follows:

E1.	$((A\to A)\to B)\to B$	E2.	$A\to B\to .\,B\to C\to .\,A\to C$
E3.	$(A\to .\,A\to B)\to A\to B$	E4.	$A\,\&\,B\to A$
E5.	$A\,\&\,B\to B$	E6.	$(A\to B)\,\&\,(A\to C)\to .\,A\to B\,\&\,C$
E7.	$NA\,\&\,NB\to N(A\,\&\,B)$	E8.	$A\to A\vee B$
E9.	$B\to A\vee B$	E10.	$(A\to C)\,\&\,(B\to C)\to .\,(A\vee B)\to C$
E11.	$A\,\&\,(B\vee C)\to (A\,\&\,B)\vee C$	E12.	$A\to \sim A\to .\sim A$
E13.	$A\to \sim B\to .\,B\to \sim A$	E14.	$\sim\sim A\to A$.

<u>Modus ponens</u> (<u>MP</u>): From A and $A\to B$ to infer B.
<u>Adjunction</u> (<u>Adj</u>): From A and B to infer $A\,\&\,B$.
The connectives '&' (symbolising conjunction) '~' (negation) and '→' (implication or entailment) are taken as primitive; '∨' (disjunction) is either taken as primitive, or defined in the full system: $A\vee B =_{Df} \sim(\sim A\,\&\sim B)$ and 'N' (necessity) is defined: $NA =_{Df} (A\to A)\to A$.
The pure implication fragment, E_I, of E, has as postulates E1 – E3 and MP; the implication-negation fragment, $E_{\overline{I}}$, the postulates E1 – E3 and E12 – E14 with MP; the ~~d~~ implication-conjunction fragment, $E_\&$, the postulates E1 – E7 with MP and Adj; the positive fragment, E^+, E1 – E11 with MP and Adj.

[illegible]

The postulates of system R are those of E together with the scheme

E0. $A\to .\,(A\to A)\to A$,

or one of its equivalents. Further each fragment of R adds E0 to the corresponding fragment of E; <u>e.g.</u> R^+ is $E^+ + E0$. Scheme E7 is however redundant wherever it occurs in R systems; and scheme E12 may be proved using E3 (or vice versa) in R systems (see [ ]).

System Rf (of Meyer [ ]) takes the propositional constant f as primitive in place of ~, and replaces negation axioms E12 – E14 of R by the single axiom: $A\to f\to f\to A$

System $\Box R$, of relevant implication with S4-necessity, results upon adding to R the new primitive '$\Box$' and these standard S4 principles (see eg. Meyer [ ]):

$\Box 1.\quad \Box A \to A$ $\qquad\qquad$ $\Box 2.\quad \Box(A \to B) \to .\ \Box A \to \Box B$

$\Box 3.\quad \Box A \,\&\, \Box B \to \Box(A \,\&\, B)$ $\qquad$ $\Box 4.\quad \Box A \to \Box\Box A$

Necessitation (Nec): From A to infer $\Box A$.

Entailment is defined thus in $\Box R$: $A \Rightarrow B =_{Df} \Box(A \to$

The $\Box R$-translation of a wff A of E is the wff A' which results on replacing each occurrence of '$\to$' in A by '$\Rightarrow$' and each occurrence of 'N' by '$\Box$'.

$\Box R$ may be reaxiomatised so as to avoid the rule of necessitation by building up on the axioms as follows ~~[illegible]~~ for each axiom Ax of the given system the new axiom $\Box Ax$ is added. For example in the reaxiomatisation ~~[illegible]~~ both $\Box A \to A$ and $\Box(\Box A \to A)$ are taken as axioms. In the reaxiomatised system the rule of necessitation is a derived rule provable by induction over proofs.

(of [ ])

System P differs from E in just these respects: in place of E1 the scheme E1'. $A \to A$ is adopted; E7 is deleted; and the permuted form, E2'. $A \to B \to .\ C \to A \to .\ C \to B$, of E2 is added. The pure implication fragment P_I of P has as postulates E1', E2, E2', and E3 and MP; the implication-negation fragment $P_{\bar{I}}$ has the postulates of P_I together with E12–E14; the implication-conjunction fragment $P\&$ the postulates of P_I together with E4–E6 and Adj; and the positive fragment P^+ the postulates of P_I together with E4–E6, E8–E11 and Adj.

System $E\Lambda$ (of [ ]) adds to E the propositional constant 'Λ' satisfying these postulates:

$\Lambda 1.\quad A \to \Lambda \to .\ {\sim}A$ $\qquad\qquad$ $\Lambda 2.\quad {\sim}(A \to A) \to \Lambda$

$E\Delta$, which is a conservative extension of E, corresponds to Ackermann's system Π' (of L J) as E corresponds to Ackermann's system Π, i.e. they have the same class of theorems.

Several other systems are singled out for attention. First, S5-modalizations of the focal systems. $E5$ $(P5)$ adds to E (P) the S5 principle

E15. $\sim NA \rightarrow N{\sim}NA$

$\Box R5$ adds to $\Box R$ the postulate: $\sim\Box A \rightarrow \Box{\sim}\Box A$, and, in necessitation form, the postulate $\sim\Box A \Rightarrow \Box{\sim}\Box A$.

Second, extensions of the focal systems by ~~...~~ a partial S5 type principle to the effect that one logically false proposition entails its necessary falsity. Thus $\Box Rf5$ adds to $\Box Rf$ the postulate: $f \rightarrow \Box f$ (and $f \Rightarrow \Box f$), and $E\Delta 5$ adds to $E\Delta$ the postulate: $\Delta \rightarrow N\Delta$

Third, non-transitive analogues of E and P. These systems E2 and P2 ~~...~~ resemble S2 in the way E resembles S3; they weaken the suffixed syllogism principles E2 and E2' to the ~~...~~ imported form: $(A\rightarrow B)\,\&\,(B\rightarrow C)\rightarrow.\ A\rightarrow C$.

Naturally compensation for the loss of an overpowerful prefix principle such as imported syllogism has to be made elsewhere. Thus P2, formulated with primitive connectives $\{\rightarrow, \sim, \&\}$, ~~...~~ replaces P as follows:

P1. $A \rightarrow A$ P2. $A\rightarrow B \,\&\, B\rightarrow C \rightarrow A\rightarrow C$

P3. $A\rightarrow(B\rightarrow C)\rightarrow A\,\&\,B\rightarrow C$ P4. $A\,\&\,B\rightarrow A$

P5. $(A\rightarrow B)\,\&\,(C\rightarrow D)\rightarrow.\ A\,\&\,C\rightarrow D\,\&\,B$ P6. $A\rightarrow A\,\&\,A$

P11–P14 are the same as E11–E14; to the rules of P is added the further rule of substitutivity of coimplicants: from $C(A)$ and $A\leftrightarrow B$, i.e. $A\rightarrow B \,\&\, B\rightarrow A$, to infer $C(B)$.

The sole pure implicational axiom P1 is of course derivable from P4 and P6.

Fourth, extensions of P^+ and E^+ by different negation postulates. Of special interest are the systems PP (P proper) and EP obtained from P and E respectively by weakening ~~dropping~~ E12; for once impossible situations are admitted as semantically valuable, the reductio principle E12 appears as an unnecessary & undesirable restriction. Moreover in the case of P it leads to anomalies; e.g. P has as a theorem $((A \vee \sim A) \to B) \to B$ though rejecting the thesis $((A \to A) \to B) \to B$ characteristic of E; yet the grounds for objecting to the second of these are also grounds for objecting to the first.

/Add
$A \to B \Rightarrow \sim(A \& \sim B)$
& what else ?

§2. The semantical systems.

§2.0 Positive models.

An E^+-model M is a structure $M = \langle \underline{G}, \underline{K}, \underline{R}, 0, \underline{N}, h \rangle$ where $\underline{K}$ is a set, $\underline{G} \in \underline{K}$; $\underline{R}$ is a reflexive and transitive relation on $\underline{K}$, $\underline{N}$ is a set of sets including the null set 0 and closed under the set union operation $+$, and h is a 2-place holding function such that for every atomic wff p and every $\underline{H} \in \underline{K}$ and $\alpha \in \underline{N}$, $h(p, \alpha, \underline{H}) = T$ or $= F$.

The holding function h is extended to all wff of E^+ as follows:—

$h(A \& B, \alpha, \underline{H}) = T$ iff $h(A, \alpha, \underline{H}) = T = h(B, \alpha, \underline{H})$

$h(A \vee B, \alpha, \underline{H}) = T$ iff $h(A, \alpha, \underline{H}) = T$ or $h(B, \alpha, \underline{H}) = T$

$h(A \to B, \alpha, \underline{H}) = T$ iff for every $\underline{H}' \in \underline{K}$ and $\beta \in \underline{N}$ if $\underline{H}\underline{R}\underline{H}'$ and $h(A, \beta, \underline{H}') = T$ then materially $h(B, \alpha + \beta, \underline{H}')$.

A wff B is true in E^+-model M iff $h(B, 0, \underline{G}) = T$, false in M iff $h(B, 0, \underline{G}) = F$; E^+-valid iff B is true in every E^+-model. E^+-model M falsifies B iff $h(B, 0, \underline{G}) = F$; M satisfies Γ iff for every wff $B \in \Gamma$, $h(B, 0, \underline{G}) = T$.

An R^+-model M is an E^+-model such that
(i) if $h(p, \alpha, \underline{H}_1) = T$ and $\underline{H}_1 \underline{R} \underline{H}_2$ then $h(p, \alpha, \underline{H}_2) = T$, for every atomic p and every $\underline{H}_1, \underline{H}_2 \in \underline{K}$ (the hereditariness requirement).

A $\Box R^+$-model M is a structure $M = \langle \underline{G}, \underline{K}, \underline{R}, 0, \underline{N}, \underline{W}, h \rangle$ where $\langle \underline{G}, \underline{K}, \underline{R}, 0, \underline{N}, h \rangle$ is an R^+-model and $\underline{W}$ is another reflexive and transitive relation on $\underline{K}$.

A P^+-model is an E^+-model where the elements of sets of $\underline{N}$ are ordered. A convenient choice is to take $\underline{N}$ as a set of sets of positive integers (or ordinals). Then, as in Anderson and Belnap [?], for $\alpha \in \underline{N}$,

§2.1 Forced regulation models.

An R-model M is a structure $M = \langle G, K, R, O, N, P, h \rangle$ where K is a set, $G \in K$, R is a reflexive and transitive relation on K, N is a set of sets including the null set O and closed under the set union operation $+$, P is a relation on elements of N and K such that

I would have been able to improve on this very awkward requirement!

(i) if, for every $\beta \in N$ and $H \in K$, $H_1 R H$ and $P(\alpha+\beta, H)$ materially imply $H_1 R H_2$ and $P(\gamma+\beta, H_2)$, then $(\alpha, H_1) = (\gamma, H_2)$ (the reduction requirement).

Finally h is a 3-place valuing (or valuation) function such that for every element p and every $H \in K$ and every $\alpha \in N$, $h(p, \alpha, H) = T$ or $= F$, and such that

(ii) for every element p and every $H_1, H_2 \in K$ and every $\alpha \in N$, if $H_1 R H_2$ and $h(p, \alpha, H_1) = T$ then $h(p, \alpha, H_2) = T$ (the Heredity requirement); and

(iii) for every element p and every $\alpha \in N$ and every $H_1 \in K$ if $h(p, \alpha, H_1) = F$ then, for some $H_2 \in K$ and $\beta \in N$, $H_1 R H_2$ and $P(\alpha+\beta, H_2)$ (the falsity requirement).

The valuing function h is extended from atomic wffs to all wffs of R as follows:

$h(A \& B, \alpha, H) = T$ iff $h(A, \alpha, H) = T = h(B, \alpha, H)$

~~$h(A \vee B, \alpha, H) = T$ iff ... $= T$ ... $h(B, \alpha, H)$~~

$h(A \to B, \alpha, H) = T$ iff, for every $H' \in K$ and every $\beta \in N$, if HRH' and $h(A, \beta, H') = T$ then, materially, $h(B, \alpha+\beta, H')$ ...

$h(\sim A, \alpha, H) = T$ iff, for every $H' \in K$ and every $\beta \in N$, if HRH' and $P(\alpha+\beta, H')$ then, materially, $h(A, \beta, ...$

The R-model may be simplified. $\underline{G}$ may be defined as $\underline{G} = \Sigma \underline{H} (\underline{H} \in \underline{K})$; and R may be eliminated (in the way explained in [ ]). If requirements (i) and (iii) are dropped a minimal logic version of R which does not validate E14 results.

Lemma. For every wff A,
if $\underline{H_1} R \underline{H_2}$ and $h(A, \alpha, \underline{H_1}) = T$ then $h(A, \alpha, \underline{H_2}) = T$.
Proof is by induction from the stipulated basis. There are 3 cases:
ad &: if $h(B \& C, \alpha, \underline{H_1}) = T$ & $\underline{H_1} R \underline{H_2}$ then $h(B \& C, \alpha, \underline{H_2}) = T$
ad →: by transitivity of $\hat{R}$ & definition of h.
ad ~: by transitivity of $\hat{R}$ and definition of h.

Lemma. For every wff A, for every $\underline{H_1} \in \underline{K}$ if $h(A, \alpha, \underline{H_1}) = F$ then, for some $H_2 \in \underline{K}$ and some $\beta \in \underline{N}$, $\underline{H_1} R \underline{H_2}$ and $\underline{P}(\alpha + \beta, \underline{H_2})$.
Proof is by induction from the stipulated basis.
ad &: if $h(B \& C, \alpha, \underline{H_1}) = F$ then either $h(B, \alpha, \underline{H_1}) = F$ or $h(C, \alpha, \underline{H_1}) = F$. In either case the desired result follows by induction hypothesis.
ad →: if $h(B \to C, \alpha, \underline{H_1}) = F$ then for some $\underline{H_3}$ and some $\gamma \in \underline{N}$, $\underline{H_1} R \underline{H_3}$ and $h(B, \gamma, \underline{H_3}) = T$ and $h(C, \alpha + \gamma, \underline{H_3}) = F$. Since $h(C, \alpha + \gamma, \underline{H_3}) = F$, by induction hypothesis, for some $\underline{H_2} \in \underline{K}$ and some $\delta \in \underline{N}$, $\underline{H_3} R \underline{H_2}$ and $\underline{P}(\alpha + \gamma + \delta, \underline{H_2})$. Thus, as $\hat{R}$ is transitive, for some $\underline{H_2}$ and some $\beta = \gamma + \delta$, $\underline{H_1} R \underline{H_2}$ and $\underline{P}(\alpha + \beta, \underline{H_2})$.
ad ~: if $h(\sim B, \alpha, \underline{H_1}) = F$ then, for some $\underline{H_2} \in \underline{K}$ and some $\beta \in \underline{N}$, $\underline{H_1} R \underline{H_2}$ and $P(\alpha + \beta, \underline{H_2})$.

NL It is simplest to use the fix yielded by this lemma in applying the reduction requirements.
A wff B is true in R-model M iff $h(B, O, \underline{G}) = T$.
B is R-valid iff B is true in every R-model. R-model M falsifies B iff $h(B, O, \underline{G}) = F$; M satisfies Γ iff for every wff $B \in \Gamma$, $h(B, O, \underline{G}) = T$.

Lemma Where ∨ is defined: $A \vee B =_{Df} \sim(\sim A \& \sim B)$,

(i) if $h(A, \alpha, \underline{H}) = T$ or $h(B, \alpha, \underline{H}) = T$ then $h(A \vee B, \alpha, \underline{H}) = T$

(ii) if $h(A \vee B, \alpha, \underline{H}) = T$ then $h(A, \alpha, \underline{H}) = T$ or $h(B, \alpha, \underline{H}) = T$, provided that for some wff C and every β $h(C, \alpha+\beta, \underline{H}) = F$.

Proof: (i) if $h(\sim(\sim A \& \sim B), \alpha, \underline{H}) = F$; then for some Y and some H_1, $\underline{P}(\alpha, Y, \underline{H_1})$ and $h(\sim A, Y, \underline{H_1}) = T = h(\sim B, Y, H_1)$. Hence since $P(\alpha+Y, \underline{H_1})$, $h(A, \alpha, \underline{H_1}) = F = h(B, \alpha, \underline{H_1})$. Since $\underline{HRH_1}$, by hereditariness, $h(A, \alpha, \underline{H}) = F = h(B, \alpha, \underline{H})$.

(ii) if $h(\sim(\sim A \& \sim B), \alpha, H_1) = T$ then for every β and $\underline{H}$, if $\underline{P}(\alpha, \beta, \underline{H})$ and $\underline{H_1} R \underline{H}$ then $h(\sim A \& \sim B, \beta, \underline{H}) = F$, i.e. either $h(\sim A, \beta, \underline{H}) = F$ or $h(\sim B, \beta, \underline{H}) = F$. Suppose further that $h(\sim A, \beta, \underline{H}) = F$. Then for some Y and some $\underline{H_2}$, $P(\beta+Y, \underline{H_2})$ and $\underline{HRH_2}$ and $h(A, Y, \underline{H_2}) = T$. By the reduction requirement then $(Y, \underline{H_2}) = (\alpha, \underline{H_1})$; so in this case $h(A, \alpha, \underline{H_1}) = T$. Similarly on the alternative assumption that $h(\sim B, \beta, \underline{H}) = F$ $h(B, \alpha, \underline{H_1}) = T$ follows. Then, using the falsity requirement to guarantee that for some β and $\underline{H}$, $\underline{H_1 R H}$ and $P(\alpha+\beta, \underline{H})$, either $h(A, \alpha, \underline{H_1}) = T$ or $h(B, \alpha, \underline{H_1}) = T$.

An $\underline{R_f\text{-model}}$ is an R-model; only the extension of h differs as follows: the clause for negated wff is replaced by this clause for f:–

$h(f, \alpha, \underline{H}) = F$ iff $P(\alpha, \underline{H})$

Thus P ~~Requirement (ii) now follows~~ may be eliminated in the case of R_f-models

Lemma. A wff A of R is R-valid iff its R_f-translation A_f, obtained by eliminating each wff part ~B using the definition $\sim B =_{Df} B \to f$, is R_f-valid

Proof. Suppose A_f is not R_f valid. Then for some R-model ~~[illegible]~~ $h^{\circ}(A_f, 0, \underline{G}) = F$ where h' is the R-extension of h it follows by induction over wellformedness of A_f that $h'(A, 0, \underline{G}) = F$. The converse half is similar.

An R-model $\mathfrak{M}$ *for wff* A is an R-model $\mathfrak{M}$ where h assigns truth values only for atomic subwff of A (and for f). Function h is extended as before for sub wff of A. Further in the case of disjunction h is extended as follows, for subwff B and C:
if $h(B, \alpha, \underline{\underline{H}}) = T$ or $h(C, \alpha, \underline{\underline{H}}) = T$ then $h(B \vee C, \alpha, \underline{\underline{H}}$
if $h(B \vee C, \alpha, \underline{\underline{H}}) = T$ and, for some subwff D of A (or f) and some β, $h(D, \alpha + \beta, \underline{\underline{H}}) = F$ then $h(B, \alpha, \underline{\underline{H}}) = T$ or $h(C, \alpha, \underline{\underline{H}}) =$

Under this definition, a wff A of R is *valid* (*c-valid*) iff A is true in every R-model for A, *i.e.* $h(A, O, \underline{G}) = T$ for every R-model for A.

~~Lemma. A is c-valid iff A is valid both definable~~

Theorem: Every theorem of R is both valid and c-va

A □R-model M is a structure
$M = \langle G, K, R, O, N, P, W, L \rangle$ where
$\langle G, K, R, O, N, P, L \rangle$ is an R-model and W is a reflexive and transitive relation on K such that

~~(□) if $H_1 R H_2$ and $H_2 W H_3$ then $H_1 W H_3$~~

(v) if $H_1 R H_2$ then $H_1 W H_2$

Hence if $H_1 R H_2$ and $H_2 W H_3$ then $H_1 W H_3$.

The holding function L is extended as for system R, and, in addition,
$L(\Box A, \alpha, H) = T$ iff either for every H_1 such that $H W H_1$ $L(A, \alpha, H_1) = T$ or, for every H_2 and β, if $H R H_2$ then not $P(\alpha + \beta, H_2)$.

Truth in M, □R-validity, □R-c-validity, etc., are defined along the same lines as before.

Lemma. For every wff A, if $H_1 R H_2$ and $L(A, \alpha, H_1) = T$ then $L(A, \alpha, H_2) = T$.

Lemma. For every wff A, if $L(A, \alpha, H_1) = F$ then, for some H_2 and some β, $H_1 R H_2$ and $P(\alpha + \beta, H_2)$.

NL.) The new induction step, for □, is immediate from the holding functions for □, & helps explain its design.

A □R4-model M is an □R-model such that

~~[illegible]~~

(vi) if $H_1 W H_2$ and $P(\alpha, H_2)$ then $P(\alpha, H_1)$

In this case, □R4, the holding function L may be simplified in the expected way for □, e.g.:
$L(\Box A, \alpha, H) = T$ iff for every H_1 such that $H W H_1$ $L(A, \alpha, H_1) = T$.

The lemmas above both hold.

□R5? = □R4 + symmetry of W

[illegible]

It follows from the $\Box R5$ modelling that entailment, necessarily evaluated as follows :-

$L(A \to B, \alpha, \underline{\underline{H_1}}) = T$ iff for every $\underline{\underline{H_2}}$ and $\underline{\underline{H_3}}$ and β, if $\underline{\underline{H_1 W H_2}}$ and $\underline{\underline{H_2 R H_3}}$ and $L(A, \beta, \underline{\underline{H_3}}) = T$ then $L(B, \alpha + \beta, \underline{\underline{H_3}}) = T$

In the case of $\Box R$-modelling the following ~~clause~~ alternative is added : or else for every $\underline{\underline{H_4}}$ and γ, if $\underline{\underline{H_1 R H_4}}$ then not $P(\alpha + \gamma, \underline{\underline{H_4}})$. In view of condition (v) and quantification logic, the main clause can be simplified to the following:

$L(A \to B, \alpha, \underline{\underline{H_1}}) = T$ iff, for every $\underline{\underline{H_3}}$ and β, if $\underline{\underline{H_1 W H_3}}$ and $L(A, \beta, \underline{\underline{H_3}}) = T$ then $L(B, \alpha + \beta, \underline{\underline{H_3}}) = T$

For, for some $\underline{\underline{H_2}}$, $\underline{\underline{H_1 W H_2}}$ and $\underline{\underline{H_2 R H_3}}$, iff $\underline{\underline{H_1 W H_3}}$, by quantification logic.

An $E\Delta 5$-model is a structure
$\mathfrak{M} = \langle \underline{\underline{G}}, \underline{\underline{K}}, \underline{\underline{R}}, O, \underline{\underline{N}}, \underline{\underline{P}}, h \rangle$,
where $\underline{\underline{K}}$ is a set ~~(of ...)~~, $\underline{\underline{G}} \in \underline{\underline{K}}$,
$\underline{\underline{R}}$ is a reflexive and transitive relation on $\underline{\underline{K}}$,
$\underline{\underline{N}}$ is a set of sets including the null set O closed under set union operation $+$, $\underline{\underline{P}}$ is a relation on elements of $\underline{\underline{K}}$ and $\underline{\underline{N}}$ such that
(i) if, for every $\beta \in \underline{\underline{N}}$ and $\underline{\underline{H}} \in \underline{\underline{K}}$, $\underline{\underline{H_1 R H}}$ and $\underline{\underline{P}}(\alpha+\beta, \underline{\underline{H}})$ materially imply $\underline{\underline{H R H_2}}$ and $\underline{\underline{P}}(\gamma+\beta, \underline{\underline{H_2}})$, then $(\alpha, \underline{\underline{H_1}}) = (\gamma, \underline{\underline{H_2}})$, for $\alpha, \gamma \in \underline{\underline{N}}$, $\underline{\underline{H_1}}, \underline{\underline{H_2}} \in \underline{\underline{K}}$.
(ii) if $\underline{\underline{P}}(\alpha, \underline{\underline{H_2}})$ and $\underline{\underline{H_1 R H_2}}$ then $\underline{\underline{P}}(\alpha, \underline{\underline{H_1}})$, for $\alpha \in \underline{\underline{N}}$ and $\underline{\underline{H_1}}, \underline{\underline{H_2}} \in \underline{\underline{K}}$.
Finally h is a 2-place holding (or valuation) function such that for every atomic p and every $\underline{\underline{H}} \in \underline{\underline{K}}$ and $\alpha \in \underline{\underline{N}}$, $h(p, \alpha, \underline{\underline{H}}) = T$ or $= F$, and such that
(iii) for every atomic p, λ, and $\alpha \in \underline{\underline{N}}$, $\underline{\underline{H_1}} \in \underline{\underline{K}}$, if $h(p, \alpha, \underline{\underline{H_1}}) = F$ then, for some $\underline{\underline{H_2}} \in \underline{\underline{K}}$ and $\beta \in \underline{\underline{N}}$, $\underline{\underline{H_1 R H_2}}$ and $\underline{\underline{P}}(\alpha+\beta, \underline{\underline{H_2}})$.

The holding function h is extended to wff of $E\Delta 5$ as follows:
$h(A \& B, \alpha, \underline{\underline{H}}) = T$ iff $h(A, \alpha, \underline{\underline{H}}) = T = h(B, \alpha, \underline{\underline{H}})$.
$h(\Lambda, \alpha, \underline{\underline{H}}) = F$ iff $\underline{\underline{P}}(\alpha, \underline{\underline{H}})$
$h(A \Rightarrow B, \alpha, \underline{\underline{H}}) = T$ iff for every $\underline{\underline{H'}} \in \underline{\underline{K}}$ and every $\beta \in \underline{\underline{N}}$, if $\underline{\underline{H R H'}}$ and $h(A, \beta, \underline{\underline{H'}}) = T$ then, materially $h(B, \beta+\alpha, \underline{\underline{H'}}) = T$.
$h(\sim A, \alpha, \underline{\underline{H}}) = F$ iff for some $\underline{\underline{H_1}} \in K$ and $\beta \in \underline{\underline{N}}$ $\underline{\underline{H}} R \underline{\underline{H_1}}$ and $P(\alpha+\beta, \underline{\underline{H_1}})$ and $h(A, \beta, \underline{\underline{H_1}}) = T$.

A wff B is true in a $E\Delta 5$-model $\mathfrak{M}$ iff $h(B, O, \underline{\underline{G}}) = T$; etc. Since the distinguishing ~~wff~~ postulate $\sim NA \rightarrow N \sim NA$ of $E5$ is $E\Delta 5$-valid, it is tempting to define $E5$-validity as $E\Delta 5$-validity of a Λ-free wff.

Lemma. For every wff A, if $h(A, \alpha, \underline{\underline{H}}) = F$ then for some $\underline{\underline{H_1}}$ and some β, $\underline{\underline{H R H_1}}$ and $\underline{\underline{P}}(\alpha+\beta, \underline{\underline{H_1}})$

FALSE

Lemma: For every wff B, if
$h(\sim B, \ldots$

$$\max(\alpha) = \begin{cases} \text{the largest element of } \alpha, \text{ if } \alpha \neq 0 \\ \text{zero} \quad \text{if } \alpha = 0 \end{cases}$$

In the case of P^+ the holding function for $\rightarrow$ is amended to the following:–

$h(A \rightarrow B, \alpha, \underline{H}) = T$ iff for every $\underline{H}' \in \underline{K}$ and $\beta \in \underline{N}$ if $\underline{H}\underline{R}\underline{H}'$ and $\max(\beta) \geq \max(\alpha)$ and $h(A, \beta, \underline{H}') = T$ then, automatically, $h(B, \alpha + \beta, \underline{H}') = T$.

A wff B is true in P^+-model M iff $h(B, 0, \underline{G}) = T$; etc.

Modellings for systems E_I, R_I, P_I, $E_{I\lambda}$, $R_{I\lambda}$ and $P_{I\lambda}$ are obtained from the modellings given by adding assumptions clauses for ~~$\rightarrow$~~ the impositive connectives.

<u>Theorem</u>. If $\vdash A$ then A is h-valid for each of the positive systems and their fragments.

An <u>$S4^T$-model</u>, also called an S4-model, is an E^+-model with $N = \{0\}$, i.e. $\alpha = 0$ for every $\alpha \in \underline{N}$. It is thus clear that by varying conditions on the relation $\underline{R}$ implicational analogues of normal modal systems can be got. For a characterisation of entailment proper there is, as Lewis ~~[illegible]~~, a case for abandoning the transitivity requirement on $\underline{R}$, and thereby cutting Expanded Syllogism, E2, back to Conjunctive Syllogism: $A \rightarrow B \,\&\, B \rightarrow C \rightarrow . A \rightarrow C$.

§2.2 *Direct negation models.*

The models so far studied cause substantial problems with respect to the assessment of formulae where negation occurs essentially (and not simply as a substitution instance of a positive wff). To reduce these problems the initial models are supplemented by models which treat negation more directly.

An E-model M is a structure $M = \langle G, K, R, O, N, h \rangle$, where K is a set of elements, including $G = H_0$, such that for every $H_i \in K$ there is a unique element $J_i \in K$; R is a transitive and reflexive relation on $M = \{ H_i : H_i \in K \}$. O and N are as before; and h is, as before, two-valued labelling function which assigns one of T or F to every atomic wff for every H_i and $J_i \in K$ and every $\alpha \in N$. But h also assigns one of T and F to every entailment for every $\alpha \in N$ and $J_i \in K$, i.e. entailments are assigned values arbitrarily in J-situations.

The symbols $I, I_1, I_2, \ldots, I'$, are used as general variables ranging over elements of K.

h is extended from atomic wff to all wff of E thus:

$h(A \& B, \alpha, I) = T$ iff $h(A, \alpha, I) = h(B, \alpha, I) = T$;

$h(A \vee B, \alpha, I) = T$ iff $h(A, \alpha, I) = T$ or $h(B, \alpha, I) = T$

$h(\sim A, \alpha, H_i) = T$ iff $h(A, \alpha, J_i) = F$

$h(\sim A, \alpha, J_i) = T$ iff $h(A, \alpha, H_i) = F$

If $h(A \to B, \alpha, H_i) = T$ then, for every $\beta \in N$ and $H_j \in K$, if $H_i R H_j$, then if $h(A, \beta, H_j) = T$ $h(B, \alpha+\beta, H_j) = T$ and if $h(A, \alpha+\beta, J_j) = T$ $h(B, \beta, J_j) = T$; further if $h(A \to B, \alpha, H_i) = T$ and $h(B, \alpha, H_i) = F$ then $h(A, \alpha, J_i) = F$. (The last condition is the *reduction* condition; the complication of the first condition is to take account of contraposition principles.)

If $h(A \to B, \alpha, H_i) = F$ then, for some $\beta \in N$ and some $H_j \in K$, $H_i R H_j$ and $h(A, \beta, H_j) = T$ and $h(B, \alpha+\beta, H_j) = F$,

$h(A, \alpha+\gamma, \underline{\underline{I_k}}) = T$ and $h(B, \gamma, \underline{\underline{I_k}}) = F$.

[NB. A single quantification, for some $\beta \in \underline{N}$, covering the whole consequent can be used in place of the separate quantifications for β and γ.]

A wff B is true in E-model M iff $h(B, 0, \underline{G}) = T$, i.e. $h(B, 0, \underline{\underline{H_0}}) = T$; etc.

An R-I-model M is an E-model M such that

(1) $\underline{R}$ is extended to $\overline{\underline{M}}$, i.e. $\underline{K} - \underline{M}$ through the equivalence: $\underline{\underline{I_1}} R \underline{\underline{I_2}}$ iff $\underline{\underline{H_2}} R \underline{\underline{H_1}}$ for every $\underline{\underline{I_1}}, \underline{\underline{I_2}} \in \underline{K}$;

(2) if $h(A, \alpha, \underline{\underline{I_1}}) = T$ and $\underline{\underline{I_1 R I_2}}$ then $h(A, \alpha, \underline{\underline{I_2}}) = T$, for every initial case, i.e. (i) for every atomic wff for every $\alpha \in \underline{N}$ and $\underline{\underline{I_1}}, \underline{\underline{I_2}} \in \underline{K}$, and (ii) for every entailment for every $\alpha \in \underline{N}$ and $\underline{\underline{I_1}}, \underline{\underline{I_2}} \in \overline{\underline{M}}$. (the hereditariness requirement).

Lemma. When M is an R-I-model, if $h(A, \alpha, \underline{\underline{I_1}}) = T$ and $\underline{\underline{I_1 R I_2}}$ then $h(A, \alpha, \underline{\underline{I_2}}) = T$, for every wff A, every $\alpha \in \underline{N}$ and $\underline{\underline{I_1}}, \underline{\underline{I_2}} \in \underline{K}$.

Proof is by induction over connectives in A.

A is R-I-valid iff A is true in every R-I-model.

A $\Box$R-I-model M is a structure $M = \langle \underline{G}, \underline{K}, \underline{R}, 0, \underline{N}, \underline{W}, h \rangle$ where $\langle \underline{G}, \underline{K}, \underline{R}, 0, \underline{N}, h \rangle$ is an R-I-model, and $\underline{W}$ is a reflexive and transitive relation on $\underline{M}$ such that if $\underline{\underline{H_1 R H_2}}$ then $\underline{\underline{H_1 W H_2}}$, and $h(\Box A, \alpha, \underline{\underline{I}})$ is a further initial case, i.e. $\Box A$ is evaluated arbitrarily in $(\alpha, \underline{\underline{I}})$ situations. The hereditariness becomes natural.

A is $\Box$R-I-valid iff A is true in every $\Box$R-I-model.

A *P-model* is simply an E-model h, where $\underline{N}$ is an ordered set; however entailmental wff are evaluated differently in $\underline{H}$-situations, *i.e.* the valuation of h differs from that for E in the following:—

If $h(A \to B, \alpha, \underline{H_i}) = T$ then, for every $\beta \in \underline{N}$ and $\underline{H_j} \in \underline{K}$, if $\underline{H_i} R \underline{H_j}$ and $max(\beta) \geq max(\alpha)$, then if $h(A, \beta, \underline{H_j}) = T$ $h(B, \alpha+\beta, \underline{H_j}) = T$ and if $h(A, \alpha+\beta, \underline{T_j}) = T$ $h(B, \beta, \underline{T_j}) = T$; further if $h(A \to B, \alpha, \underline{H_i}) = T$ then if $h(A, \alpha, \underline{T_i}) = T$ $h(B, \alpha, \underline{H_i}) = T$.

If $h(A \to B, \alpha, \underline{H_i}) = F$ then for some $\beta \in \underline{N}$ and $\underline{H_j} \in \underline{K}$ $max(\beta) \geq max(\alpha)$ and $\underline{H_i} R \underline{H_j}$ and $h(A, \beta, \underline{H_j}) = T$ and $h(B, \alpha+\beta, \underline{H_j}) = F$ and also for some $\gamma \in \underline{N}$ and $\underline{H_k} \in \underline{K}$ $max(\gamma) \geq max(\alpha)$ and $\underline{H_i} R \underline{H_k}$ and $h(A, \alpha+\gamma, \underline{T_k}) = T$ and $h(B, \gamma, \underline{T_k}) = F$.

B is *P-valid* iff B is true in every P-model, in effect P-true in every E-model; *etc.*

In the case of the positive part, P^+, of P the entailment evaluation rule simplifies to the following:
$h(A \to B, \alpha, \underline{H}) = T$ iff for every $\beta \in \underline{N}$ and $\underline{H'} \in \underline{K}$ if $\underline{H} R \underline{H'}$ and $max(\alpha) \leq max(\beta)$ and $h(A, \beta, \underline{H'}) = T$ then $h(B, \alpha+\beta, \underline{H'}) = T$.

§2.3 Simplified models for systems based on R and R*

Lemma. Every non-valid wff of R [□R etc.] has a connected R- [□R- etc.] counter-model, i.e. every R-satisfiable wff has a connected R-model (etc.).
Proof is as in Kripke [ ]. Define $\underline{K}' = \{\underline{H} \in \underline{K} : \underline{G}R_* \underline{H}\}$ where $\underline{R_*}$ is the ancestral of $\underline{R}$; $\underline{R}'$ as the restriction of $\underline{R}$ to $\underline{K}'$; and for $\underline{H} \in \underline{K}'$, $h'(p, \alpha, \underline{H}') = h(p, \alpha, \ldots)$
Then $M' = \langle \underline{G}, \underline{K}', \underline{R}', 0, \underline{N}, \underline{P}; h' \rangle$ is a connected R-model; & it follows, by induction, that for every $\underline{H} \in \underline{K}'$, $h'(B, \alpha, \underline{H}) = h(B, \alpha, \underline{H})$.

Lemma (i) For every $\underline{H} \in \underline{K}$, $h(A, \alpha, \underline{H}) = h(\sim\sim A, \alpha, \underline{H})$
(ii) If $\underline{H_1} R \underline{H_2}$, if $h(\sim A, \alpha, \underline{H_1}) = h(\sim A, \alpha, \underline{H_2})$
Hence (iii) if $\underline{H_1} R \underline{H_2}$ then $h(A, \alpha, \underline{H_1}) = h(A, \alpha, \underline{H_2})$
Therefore R is symmetric
Proof of (i) uses falsity and reduction requirements, and proof of (ii) the transitivity of $\underline{R}$.

A simplified R-model M is a structure $M = \langle 0, \underline{N}, \underline{P}, h \rangle$ where 0 and $\underline{N}$ are as before, $\underline{P}$ is a property of elements of $\underline{N}$ and h is a holding function such that for every atomic wff p and every $\alpha \in \underline{N}$, $h(p, \alpha) = T$ or $= F$, & is subject to
(i) if, for every $\beta \in \underline{N}$, $\underline{P}(\alpha + \beta)$ materially implies $\underline{P}(\gamma + \beta)$ then $\alpha = \gamma$ (the simplified reduction requirement)
(ii) for every atomic wff p and every $\alpha \in \underline{N}$ if $h(p, \alpha) = F$ then for some $\beta \in \underline{N}$ $\underline{P}(\alpha + \beta)$ (the simplified falsity requirement).
The holding function is extended in the expected way upon deletion of $\underline{H}$, viz.:
$h(A \& B, \alpha) = T$ iff $h(A, \alpha) = T = h(B, \alpha)$
$h(A \to B, \alpha) = T$ iff, for every $\beta \in \underline{N}$, if $h(A, \beta) = T$ then, materially, $h(B, \alpha + \beta) = T$
$h(\sim A, \alpha) = T$ iff for every $\beta \in \underline{N}$, if $\underline{P}(\alpha + \beta)$ then materially, $h(A, \beta) = F$
NL) A wff B is true in a simplified R-model M iff $h(B, 0) = T$;
B is R-s-valid iff ... R ...

Given symmetry and using connected cases R-models reduce to simplified R-models.

A simplified R_f model is a structure $\langle O, \underline{K}, h \rangle$ ... obtained using f.

Lemma. In R-models the ~~[illegible]~~ reduction requirement can be simplified without affecting R- or R_f-validity to the following: if, for every β, $P(\alpha+\beta, \underline{H}_1)$ implies $\underline{H}_1 R \underline{H}_2$ and $P(\beta+\gamma, \underline{H}_2)$ then $(\alpha, \underline{H}_1) = (\gamma, \underline{H}_2)$

Proof: The only postulate that requires the reduction principle $\sim\sim A \to A$ remains valid using the simpler requirement, by direct verification. The converse presupposes completeness method of §4. If $A \notin \underline{H}_1$ and $A_f \in \underline{H}_2$ then $A \to f \to f\alpha \notin \underline{H}_1$. Hence for some $\underline{H}_2$ $\underline{H}_1 R \underline{H}_2$ and $A \to f\beta \in \underline{H}_2$ and $f\alpha\beta \notin \underline{H}$. By the preceding Lemma since $\underline{H}R\underline{H}$, $A \to f_\beta \in \underline{H}_1$ and $f_{\alpha+\beta} \notin \underline{H}_1$. Remaining details in establishing the simplified reduction requirement are as for the (presupposed) proof of the reduction requirement in §4.

Theorem (i) If $\vdash_R B$ ($\vdash_{R_f} B$) then B is R-s valid (R_f-s valid)
(ii) B is R_f-s valid iff B is R_f-valid.

Proof of (i) is by induction over proof of B. As to (ii): if B is R_f-valid then B is R_f-s valid since R_f-s models are R-models with ~~[illegible]~~ $\underline{K} = \{\underline{G}\}$. Suppose, for the converse that B is not R_f-valid; then there is a connected R_f countermodel to B. Since M is connected and $\underline{R}$ is transitive, by a Lemma, for every $\underline{H} \in \underline{K}$, $h(A, \alpha, \underline{H}) = h(A, \alpha, \underline{G}) = h(A, \alpha)$ say. The reduction of $\underline{K}$ to $\{\underline{G}\}$ thus (a) provides a countermodel, etc. and hence (b) B is not R_f-s-valid.

ad (a). Since M is connected, for every $\underline{H} \in \underline{K}$, $\underline{G}R\underline{H}$. Thus $P(\gamma+\beta, \underline{G})$ implies $\underline{G}R\underline{H}$ & $P(\gamma+\beta, \underline{G})$ which implies $\underline{G}R\underline{H}$ & $P(\gamma+\beta, \underline{H})$. Hence for every β, $P(\alpha+\beta, \underline{G})$ implies $P(\gamma+\beta, \underline{G})$ implies that for every β, $P(\alpha+\beta, \underline{G})$ implies $\underline{G}R\underline{H}$ & $P(\gamma+\beta, \underline{H})$. Thus using the previous Lemma, $(\alpha, \underline{G}) = (\gamma, \underline{H})$. Since, however, $\underline{G}R\underline{H}$, for evaluation of $\vec{\;}$ [illegible], $(\alpha, \underline{G}) = (\gamma, \underline{G})$. Next if $h(p\ \alpha, \underline{G}) = F$ then for some β and $\underline{H}$, $\underline{G}R\underline{H}$ and $P(\alpha+\beta, \underline{H})$ i.e. $h(f, \alpha+\beta, \underline{H})$; hence for some β, $h(f, \alpha+\beta, \underline{G})$ i.e. $P(\alpha+\beta, \underline{G})$.

ad (b). By induction over connectives, each step which ~~[illegible]~~ shows that A holds or fails to hold in $(\alpha, \underline{H})$ may be reflected in $(\alpha, \underline{G})$. (*cf* the analogous proof in the decidability section).

An *S-model*, for system S of classical two-valued logic, is an ~~R~~ simplified R-model where $N = \{0\}$. [Thus model-wise it seems that R is related to classical logic as E is to S4: but though R includes S, E does not include S4].

A *simplified $\Box R_f$-model* is a structure $\langle \underline{G}, \underline{K}, \underline{R}, O, \underline{N}, h \rangle$ where $\langle \underline{G}, O, \underline{N}, h \rangle$ is a simplified R_f-model with $\alpha = (\alpha, \underline{G})$, and $\underline{K}$ is a set with base $\underline{G}$ and $\underline{R}$ is a reflexive and transitive relation on $\underline{K}$. Further
$h(\Box A, \alpha, \underline{H}) = T$ iff for every $\underline{H}' \in \underline{K}$ if $\underline{HRH}'$ then, materially, $h(A, \alpha, \underline{H}) = T$.

NOT PROVED ~~*Lemma*. B is $\Box R_f$-S-valid iff B is $\Box R_f$-valid~~

§3. Deduction theorems and primeness theorems

Where L is one of the systems E or R or their parts, and where, as before, $\alpha, \alpha_1, \ldots \alpha_n, \beta, \gamma, \theta$ etc. are sets (or lattice elements), 0 is the null set (or minimal element) and $\alpha+\beta$ is the set (or lattice) union of α and β, define :—

$A^1{}_{\alpha_1}, \ldots A^n{}_{\alpha_n} \Vdash_L B_\beta$ is an <u>L-proof of</u> B_β <u>from hypotheses</u> $A^1{}_{\alpha_1}, \ldots A^n{}_{\alpha_n}$ iff there is a sequence $C^1{}_{\gamma_1}, \ldots C^m{}_{\gamma_m}$ with $C^m{}_{\gamma_m} = B_\beta$, where each element of the sequence is either

(i) one of the hypotheses, or

(ii) D_0 where D is an axiom of L, or

(iii) obtained from predecessors in the sequence by application of the rule $\to E$: from A_α and $(A \to D)_\beta$ to infer $D_{\alpha+\beta}$, or

(iv) obtained from predecessors in the sequence by application of the rule $\& I$: from A_α and D_α to infer $(A \wedge D)_\alpha$.

Lc) As before $\alpha, \alpha_1 \ldots \alpha_n, \beta$ are sets, 0 is the null set and $\alpha+\beta$ is the set union of α and β

Nc) $\nabla \Vdash_L B_\beta$ iff for some $A^1{}_{\alpha_1}, \ldots A^n{}_{\alpha_n} \in \nabla$ $A^1{}_{\alpha_1}, \ldots A^n{}_{\alpha_n} \Vdash_L B_\beta$; in this case B_β is <u>L-provable from</u> ∇. ∇_α is a set of α-subscripted wff.

For systems like P and $R2$ and their parts it is necessary, once again, to use sets whose elements are ordered. Sets of ordinals are a convenient choice. For these systems the rule $\to E$ is modified by adding the proviso : <u>provided</u> $\max(\alpha) \nless \max(\beta)$, where, as before

$$\max(\alpha) = \begin{cases} \text{the largest element of } \alpha, & \text{if } \alpha \neq 0 \\ 0 & \text{if } \alpha = 0 \end{cases}$$

The first deduction theorems proved for E, R and P and their parts are given essentially in Anderson [ ] and Anderson & Belnap [ ].

Lemma: If $A^1_{\alpha_1}, \ldots A^n_{\alpha_n}, A_\Theta \Vdash_E B_\beta$ and $\Theta \neq 0$, $\Theta \not\leq \beta$ and $\Theta \not\leq \alpha_i$ for any i, $1 \leq i \leq n$ and each $A^i_{\alpha_i}$ is an entailment, i.e. of the form $(E_1 \to E_2)_{\alpha_i}$, for $1 \leq i \leq n$, then

$A^1_{\alpha_1}, \ldots A^n_{\alpha_n}, [A_\Theta] \Vdash_E N B_\beta$ — from which hypothesis A_Θ may be deleted.

Proof: Let the assumed proof sequence be represented

(α) $B^1_{\beta_1}, \ldots B^m_{\beta_m}$ with $B^m_{\beta_m} = B_\beta$

Form a new sequence

(β) $D^1_{\delta_1} \ldots D^p_{\delta_p}$ with $D^p_{\delta_p} = B_\beta$,

obtained from (α) by deleting every $B^i_{\beta_i}$ such that $\Theta \leq \beta_i$. Then (β) guarantees

$A^1_{\alpha_1}, \ldots A^n_{\alpha_n} \Vdash_E B_\beta$.

For no wff with a subscript α including Θ occurs essentially in a proof of B_β from hypotheses $A^1_{\alpha_1} \ldots A^n_{\alpha_n}$, for if it did it would follow that $\Theta \leq \beta$; for →E and &I eliminate no subscripts; and $\Theta \neq 0$, as no axioms would have a Θ subscript. Now form a new sequence

~~(γ) $D^1_{\delta_1} \ldots D^p_{\delta_p}$~~
~~$D^i_{\delta_i}$ is $ND^i_{\delta_i}$~~

(γ) $ND^1_{\delta_1}, \ldots ND^p_{\delta_p}$

[In the case of the pure calculus of entailment the same general form, $((D^i \to C) \to C)_{\delta_i}$ for arbitrary C, can displace $ND^i_{\delta_i}$ in [ ]]. Proof of the adequacy of (γ) mirrors the same proof strategy as the deduction theorem which follows. There are three cases.

Case 1: $D^i_{\delta_i}$ is one of $A^1_{\alpha_1} \ldots A^n_{\alpha_n}$, say $A^k_{\alpha_k}$. Then insert before $ND^i_{\delta_i}$ in (γ) the zero subscripted E proof sequence of [illegible] $(A^k \to NA^k)_0$, using the fact that A^k is an entailment. $ND^i_{\delta_i}$ then results by →E.

Case 2: $D^i_{\delta_i}$ is C_0 for some axiom C of system E.

<u>Case 3</u>. $D^i_{\delta_i}$ is inferred by $\to E$ from $D^j_{\delta_j}$ and $D^k_{\delta_k}$, with $j < i$, $k < i$. Then $D^j_{\delta_j}$ (say) is $(D^k \to D^i)_{\delta_j}$ and $\delta_i = \delta_j + \delta_k$. By induction hypothesis, $ND^k_{\delta_k}$ and $ND^j_{\delta_j}$ i.e. $N(D^k \to D^i)_{\delta_j}$ are available. Insert before $D^i_{\delta_i}$ a zero subscripted E-proof of $(N(D^k \to D^i) \to . ND^k \to ND^i)_0$; and $ND^i_{\delta_j + \delta_k}$ results by two applications of $\to E$.

<u>Case 4</u>. $D^i_{\delta_i}$ is inferred by $\wedge I$ from $D^j_{\delta_j}$ and $D^k_{\delta_i}$, with $j < i$, $k < i$. Then $\delta_i = \delta_j = \delta_k$ and $D^i = D^j \wedge D^k$. By induction hypothesis $ND^j_{\delta_i}$ and $ND^j_{\delta_j}$ are available in (K). Insert before $ND^i_{\delta_i}$ the ~~axiom~~ $ND^j \wedge ND^k \to . N(D^j \wedge D^k)$, and $(ND^j \wedge ND^k)_{\delta_i}$

Lemma. If $A^1_{\alpha_1}, \dots A^n_{\alpha_n} \vdash_{\Box R} B_\beta$, then $\Box A^1_{\alpha_1}, \dots \Box A^n_{\alpha_n} \vdash_{\Box R} \Box B_\beta$.

Proof. Let the given proof sequence be represented:

$$C^1_{\delta_1}, \dots, C^m_{\delta_m} = B_\beta$$

Form a new proof sequence

$$\Box C^1_{\delta_1}, \dots \Box C^m_{\delta_m}$$

; then this sequence provides a proof of $\Box B_\beta$ from hypotheses $\Box A^1_{\alpha_1}, \dots \Box A^n_{\alpha_n}$.

The cases are these at stage $C^i_{\delta_i}$:

$C^i_{\delta_i}$ is $A^j_{\alpha_j}$: then $\Box C^i_{\delta_i}$ is $\Box A^j_{\alpha_j}$;

$C^i_{\delta_i}$ is D_0 where D is an axiom; then $\Box C^i_{\delta_i}$ is $\Box D_0$, which can be introduced in a $\Box R$ proof from hypotheses;

$C^i_{\delta_i}$ is obtained by rule $\to E$ from C^j_{α} and $(C^j \to C^i)_\beta$ ~~and~~ with $\delta_i = \alpha + \beta$; then $\Box C^i_{\delta_i}$ is obtained from $\Box C^j_\alpha$ and $\Box(C^j \to C^i)_\beta$ which occur in the new sequence by the following inserted steps:

$\Box(C^j \to C^i) \to . \Box C^i \to \Box^j_{\beta}$, $\Box C^i \to \Box^j_\beta$

and use of $\to E$

$C^i_{\delta_i}$ is obtained from $C^j_{\delta_j}$ and $C^k_{\delta_k}$ by $\& I$. then $\delta_i = \delta_j = \delta_k$ and $C^i = C^j \& C^k$: then $\Box(C^j \& C^k)_{\delta_i}$ is obtained from $\Box C^j_{\delta_i}$ and $\Box C^k_{\delta_i}$ by $\& I$, $\to E$ and the following inserted steps:

$(\Box C^j \& \Box C^k)_{\delta_i}$, $(\Box C^j \& \Box C^k) \to \Box(C^j \& C^k)_{0}$

$C^i_{\delta_i}$ is ~~d~~ $\Box D_0$ where D_0 is an axiom; then $C^i_{\delta_i}$ is $\Box\Box D_0$ and is obtained by $\to E$ from the following inserted premises, $\Box D_0$, $\Box D_0 \to \Box\Box D_0$

Lemma. If $\nabla \Vdash_E B_\beta$ and each wff in ∇

N) is an entailment then $\nabla \Vdash_E \mathbf{N} B_\beta$,

N) where $\mathbf{N} B \leftrightarrow . B \to B \to B$.

Proof is like the preceding lemma; it uses the following theorems of E: $C \to D \to . N(C \to D)$; $N(C \to D) \to . NC \to ND$ $\quad NC \& ND \to N(C \& D)$

First Deduction theorems for E ~~[illegible]~~ and R and their frags.

If $A^1_{\alpha_1}, \ldots A^n_{\alpha_n}, A_\delta \vdash_L B_\beta$
and $\delta \neq 0$, $\delta \leq \beta$ but $\delta \not\leq \alpha_i$ for any i, $1 \leq i \leq n$, then
$$A^1_{\alpha_1}, \ldots A^n_{\alpha_n} \vdash_L A \to B_{\beta - \delta}$$

where (1) L is system R

(2) L is system E, and for each i, $1 \leq i \leq n$, A^i is an entailment, i.e. of the form $(D_1 \to D_2)$

~~(3) L is system [illegible] for each i, [illegible] entailment~~

Proof: By ~~the~~ assumption there is a sequence
$$B^1_{\beta_1}, \ldots B^m_{\beta_m} \text{ with } B^m_{\beta_m} = B_\beta$$
which provides a proof of B_β from hypotheses $A^1_{\alpha_1}, \ldots A_{\{k\}}$.
Form a new sequence
$$B^1_{\beta_1}{}', \ldots B^m_{\beta_m}{}'$$

REPLACE $\{k\}$ by δ throughout

where
$$B^i_{\beta_i}{}' = \begin{cases} (A \to B^i)_{\beta_i - \{k\}} & \text{if } \{k\} \leq \beta_i \\ B^i_{\beta_i} & \text{if } \{k\} \not\leq \beta_i \end{cases}$$
Then since $\{k\} \leq \beta$, $B^m_{\beta_m}{}' = (A \to B)_{\beta - \{k\}}$.
Following the proof strategy of Church [ ], pp 88-89, it is clear how to make insertions in the new sequence so that it provides a proof from hypotheses $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}$ of its last element $B^m_{\beta_m}{}'$. Suppose the insertions have been completed ~~[illegible]~~ just up to ~~the~~ the $(i-1)$th stage. At the ith stage there are three cases:

Case 1a. $B^i_{\beta_i}$ is $A_{\{k\}}$. Then $B^i_{\beta_i}{}'$ is $(A\to A)_0$. Insert before $B^i_{\beta_i}{}'$ a proof sequence of $(A\to A)_0$ using zero subscripted axioms and $\to E$.

Case 1b. $B^i_{\beta_i}$ is one of $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}$, say $A^r_{\alpha_r}$. Then $B^i_{\beta_i}{}'$ is also $A^r_{\alpha_r}$, since $\{k\} \not\leq \alpha_r$. Thus $B^i_{\beta_i}{}'$ occurs as one of the hypotheses (case (i) in an L-proof from hypotheses).

Case 2. $B^i_{\beta_i}$ is D_0 for some axiom D of L. Since $\{k\} \neq 0$, as a consequence of $\{k\} \not\leq \alpha_i$, $B^i_{\beta_i}{}'$ is also D_0. Thus $B^i_{\beta_i}{}'$ occurs as a zero subscripted axiom (case (ii) in an L-proof from hypotheses).

Case 3. $B^i_{\beta_i}$ is inferred by $\to E$ from $B^j_{\beta_j}$ and $B^k_{\beta_k}$ with $j<i$, $k<i$. Then $B^j_{\beta_j}$ (say) is $(B^k \to B^i)_{\beta_j}$ and $\beta_i = \beta_j + \beta_k$. There are 4 subcases:–

Case 3a. $\{k\} \leq \beta_j$ and $\{k\} \leq \beta_k$, so $\{k\} \leq \beta_i$. Then $B^k_{\beta_k}{}'$ is $(A \to B^k)_{\beta_k - \{k\}}$, $B^j_{\beta_j}{}'$ is $(A \to (B^k \to B^i))_{\beta_j - \{k\}}$, and $B^i_{\beta_i}{}'$ is $(A \to B^i)_{\beta_j + \beta_k - \{k\}}$. Insert before $B^i_{\beta_i}{}'$ a zero subscripted proof sequence of $(A \to (B^k \to B^i) \to . A \to B^k \to . A \to B^i)_0$; then insert $(A \to B^k \to . A \to B^i)_{\beta_j - \{k\}}$. $B^i_{\beta_i}{}'$ is inferred by $\to E$.

Case 3b. $\{k\} \leq \beta_k$ but $\{k\} \not\leq \beta_j$. Then $B^k_{\beta_k}{}'$ is $(A \to B^k)_{\beta_k - \{k\}}$ but $B^j_{\beta_j}$ is $(B^k \to B^i)_{\beta_j}$. Insert the axiom $(B^k \to B^i \to . A \to B^k \to . A \to B^i)_0$, and $(A \to B^k \to A \to B^i)_{\beta_j}$ before $B^i_{\beta_i}{}'$. Then $B^i_{\beta_i}{}'$, i.e. $(A \to B^i)_{\beta_k + \beta_j - \{k\}}$, follows by $\to E$.

Case 3c. $\{k\} \not\leq \beta_k$ and $\{k\} \not\leq \beta_j$. Then $B^k_{\beta_k}{}'$ is $B^k_{\beta_k}$, $B^j_{\beta_j}{}'$ is $(B^k \to B^i)_{\beta_j}$, and $B^i_{\beta_i}{}'$, i.e. $B^i_{\beta_j+\beta_k}$, is inferred by $\to E$.

<u>Case 3d</u>. $\{k\} \subseteq \beta_j$ but $\{k\} \not\subseteq \beta_k$. Thus $B^k{}_{\beta_k}{}'$ is $B^k{}_{\beta_k}$, $B^j{}_{\beta_j}{}'$ is $\not{k}(A \rightarrow (B^k \rightarrow B^i))_{\beta_j - \{k\}}$, and $B^i{}_{\beta_i}{}'$ is $(A \rightarrow B^i)_{\beta_j + \beta_k - \{k\}}$.

(1) L is system R. Insert before $B^i{}_{\beta_i}{}'$ a zero-subscripted proof sequence of $(A \rightarrow (B^k \rightarrow B^i) \rightarrow . B^k \rightarrow . A \rightarrow B^i)_0$, and then insert $(B^k \rightarrow . A \rightarrow B^i)_{\beta_j - \{k\}}$. $B^i{}_{\beta_i}{}'$ then results by $\rightarrow E$.

(2) L is system E. By a lemma, since $\{k\} \not\subseteq \beta_k$, there is an E-proof from hypotheses of $(NB^k)_{\beta_k}$. Insert this sequence, then insert the zero-subscripted proof sequence of $(A \rightarrow (B^k \rightarrow B^i) \rightarrow . NB^k \rightarrow . A \rightarrow B^i)_0$, and finally insert $\not{k}(NB^k \rightarrow . A \rightarrow B^i)_{\beta_j - \{k\}}$. $B^i{}_{\beta_i}$ then results by $\rightarrow E$.

Case 4. $B^i_{\beta_i}$ is inferred by &I from $B^j_{\beta_j}$ and $B^k_{\beta_k}$ with $j < i$, $k < i$. Then $\beta_i = \beta_j = \beta_k$ and B^i is $(B^j \& B^k)$. There are 2 subcases:—

Case 4a. $\delta \subseteq \beta_i$. Then $B^k_{\beta_k}{}'$ is $(A \to B^k)_{\beta_k - \delta}$, $B^j_{\beta_j}{}'$ is $(A \to B^j)_{\beta_k - \delta}$ and $B^i_{\beta_i}{}'$ is $(A \to B^j \& B^k)_{\beta_k - \delta}$. Insert before $B^i_{\beta_i}{}'$ the axiom $((A \to B^j) \& (A \to B^k) \to . A \to (B^j \& B^k))_0$ and, what is inferred by &I, $((A \to B^j) \& (A \to B^k))_{\beta_k - \delta}$. Then $B^i_{\beta_i}{}'$ is inferred by $\to$E.

Case 4b. $\delta \not\subseteq \beta_i$. Then $B^k_{\beta_k}{}'$ is $B^k_{\beta_k}$, $B^j_{\beta_j}{}'$ is $B^j_{\beta_k}$, and $B^i_{\beta_i}{}'$ is $(B^j \& B^k)_{\beta_k}$ which is inferred, as before, by &I.

This deduction theorem holds also for such extensions of E and R as E∧, □R, □R5, etc. It is not, of course, the only deduction theorem for E and R. Alternative deduction theorems for R_I are given in [ ] and [ ]*, and an alternative deduction theorem for E_I is as follows: if $A_1, \dots A_n, A \vdash_{E_I} B$ and $A_1 \dots A_n$ are entailments and A is used in the proof then $A_1 \dots A_n \vdash_E A \to B$. In order to deal with disjunction in R the following deduction theorem is needed:

A Second deduction theorem for R and E.

If $A^1_{\alpha_1}, \dots A^n_{\alpha_n}, A_\delta \Vdash_R B_\beta$

then either $A^1_{\alpha_1}, \dots A^n_{\alpha_n} \Vdash_R A \to B_{\beta - \delta}$ with $\delta \subseteq \beta$

or $A^1_{\alpha_1}, \dots A^n_{\alpha_n} \Vdash_R B_\beta$.

Similarly for E where $A^1_{\alpha_1}, \dots A^n_{\alpha_n}$ are entailments.

Proof: Let

$B^1_{\beta_1}, \dots, B^n_{\beta_n} = B_\beta$ be a proof of B_β from hypotheses $A^1_{\alpha_1} \dots A^n_{\alpha_n}$.

---

* The simple use-of-hypothesis account breaks down once again [illegible]

is shown by induction for each $B^i_{\beta_i}$ that
either (i) $A^1_{\alpha_1}, \ldots A^n_{\alpha_n} \Vdash_R A \to B^i_{\beta_i - \delta}$ and $\delta \leq \beta_i$
or (ii) $A^1_{\alpha_1} \cdots A^n_{\alpha_n} \Vdash_R B^i_{\beta_i}$

Case 1. $B^i_{\beta_i}$ is A_δ. Then $A^1_{\alpha_1} \cdots A^n_{\alpha_n} \Vdash_R A \to B^i_{\beta_i - \delta}$
using $A \to A_0$.

Case 2. $B^i_{\beta_i}$ is a zero subscripted axiom or one of
$A^1_{\alpha_1} \cdots A^n_{\alpha_n}$. Then $A^1_{\alpha_1} \cdots A^n_{\alpha_n} \Vdash B^i_{\beta_i}$.

Case 3. $B^i_{\beta_i}$ is inferred by $\to E$. The cases are
are as before. Note that $B^i_{\beta_i}$ results when and only
when both the premises are of form (i). $\delta \leq \beta_i$
follows from $\delta \leq \beta_j$ or $\delta \leq \beta_k$.

Case 4. $B^i_{\beta_i}$ is inferred by $\& I$.

~~[illegible]~~

(Permission theorem for R)

Corollary 1. If $\Gamma, A_\delta \Vdash_R B_\beta$ and $\Gamma, C_\delta \Vdash_R B_\beta$
then $\Gamma, (A \vee C)_\delta \Vdash_R B_\beta$.

Proof: Given the premises, either $\Gamma \Vdash_R B_\beta$ and so
$\Gamma, (A \vee C)_\delta \Vdash_R B_\beta$, or both $\Gamma \Vdash_R A \to B_{\beta - \delta}$ and
$\Gamma \Vdash_R C \to B_{\beta - \delta}$ and $\delta \leq \beta$. Hence, in the alternative case,
$\Gamma \Vdash_R (A \to B) \& (C \to B)_{\beta - \delta}$ with $\delta \leq \beta$. Since $\vdash_R (A \to B) \& (C \to B) \to A \vee C \to B$,
$\Gamma \Vdash_R (A \vee C \to B)_{\beta - \delta}$ with $\delta \leq \beta$, whence $\Gamma, A \vee C_\delta \Vdash_R B_{\beta - \delta + \delta}$
i.e. $\Gamma, (A \vee C)_\delta \Vdash_R B_\beta$, since $\beta \geq \delta$.

2. (Appropriate result for E). As in corollary 1 but
with Γ consisting only of entailments.

Alternative form of the first deduction theorem for R and for E.

If $A^1_{\alpha_1}, \cdots A^n_{\alpha_n}, A_\delta \vdash_R B_\beta$
and $\delta \neq 0$, $\delta \subseteq \beta$ and δ disjoint from α_i for $1 \leq i \leq n$,
then $A^1_{\alpha_1}, \cdots A^n_{\alpha_n} \vdash_R A \to B_{\beta - \delta}$

Similarly for E where $A^1_{\alpha_1} \cdots A^n_{\alpha_n}$ are entailments.

Proof. Redefine

$$B^i_{\beta_i}{}' = \begin{cases} (A \to B^i)_{\beta_i - \delta} & \text{if } \delta \text{ not disjoint } \beta_i \\ B^i_{\beta_i} & \text{if } \delta \text{ disjoint } \beta_i \end{cases}$$

Then $B^n_{\beta_n}{}'$ is $(A \to B)_{\beta - \delta}$. The proof is as before.
Note in case 3c if δ disj β_k and δ disj β_j then δ disj $(\beta_k + \beta_j)$; in 3b δ disj β_j but not disj β_k then δ not disj $(\beta_j + \beta_k)$; in 3a δ not disj β_k and not disj β_j then δ not disj $(\beta_k + \beta_j)$; in 3d then δ is not disj $(\beta_k + \beta_j)$. ~~and the ... theorem~~

First deduction theorem in P and its proof

If $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}, A_\delta \Vdash_L B_\beta$
where $\delta \neq 0$, $n = \max(\delta) \in \beta$ but
n exceeds $\max(\alpha_i)$ for each i in $1 \le i \le n$

then $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n} \Vdash_L A \to B_{\beta-\delta}$

Proof: Using the assumed sequence $B^1_{\beta_1}, \ldots, B^m_{\beta_m} = B_\beta$ form a new sequence $B^1_{\beta_1}{}', \ldots, B^m_{\beta_m}{}'$ where

$$B^i_{\beta_i}{}' = \begin{cases} (A \to B^i)_{\beta_i - \delta} & \text{if } \max(\delta) \in \beta_i \\ B^i_{\beta_i} & \text{if } \max(\delta) \notin \beta_i \end{cases}$$

Cases 1 and 2 and 4: as before.

Case 3. $B^i_{\beta_i}$ is inferred by $\to E$ from $B^k_{\beta_k}$ and $B^j_{\beta_j} = (B^k \to B^i)_{\beta_j}$ with $j < i, k < i$, $\beta_i = \beta_j + \beta_k$, and $\max(\beta_k) \not< \max(\beta_j)$

Case 3a. $n = \max(\delta) \in \beta_j$ and $n \in \beta_k$, so $n \in \beta_i$. Then $B^k_{\beta_k}{}'$ is $(A \to B^k)_{\beta_k - \delta}$, $B^j_{\beta_j}{}'$ is $(A \to (B^k \to B^i))_{\beta_j - \delta}$ and $B^i_{\beta_i}{}'$ is $(A \to B^i)_{\beta_j + \beta_k - \delta}$

Case 3a(i). $\max(\beta_k - \delta) \not< \max(\beta_j - \delta)$. Insert a zero subscripted proof sequence of $(A \to (B^k \to B^i) \to. A \to B^k \to. A \to B^i)_0$, then insert $(A \to B^k \to. A \to B^i)_{\beta_j - \delta}$. In view of the ordering condition $\to E$ may be applied to infer $B^i_{\beta_i}{}'$.

Case 3a(ii). $\max(\beta_k - \delta) < \max(\beta_j - \delta)$. Insert a proof sequence for $((A \to B^k) \to. A \to (B^k \to B^i) \to. A \to B^i)_0$, then insert $(A \to (B^k \to B^i) \to. A \to B^i)_{\beta_k - \delta}$ (since $\max(\beta_k - \delta) \geq 0$)

Case 3b. $n \in \beta_k$ but $n \notin \beta_j$; so $n \in \beta_k + \beta_j$, and $B^k_{\beta_k}{}'$ is $(A \to B^k)_{\beta_k - \delta}$, $B^i_{\beta_i}{}'$ is $(A \to B^i)_{\beta_i - \delta}$ but $B^j_{\beta_j}{}'$ is $(B^k \to B^i)_{\beta_j}$.

Case 3b(i). $\max(\beta_j) \leq \max(\beta_k - \delta)$. Insert $(B^k \to B^i \to. A \to B^k \to. A \to B^i)_0$, $(A \to B^k \to A \to B^i)_{\beta_i}$ before $B^i_{\beta_i}{}'$, and use $\to E$ twice.

Case 3b(ii). $\max(\beta_j) \not\leq \max(\beta_k - \delta)$. Insert $(A \to B^k \to. B^k \to B^i \to A \to B^i)_0$, $(B^k \to B^i \to A \to B^i)_{\beta_k - \delta}$ before $B^i_{\beta_i}{}'$, and use $\to E$ twice.

Case 3c: as before.

Case 3.d : impossible. For since $m \in \beta_j$ the largest element that can occur belongs to β_j. As $m \notin \beta_k$, $\max(\beta_j) > \max(\beta_k)$, contradicting an assumption for case 3.

__A Second deduction theorem for P and its parts.__

If $A^1_{\alpha_1}, A^2_{\alpha_2} \ldots A^n_{\alpha_n}, A_\delta \Vdash_P B_\beta$ and $m = \max(\delta)$ and $m > \max(\alpha_i)$ for every α_i, $1 \le i \le n$,

then either $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n} \Vdash_P A \to B_{\beta-\delta}$ and $m \in \beta$

or $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n} \Vdash_P B_\beta$ and $m \notin \beta$.

Proof is like the second deduction theorem for R.

This deduction theorem is not ~~so~~ sharp enough to provide ~~[illegible]~~ the basis for a disjunction rule for P. For that the following result ~~[illegible]~~ seems to be needed.

Improved second ~~[illegible]~~ deduction theorem for P (conjecture only) (unlikely?)

If $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}, A_\delta \Vdash_P B_\beta$ and $\max(\delta) \geq \max(\alpha_i)$ for every α_i, $1 \le i \le n$,

then ~~[illegible]~~

either $A^1_{\alpha_1}, \ldots A^n_{\alpha_n} \Vdash_P A \to B_{\beta-\delta}$ and $m \in \beta$

or $A^1_{\alpha_1}, \ldots A^n_{\alpha_n} \Vdash_P B_\beta$

__Corollary__ (A primeness result for P).

If $\Gamma, A_\beta \Vdash_P C_\delta$ and $\Gamma, B_\beta \Vdash_P C_\delta$ then $\Gamma, (A \vee B)_\beta \Vdash_P C_\delta$, __provided__ $\max(\beta) > \max(\gamma)$ for each $D_\gamma \in \Gamma$.

<u>Qualified primeness theorem for P and E</u>

If $\Gamma_\alpha, A_\alpha \Vdash C_\alpha$ and $\Gamma_\alpha, B_\alpha \Vdash C$
then $\Gamma_\alpha, (A \vee B)_\alpha \Vdash C_\alpha$, for any α.
Γ_α is a set of wff all subscripted with α.

Proof: (a'). If $\Gamma_\alpha, A_\alpha \Vdash C_\alpha$ then
$\Gamma_\alpha, (A \vee B)_\alpha \Vdash (C \vee B)_\alpha$.

Let given sequence in (a') be
$$A^1_{\gamma_1} \ldots A^n_{\gamma_n} = C_\alpha. \quad \text{Then}$$
~~Form a new sequence~~

$\gamma_i = 0$ or α according as A^i is a theorem or is a consequence of at least one of the hypotheses.

Form the new sequence
$$A^1_{\gamma_1}{}' \ldots A^m_{\gamma_m}{}'$$
where
$$A^i_{\gamma_i}{}' = \begin{cases} (A^i \vee B)_{\gamma_i} & \text{if } \gamma_i = \alpha \\ A^i_0 & \text{otherwise} \end{cases}$$

There are three cases:—

<u>case $\to E$</u>. $A^i_{\gamma_i}$ follows by $\to E$ from $A^k_{\gamma_k}$ and $(A^k \to A^i)_{\gamma_j} = A^j_{\gamma_j}$
and $\gamma_i = \gamma_j + \gamma_k$.

<u>Case 1</u>. $\gamma_j = \gamma_k = \alpha$.

Then $\gamma_i = \alpha$, and by hypothesis have in new sequence $(A^k \vee B)_\alpha$ and $(A^k \to A^i \vee B)_\alpha$.
Then must $(A^k \vee B \wedge (A^k \to A^i \vee B))_\alpha$
appropriate theorems leading to, in turn, to
$$([A^k \wedge (A^k \to A^i)] \vee [A^k \wedge B] \vee [B \wedge (A^k \to A^i)] \vee [B \wedge B])_\alpha$$
[redacted]
$$(A^i \vee B \vee B \vee B)_\alpha$$
[redacted]
$$(A^i \vee B)_\alpha.$$

<u>Case 2</u>. $\gamma_j = \gamma_k = 0$. Then result proved as before by $\to E$

Case 3. $\gamma_k = \alpha$ and $\gamma_j = 0$ then $\gamma_i = \alpha$.

Then and $A^k{}_{\gamma_k}{}'$ is $(A^k \vee B)_\alpha$

$A^j{}_{\gamma_j}{}'$ is $(A^k \to A^i)_0$

$A^i{}_{\gamma_i}{}'$ is $(A^i \vee B)_\alpha$.

Insert $(B \to B)_0$ $(A^k \to A^i \mathbin{\&} B \to B)_0$

$((A^k \to A^i) \mathbin{\&} (B \to B) \to_0 (A^k \vee B) \to (A^i \vee B))_0$

whence $(A^k \vee B) \to (A^i \vee B)_0$

so $(A^i \vee B)_\alpha$.

Case 4. $\gamma_k = 0$ and $\gamma_j = \alpha$; so $\gamma_i = \alpha$.

This case is impossible for P unless $\alpha = 0$, in which case the result follows as for case 1. For E, $A^k{}_{\gamma_k}{}'$ is $A^k{}_0$, $A^j{}_{\gamma_j}{}'$ is $((A^k \to A^i) \vee B)_\alpha$ and $A^i{}_{\gamma_i}{}'$ is $(A^i \vee B)_\alpha$. If $\alpha = 0$ then the result follows as for case 1 ; if $\alpha \neq 0$ then A^k must be a theorem, hence

$(A^k \to A^i) \to A^i$ is a theorem so

insert $((A^k \to A^i) \to A^i)_0$ and $B \to B_0$ then

$(B \to B \mathbin{\&} (A^k \to A^i) \to A^i)_0$, then Resolution, then

$((A^k \to A^i) \vee B \to A^i \vee B)_0$. Result by $\to E$.

Case Hyp. $A^i{}_{\gamma_i} \in \Gamma_\alpha$ or $A^i{}_{\gamma_i}$ is A_α.

then $A^i{}_{\gamma_i}{}'$ is $(A^i \vee B)_\alpha$

insert $A^i \to A^i \vee B$.

Case Axiom. $A^i{}_{\gamma_i} = D_0$

then $A^i{}_{\gamma_i}{}' = D_0$ ~~[illegible]~~ also.

Case &I. $A^i{}_{\gamma_i} = (A^j \mathbin{\&} A^k)_{\gamma_i}$ follows by &I

from $A^j{}_{\gamma_j}$ and $A^k{}_{\gamma_k}$. Then

$\gamma_j = \gamma_k = \gamma_i$

Case 1. $\gamma_i = \alpha$

so $A^k{}_{\gamma_k}{}' = (A^k \vee B)_\alpha$ $A^j{}_{\gamma_j}{}' = (A^j \vee B)_\alpha$

$A^i{}_{\gamma_i}{}' = (A^j \mathbin{\&} A^k \vee B)_\alpha$

apply &I to get $((A^j \vee B) \mathbin{\&} (A^k \vee B))_\alpha$

Then insert appropriate theorem to get

$(A^j \mathbin{\&} A^k \vee B)_\alpha$

Case 2. $\gamma_i = 0$

then $A^i{}_{\gamma_i}{}' = A^i{}_0$, $A^j{}_{\gamma_j}{}' = A^j{}_0$, follows

$A^k{}_{\gamma_k}{}' = A^k{}_0$; $A^i = A^j \mathbin{\&} A^k$

(b′) If $\Gamma_\alpha, B_\alpha \vdash C_\alpha$ then $\Gamma_\alpha, (C \vee B)_\alpha \vdash (C \vee C)_\alpha$.

Proof is similar to (b′).

(c′) ~~If~~ $\Gamma_\alpha, (C \vee C)_\alpha \vdash C_\alpha$. For $\vdash C \vee C \to C$.

The theorem then follows on combining (a′), (b′) and (c′).

§4. Completeness by maximal set methods.

∇ is L-consistent w.r.t. $\underline{\underline{N}}$ iff, for some $\delta \in \underline{\underline{N}}$ and $D_\delta \in \nabla$, D_δ is not L-provable from ∇.

∇ is an L-ok set w.r.t. $\underline{\underline{N}}$ (where $\underline{\underline{N}}$ is a set closed under + and including 0) iff

(i) ∇ is L-consistent w.r.t. $\underline{\underline{N}}$,

(ii) $A_0 \in \nabla$ for every axiom A of L,

(iii) for every $\alpha \in \underline{\underline{N}}$, if $A_\alpha \in \nabla$ and $B_\alpha \in \nabla$ then $(A \& B)_\alpha \in \nabla$,

(iv) for every $\alpha, \beta \in \underline{\underline{N}}$, if $B_\beta \in \nabla$ and $(B \to C)_\alpha \in \nabla$ then $C_{\alpha+\beta} \in \underline{\underline{N}}$ [provided $\max(\beta) \neq \max(\alpha)$, in the case of P systems].

Lemma. If ∇ is an L-ok set w.r.t. $\underline{\underline{N}}$ then

(i) for every theorem A of L, $A_0 \in \nabla$,

(ii) for $\alpha \in \underline{\underline{N}}$, $(A \& B)_\alpha \in \nabla$ iff $A_\alpha \in \nabla$ and $B_\alpha \in \nabla$

(iii) for $\alpha \in \underline{\underline{N}}$, $A_\alpha \in \nabla$ iff $\nabla \vdash_L A_\alpha$

An L-ok set ∇ w.r.t. $\underline{\underline{N}}$ is prime iff for every $\alpha \in \underline{\underline{N}}$ if $(A \vee B)_\alpha \in \nabla$ then either $A_\alpha \in \nabla$ or $B_\alpha \in \nabla$. If ∇ is prime then $A \vee B_\alpha \in \nabla$ iff $A_\alpha \in \nabla$ or $B_\alpha \in \nabla$.

<u>Lemma</u>: If $(B \to C)_\alpha \notin \nabla$ where ∇ is an E ok set, and ∇' is a set whose elements comprise every subscripted entailment $(D_1 \to D_2)_{\delta_i}$ in ∇ and B_δ for any $\delta \not\subseteq \alpha$, $\not\subseteq \delta_i$ for $(D_1 \to D_2)_{\delta_i} \in \nabla$, and $\neq 0$,

then $C_{\alpha+\delta}$ is not E-provable from ∇'.

Proof: Suppose on the contrary, $\nabla' \Vdash_E C_{\alpha+\delta}$. Then for some entailments $D^1{}_{\delta_1}, \cdots D^n{}_{\delta_n} \in \nabla$, and therefore in ∇', $D^1{}_{\delta_1}, D^2{}_{\delta_2}, \cdots D^n{}_{\delta_n}, B_\delta \Vdash_E C_{\alpha+\delta}$. Since $\delta \not\subseteq \delta_i$ for $1 \leq i \leq n$, B_δ must occur among the hypotheses. The conditions for the subscripted deduction theorem are satisfied; thus $D^1{}_{\delta_1}, \cdots D^n{}_{\delta_n} \Vdash_E (B \to C)_\alpha$. Since, however, $D^1{}_{\delta_1}, \cdots D^n{}_{\delta_n} \in \nabla$ and ∇ is E ok, $(B \to C)_\alpha \in \nabla$, contradicting the hypothesis.

<u>Lemma</u>: If $(B \to C)_\alpha \notin \nabla$ where ∇ is an R-ok set [$\square$R-ok set], and ∇' is a set whose elements comprise those of ∇ and B_δ for any $\delta \not\subseteq \alpha$, $\not\subseteq \beta$ for $D_\beta \in \nabla$, and $\neq 0$,

then $C_{\alpha+\delta}$ is not R-provable [$\square$R-provable] from ∇'.

A suitable δ can always be got by choosing a new δ.

<u>Lemma</u> If $(B \to C)_\alpha \notin \nabla$ where ∇ is a P-ok set [P2-ok set], and ∇' is a set whose elements comprise every subscripted entailment $(D_1 \to D_2)_{\delta_i}$ in ∇

$\{m\} = \gamma$ and $B_{\{m\}}$ {for any where $max(\gamma)$ is element m} greater than all elements of α and of δ_i for $(D_1 \to D_2)_{\delta_i} \in \nabla$

$C_{\alpha+\gamma}$ then $C_{\alpha+\{m\}}$ is not P-provable [P2-provable] from ∇'.

Lemma. If C_δ is not $\mathbf{E}$-provable from ∇ then there is an countable $\mathbf{E}$-ok extension ∇^+ of ∇ w.r.t. any set N which includes all subscripts of ∇ such that $C_\delta \notin \nabla^+$.

Proof: Enumerate N and ~~then~~ enumerate the wff of $\mathbf{E}$, and then enumerate the wff of $\mathbf{E}$ with respect to the subscripts of N. Let the resulting enumeration of subscripted wff be represented:

$$D^0, D^1, \dots D^j, \dots$$

Define:

$$\nabla^0 = \nabla$$
$$\nabla^{j+1} = \nabla^j \quad \text{if } \nabla^j, D^j \vdash_{\mathbf{E}} C_\delta \text{, and}$$
$$= \nabla^j \cup \{D^j\} \quad \text{otherwise}$$
$$\nabla^+ = \bigcup_{j<\omega} \nabla^j .$$

(i) C_δ is not $\mathbf{E}$-provable from ∇^+. Proof is by induction over j from the given basis. The induction step is a consequence of the construction.

(ii) $C_\delta \notin \nabla^+$, by (i).

(iii) $D_0 \in \nabla^+$ where D is an axiom of system $\mathbf{E}$.

(iv) ∇^+ is closed under $\rightarrow E$. Suppose, otherwise, that $B_\gamma \in \nabla^+$, $(B \rightarrow D)_\beta \in \nabla^+$ but $D_{\gamma+\beta} \notin \nabla^+$ (and for P: $max(\gamma) \neq max(\beta)$). Then $\nabla^+, D_{\gamma+\beta} \vdash_{\mathbf{E}} C_\delta$. But then since $\nabla^+ \vdash_{\mathbf{E}} B_\gamma$ and $\nabla^+ \vdash_{\mathbf{E}} (B \rightarrow D)_\delta$, $\nabla^+ \vdash_{\mathbf{E}} D_{\gamma+\delta}$, whence $\nabla^+ \vdash_{\mathbf{E}} C_\delta$ contradicting (i).

(v) ∇^+ is closed under $\& I$. Suppose $B_\gamma \in \nabla^+$, $D_\gamma \in \nabla^+$ but $(B \& D)_\gamma \notin \nabla^+$. Then $\nabla^+, (B \& D)_\gamma \vdash_L C_\delta$, and (much as in (iv)) $\nabla^+ \vdash_L C_\delta$ contradicting (i).

Lemma: If, further, L is system R then ∇^+ is prime.

Proof: Suppose $A \vee B_\alpha \in \nabla^+$, $A_\alpha \notin \nabla^+$ and $B_\alpha \notin \nabla^+$. Then for some p,
$\nabla^p, A_\alpha \Vdash_R C_\delta$, $\nabla^p, B_\alpha \Vdash_R C_\delta$ but not
$\nabla^p, (A \vee B)_\alpha \Vdash_R C_\delta$. By the ~~[illegible]~~ second deduction theorem either $\nabla^p \Vdash_R A \to C_{\delta-\alpha}$ and $\nabla^p \Vdash_R B \to C_{\delta-\alpha}$ and $\alpha \subseteq \delta$ or else $\nabla^p \Vdash_R C_\delta$. Since the second is impossible, $\nabla^p \Vdash_R (A \to C) \& (B \to C)_{\delta-\alpha}$.
Hence since $\vdash_R (A \to C) \& (B \to C) \to . A \vee B \to C$,
$\nabla^p \Vdash_R A \vee B \to C_{\delta-\alpha}$, and so $\nabla^p, A \vee B_\alpha \Vdash_R C_\delta$.

Lemma: If no wff in non-null set Σ is L-provable from ∇ then there is an L-ok extension ∇^+ of ∇ w.r.t any set $\underline{N}$ which includes all subscripts of ∇ such that no wff in Σ belongs to ∇^+.
Proof is like the similar lemma where $\Sigma = \{C_\delta\}$ except that ∇^+ is redefined, as follows:

$$\nabla^0 = \nabla$$
$$\nabla^{j+1} = \nabla^j \text{ if } \nabla^j, D^j \Vdash_L C_\delta \text{ for some } C_\delta \in \Sigma$$
$$= \nabla^j \cup \{D^j\} \text{ otherwise}$$
$$\nabla^+ = \bigcup_{j<\omega} \nabla^j$$

Then no wff $C_\delta \in \Sigma$ is L-provable from ∇^+, and ∇^+ is L-ok.

Lemma. If $\Box B_\alpha \notin \nabla$ where ∇ is an $\Box R$-ok set [$\Box R4$-ok set] then there is a set ∇', whose elements comprise each C_δ such that $\Box C_\delta \in \nabla$, such that B_α is not $\Box R$-provable [$\Box R4$-provable] from ∇.

Proof. Suppose $C'_{\delta_1}, \dots C^n_{\delta_n} \Vdash_{\Box R} B_\alpha$. Then by a Lemma $\Box C'_{\delta_1}, \dots \Box C^n_{\delta_n} \Vdash_{\Box R} \Box B_\alpha$. Hence since ∇ is $\Box R$-ok, $\Box B_\alpha \in \nabla$.

Lemma. If C_δ is not $\Box R$-provable from ∇ then there is an $\Box R$-ok extension ∇^+ of ∇ w.r.t. countable set N which includes all subscripts of ∇ and that $C_\delta \notin \nabla^+$.

Lemma. If $\sim A_\alpha \in \nabla$ where ∇ is E ok but $\sim(A \to A)_{\alpha+\beta} \notin \nabla$ (or $\Delta_{\alpha+\beta} \notin \nabla$), and ∇' is any set containing every entailment in ∇ then A_β is not E-provable from ∇'.

Proof: Suppose $\nabla' \Vdash_E A_\beta$. Then by a Lemma since each member of ∇' is an entailment $\nabla' \Vdash_E A \to A \to A_\beta$; so $\nabla' \Vdash_E \sim A \to \sim(A \to A)_\beta$, and $\nabla', \sim A_\alpha \Vdash \sim(A \to A)_{\alpha+\beta}$. Hence since ∇ is E ok $\sim(A \to A)_{\alpha+\beta} \in \nabla$, contradicting the hypothesis (In the case of $\Delta_{\alpha+\beta}$, use the principle $\sim(A \to A) \to \Delta$).

~~Corollary: If $\sim A_\alpha \in \nabla$ with ∇ E ok but $\sim(A \to A)_{\alpha+\beta} \notin \nabla$ (or $\Delta_{\alpha+\beta} \notin \nabla$) then for any E ok set~~

Lemma: If $(B \to C)_\alpha \notin \nabla$ where ∇ is an Rok [$\Box$R-ok] set, then there is an Rok [$\Box$R-ok] set Σ which includes ∇ such that $B_{\{k\}} \in \Sigma$ but $C_{\alpha+\{k\}} \notin \Sigma$ for some $\{k\}$.
Proof combines previous lemmata.

Lemma: If $(B \to C)_\alpha \notin \nabla$ where ∇ is an E-ok set then (i) there is an E-ok set Σ such that for some $\{k\}$ $B_{\{k\}} \in \Sigma$ and $(D_1 \to D_2)_\delta \in \Sigma$ if $(D_1 \to D_2)_\delta \in \nabla$ but $C_{\alpha+\{k\}} \notin \Sigma$

(ii) there is an E-ok set Ⓗ which contains every subscripted entailment in ∇ and does not contain any subscripted negation not in ∇ (i.e. if $\sim D_\delta \notin \nabla$ then $\sim D_\delta \notin$ Ⓗ) such that for some $\{k\}$ $B_{\{k\}} \in$ Ⓗ but $C_{\alpha+\{k\}} \notin$ Ⓗ.

OMIT

Proof of (ii): Given ∇, there is a set ∇', whose elements comprise every subscripted entailment in ∇ and $B_{\{k\}}$ for suitable $\{k\}$, such that $C_{\alpha+\{k\}}$ is not E-provable from ∇'. Suppose, for some $\sim D_\delta$ not in ∇, $\nabla', B_{\{k\}} \Vdash_E \sim D_\delta$. Then $B_{\{k\}}$ must be used in the proof since $\sim D_\delta$ is not E-provable from ∇'; hence $k \in \delta$. Now choose any k such that for each $\sim D_\delta$ not in ∇ with $B \to \sim D_{\delta-\alpha} \in \nabla$ for some α, $k \notin \delta$. (Any new k will satisfy these conditions).

Let Σ be the set consisting of $C_{\alpha+\{k\}}$ and every subscripted negation $\sim D_\delta$ not in ∇. Then Σ is not null and no element of Σ is E-provable from ∇'. Hence, by a lemma ∇' has an E-ok extension, Ⓗ say, such that no element of Σ belongs to Ⓗ.

<u>Completeness theorems for Rpure R and R^+</u>

(i) If A_0 is not Rf provable from Γ_0 then there is an R-model $M = \langle \underline{G}, \underline{K}, \underline{R}, O, \underline{N}, \underline{P}, h \rangle$ with $\underline{K}$ and $\underline{N}$ denumerable, which satisfies Γ and falsifies A. Hence every Rf consistent set is satisfiable in a denumerable model.

(ii) If A is Rf-valid then $\vdash_{Rf} A$

(iii) If A is R-valid then $\vdash_R A$

(iv) If A is R^+-valid then $\vdash_{R^+} A$.

<u>Proof</u>: (i) By a lemma there is an Rf ok set, $\underline{G}$ say, wrt $\{0\}$, which includes Γ_0 but excludes A_0. Define a canonical R-model M, with base $\underline{G}$, as follows:—

$\underline{K}$ and $\underline{N}$ are defined by a joint inductive definition:

(i) $\underline{G} \in \underline{K}$ and $0 \in \underline{N}$

(ii) If for $\underline{H_1} \in \underline{K}$ and $\beta \in \underline{N}$, $(B \to C)_\beta \notin \underline{H_1}$ then by a lemma there is a new (singleton) subscript set γ and an Rf ok set $\underline{H_2}$, which includes $\underline{H_1}$, such that $B_\gamma \in \underline{H_2}$ and $C_{\beta \cup \gamma} \notin \underline{H_2}$; put $\underline{H_2} \in \underline{K}$ and $\gamma \in \underline{N}$. (A convenient choice of γ is as the ~~set~~ set consisting of the first positive integer k not already in $\underline{N}$).

(iii) $\underline{K}$ is the set consisting of $\underline{G}$ and its successors.

(iv) $\underline{N}$ is the closure under set union of the sets so generated. It follows, using set theory, that both $\underline{K}$ and $\underline{N}$ are denumerable. The remaining components of the canonical model are defined thus:

$\underline{H_1}\,\underline{R}\,\underline{H_2}$ iff for every $\beta \in \underline{N}$ and every wff C, if $C_\beta \in \underline{H_1}$ then $C_\beta \in \underline{H_2}$, i.e. iff $\underline{H_1} \subseteq \underline{H_2}$;

$\underline{P}(\alpha, \underline{H})$ iff $f_\alpha \notin \underline{H}$, for $\alpha \in \underline{N}$ and $\underline{H} \in \underline{K}$;

$h(p, \alpha, \underline{H}) = T$ iff $p_\alpha \in \underline{H}$, for every atomic wff p, every $\alpha \in \underline{N}$ and every $\underline{H} \in \underline{K}$.

(*) $h(A, \alpha, \underline{H}) = T$ iff $A_\alpha \in \underline{H}$, for every $\alpha \in \underline{N}$ and $\underline{H} \in \underline{K}$.

Proof is by induction from the specific basis.

<u>wff f</u>: $h(f, \alpha, \underline{H}) = T$ iff $\sim \underline{P}(\alpha, \underline{H})$ iff $f_\alpha \in \underline{H}$.

ad &: $h(A \& B, \alpha, \underline{H}) = T$ iff $h(A, \alpha, \underline{H}) = T = h(B, \alpha, \underline{H})$
iff $A_\alpha \in \underline{H}$ and $B_\alpha \in \underline{H}$, by induction hypothesis
iff $A \& B_\alpha \in \underline{H}$, since $\underline{H}$ is R_f-ok.

ad $\to$: (1) if $B \to C_\alpha \in \underline{H}$ and $\underline{H} R \underline{H}'$ then $B \to C_\alpha \in \underline{H}'$; so if $B_\beta \in \underline{H}'$ then, since $\underline{H}'$ is R_f-ok, $C_{\alpha+\beta} \in \underline{H}'$ for any $\alpha, \beta \in \underline{N}$. Then if $B \to C_\alpha \in \underline{H}$, then $h(B \to C, \alpha, \underline{H}) = T$, using the induction hypotheses and applying quantification logic.

(2) if $B \to C_\alpha \notin \underline{H}$ for $\alpha \in \underline{N}$ then by the construction above there is an $\underline{H}' \in \underline{K}$ and $\beta \in \underline{N}$ such that $\underline{H} R \underline{H}'$ and $B_\beta \in \underline{H}'$ while $C_{\alpha+\beta} \notin \underline{H}'$. Hence, using the induction hypothesis, $h(B \to C, \alpha, \underline{H}) \neq T$.

(†) M is an R-model.

It is immediate that $\underline{G} \in \underline{K}$, $0 \in \underline{N}$ and that $\underline{N}$ is a set of well-ordered [illegible]. Moreover since $\underline{R}$ is an inclusion relation $\underline{R}$ is reflexive and transitive and the hereditariness requirement is satisfied. As to the falsity requirement, suppose $A_\alpha \notin \underline{H}$. Then $(A \to f \to f)_\alpha \notin \underline{H}$, so for some $\beta \in \underline{N}$ and $\underline{H}_1 \in \underline{K}$ $\underline{H} R \underline{H}_1$ and $f_{\alpha+\beta} \notin \underline{H}_1$; thus for some β and $\underline{H}_1$ $\underline{H} R \underline{H}_1$ and $P(\alpha+\beta, \underline{H}_1)$. Finally as to the reduction requirement, suppose $(\alpha, \underline{H}_1) \neq (\gamma, \underline{H}_2)$, then for some wff A, $A_\alpha \notin \underline{H}_1$ and $A_\gamma \in \underline{H}_2$ (say; the other case is similar). Accordingly $(A \to f \to f)_\alpha \notin \underline{H}_1$, whence for some $\beta \in \underline{N}$ and $\underline{H} \in \underline{K}$, $\underline{H}_1 R \underline{H}$ and $A \to f_\beta \in \underline{H}$ and $f_{\alpha+\beta} \notin \underline{H}$. Since $A \to f_\beta \in \underline{H}$ and $A_\gamma \in \underline{H}_2$ either $\sim \underline{H} R \underline{H}_2$ or $f_{\beta+\gamma} \in \underline{H}_2$. Thus it is false for every β and every $\underline{H}$ that if $\underline{H}_1 R \underline{H}$ and $P(\alpha+\beta, \underline{H})$ then $\underline{H} R \underline{H}_2$ and $P(\beta+\gamma, \underline{H}_2)$.

Applying (*) since $A_0 \notin \underline{G}$, $h(A, 0, \underline{G}) = F$, and for $B_0 \in \Gamma$, $B_0 \in \underline{G}$, so $h(B, 0, \underline{G}) = T$. Hence, since by (†) the canonical model M is an R-model, M satisfies Γ and falsifies A.

(ii) If A is not a theorem of R_f then A_0 is not R_f provable from the null set of hypotheses Δ_0. Hence, by (i), there is an R-model M which falsifies A, so A is not R_f-valid.

(iii) If wff A of R is R-valid then (see §2) A is R_f-valid, so, by (ii), A is a theorem of R_f. Hence, since A is a wff of R and R is a conservative extension of R_f (see Meyer [ ]), A is a theorem of R.

(iv) As for (ii) but ~~all~~ all statements and conditions concerning f and $\underline{P}$ are deleted.

A direct proof of the completeness of R may be got as follows:

~~Theorem~~

Completeness & Skolem-Lowenheim theorems for R.

(i) If A_0 is not R-provable from Γ_0 then there is an R-model $M = \langle \underline{G}, \underline{K}, \underline{R}, 0, \underline{N}, \underline{P}, h \rangle$ with $\underline{K}$ and $\underline{N}$ denumerable which satisfies Γ and falsifies A.

(ii) If A is R-valid then $\vdash_R A$.

Proof of (i) varies the proof of the preceding theorem at these points. Since the primitive set $\{\rightarrow, \sim, \&\}$ replaces the primitive set $\{\rightarrow, f, \&\}$ of R_f, f is not a wff of R. $\underline{P}$ is redefined thus:

$\underline{P}(\alpha, \underline{H})$ iff for every wff C, $\sim(C \rightarrow C)_\alpha \notin \underline{H}$

& In terms of R, $f = \sim(p)(p \rightarrow p)$.

The induction step for $\sim$ in (*) is proved as follows:

(1) Suppose $\sim A^{*}_{\alpha} \in \underline{H}$ and $\underline{H} R \underline{H_1}$ and $A_\beta \in \underline{H_1}$.
Then since $\vdash_R \sim A \rightarrow . A \rightarrow \sim(A \rightarrow A)$, $A \rightarrow \sim(A \rightarrow A)_\alpha \in H$.
Since $\underline{H} R \underline{H_1}$ and $A_\beta \in \underline{H_1}$, $\sim(A \rightarrow A)_{\alpha+\beta} \in \underline{H_1}$; hence $\sim P(\alpha+\beta, \underline{H_1})$.
Hence $\sim A_\alpha \in \underline{H} \supset . \ \underline{H} R \underline{H_1} \ \& \ P(\alpha+\beta, \underline{H_1}) \supset . \ A_\beta \notin \underline{H_1}$,
whence by the induction hypothesis and quantification logic,
$\sim A_\alpha \in \underline{H} \supset . \ L(\sim A, \alpha, \underline{H}) = T$.

(2) Suppose $\sim A_\alpha \notin \underline{H}$; then $A \rightarrow \sim(D \rightarrow D)_\alpha \notin \underline{H}$ for arbitrary D.
Hence for some $\underline{H_1}$ and β, $\underline{H} R \underline{H_1}$ and $A_\beta \in \underline{H_1}$ and
$\sim(D \rightarrow D)_{\alpha+\beta} \notin \underline{H_1}$, i.e. $\underline{P}(\alpha+\beta, \underline{H_1})$. Hence
$L(\sim A, \alpha, \underline{H}) = F$. [The argument requires that for some $\underline{H_1}$ and β for every D — not for every D there is some $\underline{H_1}$ and β — so its validity may be questioned. I think the substitution of $A \rightarrow \sim(p)(p \rightarrow p) \rightarrow . \sim A$ for $A \rightarrow \sim(D \rightarrow D) \rightarrow . \sim A$ makes it plain that the argument is satisfactory. For the sceptical the semantics may be complicated by adding to $\mathfrak{M}$ a class $\underline{X}$ of individuals; by replacing $\underline{P}(\alpha, \underline{H})$ by $\underline{P}(\alpha, \underline{H}, C)$ where $C \in \underline{X}$, and by complicating appropriately the conditions on $\underline{P}$. In the canonical model $\mathfrak{M}$, $\underline{X}$ is defined as the class of all wff and
$\underline{P}(\alpha, \underline{H}, C)$ iff $\sim(C \rightarrow C)_\alpha \notin \underline{H}$.]

Falsity and reduction requirements are reestablished as follows: Suppose $A_\alpha \notin \underline{H}$, then $\sim\sim A_\alpha \notin \underline{H}$, so for some $\beta \in \underline{H}$ and some $\underline{H_1}$, $\underline{H} R \underline{H_1}$ and $P(\alpha+\beta, \underline{H_1})$, as required. Suppose not $(\alpha, \underline{H_1}) \neq (\gamma, \underline{H_3})$. Then $A_\alpha \notin \underline{H_1}$ and $A_\gamma \in \underline{H_3}$ say. Since $A_\alpha \notin \underline{H_1}$, $\sim\sim A_\alpha \notin H_1$, so by (2), for some H and β, $\underline{H_1} R \underline{H}$ and $P(\alpha+\beta, \underline{H})$ but $\sim A_\beta \in \underline{H}$. By (1) then for $\underline{H_3}$ such that $\underline{H} R \underline{H_3}$ and $\underline{P}(\beta+\gamma, \underline{H_3})$, $A_\gamma \notin \underline{H_3}$ contradicting the supposition. In sum, for some H and β, $\underline{H} R \underline{H}$ and $P(\alpha+\beta, \underline{H_1})$ but it is not the case that both $\underline{H} R \underline{H_3}$ and $P(\beta+\gamma, \underline{H_3})$, as required.

Corollary 1. R_f is a conservative extension of R.

A *normalised* R-model M is an R-model M such that $\underline{P}(0, \underline{G})$.

Corollary 2: $\vdash_R A$ $(\vdash_{R_f} A)$ iff A is true in all normalised R-models.

Proof. One half is immediate, by specialisation. For the other, suppose A is not a theorem of R_f (or R). Then A_0 is not R_f-provable from Δ_0. But also $\sim\vdash_{R_f} f$ $(\sim\vdash_R \sim(D \to D)$ for any $D)$; hence f_0 is not R_f-provable from Δ_0. Now let $\underline{G}$ be an R_f-ok set extending Δ_0 which excludes both A_0 and f_0. The remainder of the completeness is just as before: an ~~R_f~~ canonical R-model M with base $\underline{G}$ is constructed. Moreover M is normalised; since $f_0 \notin \underline{G}$, $\underline{P}(0, \underline{G})$.

Corollary 3: (*Meyer-Dunn theorem for R*). Material detachment is admissible for R, i.e. if $\vdash_R A$ and $\vdash_R \sim(A \& \sim B)$ then $\vdash_R B$.

Proof: Suppose A and $\sim(A \& \sim B)$ are theorems of R but B is not. Then there is a normalised R-model M such that $h(A, 0, \underline{G}) = T = h(\sim(A \& \sim B), 0, \underline{G})$ but $h(B, 0, \underline{G}) \neq T$. ~~Since $\underline{P}(0, \underline{G})$, $P(0+0, \underline{G})$, and $h(A \& \sim B, 0, \underline{G}) = F$. Hence $h(A, 0, \underline{G}) = F$, which is impossible, or $h(\sim B, 0, \underline{G}) = F$.~~ Since $h(\sim(A \& \sim B), 0, \underline{G}) = T$, by an earlier lemma either $h(\sim A, 0, \underline{G}) = T$ or $h(B, 0, \underline{G}) = T$. As the second case is impossible, $h(\sim A, 0, \underline{G}) = T$. Since, however, $\underline{P}(0, \underline{G})$, $P(0+0, \underline{G})$, and $h(A, 0, \underline{G}) \neq T$, which is also impossible.

[incomplete: breaks down for +ve pts] 12

<u>Separation theorems for R formulated with</u> $\{\to, \sim, \&\}$ <u>and R_f formulated with</u> $\{\to, f, \&\}$. If A is a theorem of R, or of R_f, and L is a fragment of R ($\{\to\}$, $\{\to \sim\}$, $\{\to \&\}$ fragments), or of R_f ($\{\to\}$, $\{\to, f\}$, $\{\to, \&\}$ fragments) then A is a theorem of L iff A is a wff of L.

~~Sup~~

Proof: Suppose, for the nontrivial half, that A is a wff of L and a theorem of R. Then A is R-valid and, since a wff of L, also L-valid. Hence, by the second part of the completeness theorem, A is a theorem of L, <u>provided</u> L is a ~~positive~~ negative fragment of R.

~~[illegible]~~

Completeness theorem for R^+ and separation theorems for R formulated with $\{\to, \sim, \&, \vee\}$ and R_f formulated with $\{\to, f, \&, \vee\}$. If ~~[illegible]~~ L is a NEGATION fragment of R, of $(\{\to\}, \{\to, \sim\}, \{\to, f\}, \{\to, \&\}, \{\to, \&, \vee\})$ then

(i) if A is a theorem of L, A is L-valid

(ii) if A is a theorem of R, then A is a theorem of L iff A is a wff of L.

(iii) if A is a theorem of R^+ then where L is a fragment of R^+ A is a theorem of L iff A is a wff of L.

Proof of (i). By a lemma there is a prime L-ok set G which includes B_0 w.r.t. but excludes A_0 (delete the requirements which do not apply). Define a canonical model M with base G as before, except for the following points:— $(B \to C)_\beta$ is only considered in case $(B \to C)$ is a subformula of A; and when a new set $H \circ K$ is introduced it is required that the set be a prime L-ok set — which it is guaranteed by Corollary. $h(p, \alpha, H) = T$ iff $p\alpha \in H$ for every atomic component p of A and for f, and for every $\alpha \in H$ and $A \in K$. (*) is proved for subformulae of A. The new step for disjunction follows using primeness.

Proof of (ii) is as before.

It appears from the completeness proof that the ~~[illegible]~~ qualification on the disjunction adding function can be lifted; for at each stage of the construction there is a suitable wff $C \vee p$ (or $f \vee p$) not in H.

Corollary 1. Church's theory of weak implication, R_I, is complete.

2. R_I is the pure implicational part of R.

As to 2. If A is a theorem of R and a wff of R_I then A is a theorem of R_I by the preceding theorem. But it then follows, using a Gentzen formulation of R_I (got from the Kripke formulation of E_I in Belnap & Wallace [ ] by dropping the restriction to entailments on the left of $\vdash\to$; see also Meyer [ ]), Completeness theorem & Skolem-Lowenheim theorem for $\Box R$

(i) If A_0 is not $\Box R$-provable from Γ_0 then there is an $\Box R$-model ~~[illegible]~~ $M = \langle G, K, R, O, N, P, W, L \rangle$ with K and N countable which satisfies Γ and falsifies A.

(ii) If A is $\Box R$-valid then $\vdash_{\Box R} A$.

(iii) $\vdash_{\Box R} A$ iff A is true in every normalized $\Box R$-model.

Proof is like that for R, but K is enlarged as follows:

N1) if for $H_1 \in K$ and $\beta \in N$, $\Box A_\beta \notin H_1$ then by a Lemma there is an $\Box R$-ok set H_2 which contains every wff B_γ such that $\Box B_\gamma \in H_1$ and such that $A_\beta \notin H_2$. $\Box R$-ok sets are of course used in place of R-ok sets. Further

N2) $H_1 W H_2$ iff for every $\alpha \in N$ and every wff B, if $\Box B_\alpha \in H_1$ then, materially, $B_\alpha \in H_2$.

(*) $L(A, \alpha, H) = T$ iff $A_\alpha \in H$, for $\alpha \in N$ and $H \in K$ and $\Box$. If $\Box A_\alpha \in H$ then $L(\Box A, \alpha, H) = T$ by the definition of L and W and by quantification logic. Conversely if $\Box A_\alpha \notin H$ then, by construction, for some H_1, $H W H_1$ and $A_\alpha \notin H_1$, i.e., by the induction hypothesis, $L(A, \alpha, H_1) = F$. Furthermore if $\Box A_\alpha \notin H$ then since H is $\Box R$-ok, $\Box A \to f \to f_\alpha \notin H$ (or $\sim\sim\Box A_\alpha \notin H$), hence for some β and some H_2, $H R H_2$ and $f_{\alpha+\beta} \notin H_2$ i.e. $P(\alpha+\beta, H_2) = T$.

(†) M is an $\Box R$-model.
For W is reflexive and, since $\vdash_{\Box R} \Box A \to \Box\Box A$, transitive. Since R is an inclusion relation, $H_1 R H_2$ and $H_2 W H_3$ imply $H_1 W H_3$.

The remainder of the proof is like that for Rp (or R).

Similar results can be proved for $\Box R5$. In particular using $f \to \Box f$, if $H_1 W H_2$ & $f_\alpha \notin H_2$ then $f_\alpha \notin H_1$, as required.

~~[illegible]~~ The admissibility of Material Detachment ~~[illegible]~~ follows, as before, for both $\Box R$ and $\Box R5$. ~~[illegible] follows as before for $\Box R5$~~. In the case of $\Box R$ however there is one further note because of the presence of $\underline{P}$ in the evaluation function for $\Box$.

<u>Separation theorems for $\Box R$</u>, formulated with $\{\rightarrow, \Box, f, \&\}$ $\{\rightarrow, \Box, \sim, \&\}$, $\{\rightarrow, \Box, f, \&, \vee\}$, $\{\rightarrow, \Box, \sim, \&, \vee\}$. Disjunction is only considered in the fragments $\{\rightarrow, \Box, \&, \vee\}$ and $\{\rightarrow, \&, \vee\}$; otherwise all proper fragments are considered. If A is a theorem of $\Box R$ and L is one of the above fragments of $\Box R$ then A is a theorem of L iff A is a wff of L

Proof:

<u>Case 1</u>. L is a fragment including negation or falsity. Then the proof is as usual.

<u>Case 2</u>. L is a fragment not including negation or falsity

?

Completeness and Skolem-Löwenheim theorems for $E\Delta 5$.

(i) If A_0 is not $E\Delta 5$-provable from Γ_0 then there is an $E\Delta 5$-model $M = \langle \underline{\underline{G}}, \underline{\underline{K}}, \underline{\underline{R}}; \underline{\underline{O}}, \underline{\underline{N}}, \underline{\underline{P}}, \underline{L} \rangle$ with $\underline{\underline{K}}$ and $\underline{\underline{N}}$ denumerable which satisfies Γ and falsifies A

(ii) If A is $E\Delta 5$-valid then $\vdash_{E\Delta 5} A$.

Proof of (i) varies the corresponding result for R_f at these points :- In the construction of $\underline{\underline{K}}$ each new set $\underline{\underline{H_2}}$, which is introduced in order to falsify the assumption off $(B \to C)_\beta$ which is not in $\underline{\underline{H_1}}$, is related to $\underline{\underline{H_1}}$ as follows : if $(D_1 \to D_2)_\delta \in \underline{\underline{H_1}}$ then $(D_1 \to D_2)_\delta \in \underline{\underline{H_2}}$.

Correspondingly $\underline{\underline{R}}$ is defined thus :

$\underline{\underline{H_1 R H_2}}$ iff for every $\beta \in \underline{\underline{N}}$ and every wff $(D_1 \to D_2)$ if $(D_1 \to D_2)_\beta \in \underline{\underline{H_1}}$ then $(D_1 \to D_2)_\beta \in \underline{\underline{H_2}}$. Furthermore $\underline{\underline{P}}(\alpha, \underline{\underline{H}})$ iff $\Delta_\alpha \notin \underline{\underline{H}}$.

(*) $L(A, \alpha, \underline{\underline{H}})$ iff $A_\alpha \in \underline{\underline{H}}$ for $\alpha \in \underline{\underline{N}}$ and $\underline{\underline{H}} \in \underline{\underline{K}}$,

$\underline{\text{add}\ \sim}$. (1) Suppose $\sim A_\alpha \notin \underline{\underline{H}}$. Then $A \to \Delta_\alpha \notin \underline{\underline{H}}$. Hence for some $\underline{\underline{H_1}}$ and β, $\underline{\underline{H R H_1}}$ and $A_\beta \in \underline{\underline{H_1}}$ and $\Delta_{\alpha+\beta} \notin \underline{\underline{H_1}}$, i.e. $\underline{\underline{P}}(\alpha+\beta, H_1)$. Hence $L(\sim A, \alpha, \underline{\underline{H}}) \neq T$

(2) Suppose $\sim A_\alpha \in \underline{\underline{H}}$ and $\underline{\underline{H R H_1}}$ and $\underline{\underline{P}}(\alpha+\beta, \underline{\underline{H_1}})$. Then $\Delta_{\alpha+\beta} \notin \underline{\underline{H_1}}$; and, since $\underline{\underline{H R H_1}}$, $N\Delta_{\alpha+\beta} \notin \underline{\underline{H}}$. For if $NA_\gamma \in \underline{\underline{H}}$ then $(A \to A) \to A_\gamma \in \underline{\underline{H}}$; so if $\underline{\underline{H R H_1}}$ then if $A \to A_0 \in \underline{\underline{H_1}}$, as it does, $A_\gamma \in \underline{\underline{H_1}}$. Finally then $\Delta_{\alpha+\beta} \in \underline{\underline{H}}$, since $\vdash_{E\Delta 5} \Delta \to N\Delta$. The conditions are satisfied to apply a lemma which asserts that A_β is not $E\Delta 5$-provable from any set ∇_1 comprising every entailment in $\underline{\underline{H}}$. By the construction of $\underline{\underline{K}}$ the only $\underline{\underline{H}}$ in $\underline{\underline{K}}$ are obtained by applying a single extension lemma. Hence for any $\underline{\underline{H_1}} \supseteq \nabla_1$ in $\underline{\underline{K}}$, $A_\beta \notin \underline{\underline{H_1}}$. In sum, $\sim A_\alpha \in \underline{\underline{H}}$ implies $L(\sim A, \alpha, \underline{\underline{H}}) = T$.

(+) M is an $E\Delta 5$-model.

Since $\underline{R}$ is an inclusion of entailments relation it is reflexive and transitive. That $P(\alpha, H_2)$ and $H_1 R H_2$ imply $P(\alpha, H_1)$ follows as in (2) above

using $\Delta \to N\Delta$ and $^{\text{the}}_{+}\ NA_p \in \underline{H}_1$ and $\underline{H}_1 R \underline{H}_2$ imply $A_p \in \underline{H}_2$.
Falsity and refutation requirements follow, using the theorem $\sim\sim A \to A$; just as in the case of the system R.

A normalised $E\Delta\mathfrak{F}$-model is an $E\Delta\mathfrak{F}$-model such that $\underline{P}(O, \underline{G})$

Corollaries: $\vdash_{E\Delta\mathfrak{F}} A$ iff A is true in every normalised $E\Delta\mathfrak{F}$-model. Material detachment is admissible for $E\Delta\mathfrak{F}$.

~~A separation theorem for $E\Delta\mathfrak{F}$. If L is one of the following fragments of $E\Delta\mathfrak{F}$; $\{\to\}$, $\{\to, \Delta\}$ $\{\to, \sim, \Delta\}$, $\{\to, \&\}$ then A is a theorem of L iff A is a wff of L.~~

[Premisses presupposed : also presupposed in dependent theorems] 17.

Completeness and Skolem-Löwenheim theorems for E and E^+

(i) If A_0 is not E-provable from Γ_0 then there is an E-model $\mathcal{M} = \langle \underline{G}, \underline{K}, \underline{R}, O, \underline{N}, L \rangle$ with $\underline{K}$ and $\underline{N}$ denumerable which satisfies Γ and falsifies A. Similarly with E^+ for E.

(ii) If A is E-valid then $\vdash_E A$.

Proof of (i) follows the same lines as earlier proofs. $\underline{G} = H_0$ is an the E-ok extension of Γ_0 w.r.t. $\{0\}$ which excludes A_0. Then $\underline{M}$ and $\underline{N}$ are defined jointly, thus:

(i) $\underline{G} \in \underline{M}$ and $\{0\} \in \underline{N}$.

(ii) if for $\underline{H_1} \in \underline{M}$ and $\beta \in \underline{N}$, $(B \to C)_\beta \notin \underline{H_1}$ then by a lemma there is a new (singleton) subscript γ and an E-ok set $\underline{H_2}$ such that $B_\gamma \in \underline{H_2}$, $C_{\beta+\gamma} \notin H_2$ and such that if $(D_1 \to D_2)_\alpha \in \underline{H_1}$ then $(D_1 \to D_2)_\alpha \in \underline{H_2}$: put $\underline{H_2} \in \underline{M}$ and $\gamma \in \underline{N}$.

(iii) $\underline{M}$ is the set consisting of $\underline{G}$ and its successors under (ii)

(iv) $\underline{N}$ is the closure under set union of elements assigned to it.

Further: if $\underline{H_i} \in \underline{M}$ then $\underline{T_i} \in \overline{\underline{M}}$, where for every $\beta \in \underline{N}$ ~~A_β~~ and every wff A, $A_\beta \in \underline{T_i}$ iff $\sim A_\beta \notin \underline{H_i}$.

$\underline{K} = \underline{M} \cup \overline{\underline{M}}$

$\underline{H_1} \underline{R} \underline{H_2}$ iff for every $\beta \in \underline{N}$ and every wff $(D_1 \to D_2)$ if $(D_1 \to D_2)_\beta \in \underline{H_1}$ then $(D_1 \to D_2) \in \underline{H_2}$.

$L(p, \alpha, \underline{I}) = T$ iff $p_\alpha \in \underline{I}$ for every atomic wff p, every $\alpha \in \underline{N}$ and $\underline{I} \in \underline{K}$; and $L(D_1 \to D_2, \alpha, \underline{J}) = T$ iff $(D_1 \to D_2)_\alpha \in \underline{J}$ for every wff $D_1 \to D_2$, every $\alpha \in \underline{N}$ and $\underline{J} \in \overline{\underline{M}}$.

(*) $h(A, \alpha, \underline{\underline{I}}) = T$ iff $A_\alpha \in \underline{\underline{I}}$ for every wff A, every $\alpha \in \underline{N}$ and $\underline{\underline{I}} \in \underline{K}$.

Proof is by induction from the specified initial cases.

ad &. $h(B \& C, \alpha, \underline{H_i}) = T$ iff $h(B,\alpha,\underline{H_i}) = h(C,\alpha,\underline{H_i}) = T$ is proved as before using induction hypotheses & properties of E-theories.

$h(B \& C, \alpha, \underline{T_i}) = T$ iff $h(B,\alpha,\underline{T_i}) = T = h(C,\alpha,\underline{T_i})$

iff $B_\alpha \in \underline{T_i}$ and $C_\alpha \in \underline{T_i}$

iff $\sim B_\alpha \notin \underline{H_i}$ and $\sim C_\alpha \notin \underline{H_i}$

iff $\sim B_\alpha \vee \sim C_\alpha \notin \underline{H_i}$ by primeness

iff $\sim(B \& C) \notin \underline{H_i}$ since $\vdash_E \sim B \vee \sim C \leftrightarrow \sim(B \& C)$

iff $B \& C \in \underline{T_i}$

ad ∨. Similar in principle to the &-case.

ad ~. $h(\sim A, \alpha, \underline{H_i}) = T$ iff $h(A,\alpha,\underline{T_i}) \neq T$

iff $A_\alpha \notin \underline{T_i}$

iff $\sim A_\alpha \in \underline{H_i}$

$h(\sim A, \alpha, \underline{T_i}) = T$ iff $A_\alpha \in \underline{H_i}$

iff $\sim\sim A_\alpha \in \underline{H_i}$ by $\vdash_E \sim\sim A \leftrightarrow A$

iff $\sim A_\alpha \notin \underline{T_i}$

ad →. $h(B \to C, \alpha, \underline{I}) = T$ iff $B \to C_\alpha \in \underline{I}$, by [illegible]

(1) If $B \to C_\alpha \in \underline{H}$, then, if $\underline{H} R \underline{H'}$ and $A_\beta \in \underline{H'}$ then $C_{\alpha+\beta} \in \underline{H'}$ — is proved as before. Also if $B \to C_\alpha \in \underline{H}$ then $\sim C \to \sim B_\alpha \in \underline{H}$ since $\vdash_E B \to C \to. \sim C \to \sim B$. So similarly if $\underline{H} R \underline{H'}$ then $\sim C_\beta \in \underline{H'}$ materially implies $\sim B_{\alpha+\beta} \in \underline{H'}$, i.e. $B_{\alpha+\beta} \in \underline{T'}$ materially implies $C_\beta \in \underline{T'}$. Finally, since $\vdash_E B \to C \to. \sim B \vee C$, if $B \to C_\alpha \in \underline{H}$ then $\sim B \vee C_\alpha \in \underline{H}$, so $\sim B_\alpha \in \underline{H}$ or $C_\alpha \in \underline{H}$, whence $B_\alpha \notin \underline{T}$ or $C_\alpha \in \underline{H}$ and $h(B,\alpha,\underline{T}) \neq T$ or $h(C,\alpha,\underline{H}) = T$.

(2) If $B \to C_\alpha \notin \underline{H}$ then by construction for some $\underline{H'} \in \underline{K}$ and $\beta \in \underline{N}$ $B_\beta \in \underline{H'}$, $C_{\alpha+\beta} \notin \underline{H'}$ and $\underline{H} R \underline{H'}$. Also if $B \to C_\alpha \notin \underline{H}$ then $\sim C \to \sim B_\alpha \notin \underline{H}$; thus, by the construction, for some $\underline{H''} \in \underline{K}$ and $\gamma \in \underline{N}$, $\underline{H} R \underline{H''}$, $\sim C_\gamma \in \underline{H''}$ and $\sim B_{\alpha+\gamma} \notin \underline{H''}$, i.e. $B_{\alpha+\gamma} \in \underline{T''}$ and $C_\gamma \notin \underline{T''}$.

Much as before

(+) $\mathfrak{M}$ is an E-model.

<u>Completeness and Skolem-Löwenheim theorems for R using R-I-models</u>

Statement & proof are like the preceding result; but note :— Step (ii) in the construction of $\underline{\underline{M}}$ is carried out as for R_+. $\underline{\underline{H_1 R H_2}}$ iff, for every β and every wff C, if $C\beta \in \underline{\underline{H_1}}$ then $C\beta \in \underline{\underline{H_2}}$; $\underline{\underline{T_1 R T_2}}$ iff $\underline{\underline{H_1 R H_2}}$. The model is an R-I-model, since $C\beta \in \underline{\underline{H_1}}$ and $\underline{\underline{H_1 R H_2}}$ automatically implies $C\beta \in \underline{\underline{H_2}}$ in virtue of the definition of $\underline{\underline{R}}$! As for the T-case, if $C\beta \in \underline{\underline{T_2}}$ and $\underline{\underline{T_1 R T_2}}$ then $\sim C\beta \notin \underline{\underline{H_2}}$ and $\underline{\underline{H_1 R H_1}}$, so $\sim C\beta \notin \underline{\underline{H_1}}$, i.e. $C\beta \in \underline{\underline{T_1}}$.

<u>Completeness and Skolem-Löwenheim theorems for $\Box R$ using $\Box R$-I-models</u>

Use $\Box R$-ok sets in place of the R-ok sets of the preceding result, and extend $\underline{\underline{M}}$ by the following step: if $\Box A\beta \notin \underline{\underline{H_1}}$ for $\beta \in \underline{\underline{K}}$ and $\underline{\underline{H_1}} \in \underline{\underline{M}}$, then there is a $\Box R$-ok set $\underline{\underline{H_2}}$, which contains every wff B_γ such that $\Box B_\gamma \in \underline{\underline{H_1}}$, such that $B_\gamma \notin \underline{\underline{H_2}}$: put $\underline{\underline{H_2}} \in \underline{\underline{M}}$.

<u>Translation theorem 1</u>. A is a theorem of E iff its $\Box R$-translation A^+ is a theorem of $\Box R$.
Proof: One half, if $\vdash_E A$ then $\vdash_{\Box R} A^+$, follows by induction over the E-proof of A. As to the other, suppose $\sim\vdash_E A$; then there is an E-model $M = \langle \underline{G}, \underline{K}, \underline{W}, O, \underline{N}, h \rangle$ such that $h(A, O, \underline{G}) = F$. Form a new model $M_1 = \langle \underline{G}, \underline{K}, \underline{Id}, O, \underline{N}, \underline{W}, h \rangle$ where $\underline{Id}$ is the identity relation on $\underline{K}$ and remaining elements are as before. Then M_1 is a $\Box R$-I-model which falsifies A^+; hence $\sim\vdash_{\Box R} A^+$.

<u>Corollary</u> A is a theorem of E_I iff its $\Box R$-translation A^+ is a theorem of $\Box R_I$. (Correct, but proved in Meyer [?])

<u>Translation theorem 2</u> A is a theorem of E^+ iff its $\Box R^+$-translation A^+ is a theorem of $\Box R^+$.

completeness and Skolem-Lowenheim theorems for P and P^+.

Statement & proof is like for that E_λ and E^+ except at the following points:
At point (iii) in the construction of $\underline{\underline{M}}$ it required that $n = \max(Y)$ exceeds every element of β and of α for $(B_1 \to B_2)_\alpha \in \underline{\underline{H}}$.

ad $\to$. If $B \to C_\alpha \in \underline{\underline{H}}$ and $\underline{\underline{H}} R \underline{\underline{H'}}$ and $\max(\beta) \geq \max(\alpha)$ and $A_\beta \in \underline{\underline{H'}}$ then since $B \to C_\alpha \in \underline{\underline{H'}}$, $C_{\alpha\cup\beta} \in \underline{\underline{H'}}$ by the $\to E$ rule for P since $\underline{\underline{H'}}$ is P-ok. The remainder is much as before but taking account of maximization requirements.

Meyer-Dunn theorem for E and P.

(γ) is admissible: i.e. if $\vdash_L A$ and $\vdash_L \sim A \vee B$ then $\vdash_L B$, for where L is P or E.

Proof: Suppose otherwise that in some L-model $h(A, 0, \underline{\underline{G}}) = T = h(\sim A \vee B, 0, \underline{\underline{G}})$ and $h(B, 0, \underline{\underline{G}}) \neq T$. Since $h(\sim A, 0, \underline{\underline{G}}) = T$ or $h(B, 0, \underline{\underline{G}}) = T$, $h(\sim A, 0, \underline{\underline{G}}) = T$, so $h(A, 0, \underline{\underline{T_0}}) = F$. But $h(A \to A, 0, \underline{\underline{G}}) = T$, so that if $h(A, 0, \underline{\underline{T_0}}) = F$ then $h(A, 0, \underline{\underline{G}}) = F$ by the reduction condition. Hence $h(A, 0, \underline{\underline{G}}) = F$, contradicting $h(A, 0, \underline{\underline{G}}) = T$.

Because of the unfortunate way negation and disjunction features are used in showing that the implication relation is correct in the canonical model, a separation theorem is not an immediate corollary of completeness theorems. However some partial results may be obtained by building on independently established results.

Separation theorem for $\{\to,\sim\}$ part, $E_{\overline{I}}$, of E.
If A is a theorem of E then A is a theorem of $E_{\overline{I}}$ iff A is a wff of $E_{\overline{I}}$.
Proof: Suppose A is a theorem of E and a wff of $E_{\overline{I}}$. Then its $\Box R$-translation A^+ is a theorem of $\Box R$; but A^+ is a wff whose only connectives are $\to$, $\sim$ and $\Box$, hence by the separation theorem of $\Box R$, A^+ is a theorem of the $\{\to,\sim,\Box\}$ fragment of $\Box R$, viz of $\Box R_{\overline{I}}$. Then, however, by a result of Meyer [ ], A is a theorem of $E_{\overline{I}}$.

Separation theorem for the pure entailment part, E_I, of E.
If A is a theorem of E then A is a theorem of E_I iff A is a wff of E_I.
Proof: By the previous theorem if A is a theorem of E and a wff of E_I then A is a theorem of $E_{\overline{I}}$. But it follows using the Belnap-Wallace Gentzen formulations of $E_{\overline{I}}$ (in [ ]) that if A is a wff of E_I and a theorem of $E_{\overline{I}}$, A is also a theorem of E_I.

Separation theorem for E^+ and P^+. If A is a theorem of E^+ (P^+) and L is one of these fragments of E (P^+) – $\{\to\}$, $\{\to,\&\}$ – then A is a theorem of L iff A is a wff of L.

§ 5 Decidability

§5.1 . An equivalence class method is used to show that the systems studied have the finite model property (For further details see [ ] and [ ]).

(L) Where Ψ is a set of wff closed under subformulas, define: $(\alpha_1, \underline{\underline{I}}_1) \equiv_{\Psi} (\alpha_2, \underline{\underline{I}}_2)$ iff, for every wff B in Ψ, $\lambda(B, \alpha_1, \underline{\underline{I}}_1) = T$ iff $\lambda(B, \alpha_2, \underline{\underline{I}}_2) = T$. Then $\equiv_{\Psi}$ is an equivalence relation which partitions situations $(\alpha, \underline{\underline{I}})$ into equivalence classes; and there are finitely many equivalence classes when Ψ is finite. Next $(\hat{\underline{\underline{I}}})_{\Psi} =_{Df} \{\underline{\underline{I}}' : (U\alpha \in \underline{N}).\ (\alpha, \underline{\underline{I}}') \equiv_{\Psi} (\alpha, \underline{\underline{I}})\}$ and $(\hat{\alpha})_{\Psi} =_{Df} \{\alpha' : (U\underline{\underline{I}} \in \underline{K}).\ (\alpha', \underline{\underline{I}}) \equiv_{\Psi} (\alpha, \underline{\underline{I}})\}$.

Then relative to a given Ψ, $\hat{\underline{K}} = \{\hat{\underline{\underline{I}}} : \underline{\underline{I}} \in \underline{K}\}$, $\hat{\underline{N}} = \{\hat{\alpha} : \alpha \in \underline{N}\}$. Also $\hat{\lambda}(A, \hat{\alpha}, \hat{\underline{\underline{I}}}) = T$ iff $\lambda(A, \alpha, \underline{\underline{I}}) = T$ and $A \in \Psi$, for every initial case (and thus for every atomic wff A). In the case of system R, $\hat{\underline{\underline{H}}}_1 \hat{R} \hat{\underline{\underline{H}}}_2$ iff for every wff $B \in \Psi$ and every $\alpha \in \underline{N}$, if $\lambda(B, \alpha, \underline{\underline{H}}_1) = T$ then, materially, $\lambda(B, \alpha, \underline{\underline{H}}_2) = T$ and $\hat{\underline{\underline{I}}}_1 \hat{\hat{R}} \hat{\underline{\underline{I}}}_2$ iff $\hat{\underline{\underline{H}}}_2 \hat{R} \hat{\underline{\underline{H}}}_1$. For $\Box R$, $\hat{\underline{\underline{H}}}_1 \hat{\underline{W}} \hat{\underline{\underline{H}}}_2$ iff for every wff $B \in \Psi$ and $\alpha \in \underline{N}$ if $\lambda(\Box B, \alpha, \underline{\underline{H}}_1) = T$ then, materially, $\lambda(B, \alpha, \underline{\underline{H}}_2) = T$. For E and P, $\hat{\underline{\underline{H}}}_1 \hat{R} \hat{\underline{\underline{H}}}_2$ iff for every ~~wff~~ ~~wff~~ $B, C \in \Psi$ and every $\alpha \in \underline{N}$ if $\lambda(B \to C, \alpha, \underline{\underline{H}}_1) = T$ then, materially, $\lambda(B \to C, \alpha, \underline{\underline{H}}_2) = T$. This specification defines a *filtration* $\hat{M} = \langle \hat{G}, \hat{K}, \hat{R}, \hat{O}, \hat{N}, [\hat{W}], \hat{\lambda} \rangle$ of L-model M *through* Ψ, written $\hat{M} = M/\Psi$.

Lemma. Where M is an L-model (for L = R, ΟR, E, or P(?)), then

(i) if $\underline{\underline{H}}_1 R \underline{\underline{H}}_2$ then $\hat{\underline{\underline{H}}}_1 \hat{R} \hat{\underline{\underline{H}}}_2$;

(ii) if $\underline{\underline{H}}_1 W \underline{\underline{H}}_2$ then $\hat{\underline{\underline{H}}}_1 \hat{W} \hat{\underline{\underline{H}}}_2$;

(iii) $\hat{R}$ is reflexive and transitive;

(iv) $\hat{W}$ is reflexive and connective;

(v) where L is R or ΟR and $A \in \Psi$, if $\hat{\underline{\underline{H}}}_1 \hat{R} \hat{\underline{\underline{H}}}_2$ and $\hat{\lambda}(A, \hat{\alpha}, \hat{\underline{\underline{H}}}_1) = T$ then $\hat{\lambda}(A, \hat{\alpha}, \hat{\underline{\underline{H}}}_2) = T$ ~~ff~~ (proving

[illegible]

hence $\hat{M}$ is also an L-model.

Lemma: For every wff $A \in \Phi$, for every $\underline{I} \in \underline{K}$,
$\hat{L}(A, \hat{\alpha}, \hat{\underline{H}}) = T$ iff $L(A, \alpha, \underline{I}) = T$

Proof is by induction on the number of connectives in A. The basis for initial cases is immediate, & the induction steps for $^3\&$, $^{\pm}\vee^+$ and $^{\pm}\sim^+$ are straightforward. The step for $\to$ is based on the fact that $L(A \to B, \alpha, \underline{H}) = T$ iff for every β and $\underline{H}'$, if $\underline{H}R\underline{H}'$ and $L(A, \beta, \underline{H}') = T$ [and $\max(\beta) \geq \max(\alpha)$] then $L(B, \alpha+\beta, \underline{H}') = T$ and similarly for $\hat{R}$. The case for $\underline{T}$ selections is an initial case. If $\hat{L}(B \to C, \hat{\alpha}, \hat{\underline{H}}) = T$ then $L(B \to C, \alpha, \underline{H}) = T$, since $\underline{H}R\underline{H}'$ implies $\hat{\underline{H}}\hat{R}\hat{\underline{H}'}$. Conversely, suppose $\hat{L}(B \to C, \hat{\alpha}, \hat{\underline{H}}) \neq T$. Then for some $\hat{\underline{H}'}$ and some $\hat{\beta}$, $\hat{\underline{H}}\hat{R}\hat{\underline{H}'}$ and $\hat{L}(B, \hat{\beta}, \hat{\underline{H}'}) = T$ and $\hat{L}(C, \widehat{\alpha+\beta}, \hat{\underline{H}'}) = F$, whence $\hat{\underline{H}}\hat{R}\hat{\underline{H}'}$ and $L(B, \beta, \underline{H}') = T$ and $L(C, \alpha+\beta, \underline{H}') = F$, by the induction hypothesis. Hence, using the definition of $\hat{R}$, $L(B \to C, \alpha, \underline{H}) \neq T$.

Decidability theorems.

(i) If wff A is false in L-model M then, where (H) is the subformulas closure of A, A is false in L-model $M/(H)$;

(ii) L has the finite model property, and accordingly is decidable, and therefore

(iii) E, P and R and their sizable fragments are decidable. (iv) E^+, P^+ and R^+ and their fragments are decidable.

Proof of (i). Applying previous results $\hat{M} = M/(H)$ is an L-model, and $\hat{L}(A, \hat{0}, \hat{\underline{G}}) = F$. Further $\hat{R}$ and $\hat{D}$ are finite since there are only finitely many equivalence classes $\langle \alpha, \underline{I} \rangle$ when (H) is finite

How convincing!

§5.3 Decidability using

b. Simplified R_f

Where Ψ is a set of formulae closed under subformulae and including f, define

$\alpha_1 \equiv_\Psi \alpha_2$ iff for every wff $B \in \Psi$, $h(B,\alpha_1) = h(B,\alpha_2)$;

$(\hat{\alpha})_\Psi = \{\alpha' : \alpha' \equiv_\Psi \alpha\}$.

A filtration $\hat{M} = M/\Psi = \langle \hat{\underline{N}}, \hat{0}, \hat{\underline{P}}, \hat{h} \rangle$ ~~[illegible]~~ of M through Ψ is defined as follows (relative to a given Ψ).

$\hat{N} = \{\hat{\alpha} : \alpha \in \underline{N}\}$; $\hat{\alpha} \hat{+} \hat{\beta} = \widehat{\alpha+\beta}$.

$\hat{P}(\hat{\alpha})$ iff $h(f, \alpha) = F$? (since $f \in \Psi$ always);

$\hat{h}(p, \hat{\alpha}) = T$ iff $h(p,\alpha) = T$ & $p \in \Psi$

Lemma. Where M is a simplified R-model, $\hat{M}$ is also.

Lemma: For every wff $A \in \Psi$,

$\hat{h}(A, \hat{\alpha}) = h(A, \alpha)$.

Proof is by induction from the following dual basis;

$\hat{h}(p, \hat{\alpha}) = T$ iff $h(p,\alpha) = T$ & $p \in \Psi$ iff $h(p,\alpha) = T$

$\hat{h}(f, \hat{\alpha}) = T$ iff $\sim\hat{P}(\hat{\alpha})$ iff $h(f,\alpha) = T$

& step is immediate;

$\rightarrow$ $\hat{h}(A \rightarrow B; \hat{\alpha}) = T$ iff, for every $\hat{\beta} \in \hat{N}$, $\hat{h}(A,\hat{\beta}) = T \supset \hat{h}(B, \widehat{\alpha+\beta}) = T$

iff for every $\beta \in N$, $h(A,\beta) = T \supset h(B, \alpha+\beta) = T$

by applying the induction hypothesis,

iff $h(A \rightarrow B, \alpha) = T$.

Theorem (Decidability for R_f and R)

§ 8. Semantic tableaux for the systems.

A tableau construction for a wff A (i.e. $A_{\square}$) is begun by putting $A_{\square}$ in the right column of the two columns of the main tableau G of the construction (The exposition presupposes the work of Kripke; see especially [ ], p. 72 ff). The construction is continued, in the case of wff of E (in form $\square R$) and its fragments, by applying the following rules for any tableau H and any subscript α :—

&l: If $(A \& B)_\alpha$ is on the left of H, put both A_α and B_α on the left of H.

&r. If $(A \& B)_\alpha$ is on the right of H, put either A_α on the right of H or B_α on the right of H. In such a case the tableau is replaced by alternative cases (in a way well explained in [ ]).

vl. If $(A \vee B)_\alpha$ is on the left of H, put either A_α on the left of H or B_α on the left of H.

vr. If $(A \vee B)_\alpha$ is on the right of H, put ~~either A_α on the the~~ both A_α and B_α on the right of H.

→l. If $(A \to B)_\alpha$ is on the ~~right~~ left of H, for every tableau H' such that HRH', put either A_β on the right of H' or $B_{\alpha+\beta}$ on the left of H', for every subscript β in N.

→r. If $(A \to B)_\alpha$ is on the right of H, begin a new tableau H', with A_β (for new subscript $\beta \in N$) on the left of H' and $B_{\alpha+\beta}$ on the right of H', such that HRH'.

NB These negation rules are not adequate for negation in combination with disjunction: try $B \to C \to . \sim B \vee C$.

<u>$\sim l$</u>. If $\sim A_\alpha$ is on the left of H, put A_γ on the right of H for every γ in N such that $P(\alpha + \gamma, H)$.

<u>$\sim r$</u>. If $\sim A_\alpha$ is on the right of H, put A_γ, with new subscript $\gamma \in N$, on the left of H, and set $P(\alpha + \gamma, H)$ to the left of A_γ.

<u>$\sim\sim r$</u>. If A_α is on the right of H, put $\sim\sim A_\alpha$ on the right of H.

For an E-construction, tableau relation R is assumed to have the same properties as modelling relation $\underline{R}$, i.e. to be reflexive and transitive, P to have the same properties as $\underline{P}$, and subscript operations $+$ to satisfy the same conditions as its modelling correlates. Subscript set N is of course determined by the construction, beginning with element 0 and being enlarged through applications of $\to r$ and $\sim r$; N is closed under operation $+$. For constructions for fragments of E the inapplicable rules are simply deleted. Negation-free constructions have the subformula property. Negation operations are illustrated in the following example:

		$\sim A \to \sim B \to . B \to A_0$		
		G $\downarrow R$		
	$\sim A \to \sim B_\alpha$	$B \to A_\alpha$		by $\to r$
		H_1 $\downarrow R$		
	B_β	$A_{\alpha+\beta}$		by $\to r$
	A	$\sim\sim A_{\alpha+\beta}$		by $\sim\sim r$
$P(\gamma+\alpha+\beta)$	$\sim A_\gamma$			by $\sim r$
1.	2.	1.	2.	
	$\sim B_{\alpha+\gamma}$	$\sim A_\gamma$		by $\to l$
			R	

Beth's way of setting out alternatives and showing closure have been adopted in the examples, but for more complicated examples it is useful to combine Beth's method with Kripke's method of recopying alternatives (see [ ], p. 74 ff).

For P-constructions, and constructions for systems which eliminate implicative suppression, the implication rules are amended by replacing 'such that HRH'' in each case by 'such that HRH' and max (β) ≥ max (α)'. In the case of wff of R, tableaux constructions corresponding to simplified models are easier. In this case the main tableau G (together with its alternatives) is the only tableau. The rules for conjunction, disjunction and negation remain as before except that 'H' is replaced throughout by 'the tableau' and deleted from P(Y,H). The implication rules are as follows:

$\underline{\to l}$. If $(A\to B)_\alpha$ is on the left of the tableau put either A_β on the right of the tableau or $B_{\alpha\cup\beta}$ on the left of the tableau, for every β in N.

$\underline{\to r}$. If $(A\to B)_\alpha$ is on the right of the tableau, choose a new subscript β, put A_β on the left of the tableau and $B_{\alpha\cup\beta}$ on the right of the tableau, and put β in N.

A tableau is closed iff some subscripted wff A_β appears on both sides of the tableau, a set of tableaux is closed iff some tableau in it is closed, a system of tableaux iff each of its alternative sets is closed; and a subscripted construction is closed iff at some stage of the construction a closed system of alternative sets appears. To facilitate closure it is required that rules are not applied to subscripted wff occurring in a closed tableau and are not applied in case their result is repetitive, i.e. only repeats an application of the rules that has already been made (perhaps with relettered notation). The presence of A_α on the right of a tableau and A_β on the left does not ensure closure unless $\alpha = \beta$.

It is advantageous to reformulate the rules so that the constructions may be based on a tree relation S instead of on relation R: constructions and formulations based on S are called 'S-constructions' and 'S-formulations'. These reformulations do not, of course, apply to constructions for wff of systems like R when there is no relation R altogether. Consider then an E-S-construction. The construction is begun as before; all the rules are as before except for →r where 'S' replaces 'R' and →l which is altered to:

Small 'S'

$\rightarrow l$. If $(A \rightarrow B)_\alpha$ is on the left of H, put either A_β on the right of H or $B_{\alpha+\beta}$ on the left of H, for each subscript β in N, and put $(A \rightarrow B)_\alpha$ on the left of H′ for any H′ such that H S H′.

The →l rule for S-constructions for other systems where R is both reflexive and transitive is similarly reformulated. The statement as to what is meant by 'alternative sets' also has to be reformulated with 'S' in place of 'R' (for a brief statement see Kripke [ ], p.121). For systems like ET, ET an

D where relation R is reflexive but not transitive the s-formulation is a similar modification of the ~~the~~ original formulation except for the rule $\rightarrow l$ which is altered to:

$\underline{\rightarrow l}$. If $(A \rightarrow B)_\alpha$ is on the left of H, put either A_β on the right of H or $B_{\alpha * \beta}$ on the left of H and of any tableau H' such that HSH', for each subscript β in N.

<u>Lemma</u> The L-s-construction for A is closed iff the L-construction for A is closed, for each system L for which both constructions have been introduced.

Proof consists in showing that one construction can be transformed into the other, and vice versa.

Note how we split $\rightarrow l$ into two parts — as for intuitionistic logic

~~[illegible]~~

Theorem. The L-construction for A is closed iff A is L-valid, for each semantical system L ~~[illegible]~~ introduced.

Proof reduces to two Lemmas, and in each of these the arguments of Kripke ([ ], ~~on~~ pp. 76-79) are ~~[illegible]~~ adapted.

Lemma. If the L-construction for A is closed then A is L-valid.

Proof: Suppose the L-construction is closed but A is not L-valid. Then there is an L-model $M = \langle \underline{G}, \underline{K}, \underline{R}, \underline{N}, h \rangle$ such that $h(A, \{\}, \underline{G}) = F$. Also for each n, at the n^{th} stage of the closed L-construction, there is an alternative set S of the construction and a mapping θ, mapping tableau of S into elements of $\underline{K}$ ~~[illegible]~~ such that

(*) If H is a tableau of S, $\underline{H} = \theta(H)$ and B_α is any wff occurring on the left (right) of H then $h(B, \alpha, \underline{H}) = T$ (F). Furthermore, if H_1 and H_2 are in S and $\underline{H_1} = \theta(H_1)$ and $\underline{H_2} = \theta(H_2)$ then $H_1 R H_2$ implies $\underline{H_1}\ \underline{R}\ \underline{H_2}$.

NB) Proof is by induction on n. For $n = 1$ there is only one tableau G with $A_{\{\}}$ on the right; but $h(A, \{\}, \underline{G}) = F$, as required. For the induction step suppose that, in reaching the $(n+1)$ stage, one of the rules is applied to some tableau H of S, and that (*) holds for all stages up to & including the n^{th}.

(incomplete in detail)

GENERALLY EACH SEMANTICAL SYSTEM YIELDS A CORRESPONDING TABLEAUX SYSTEM.

§7. Deductive tableaux and natural deduction and an alternative route to completeness.

Deductive tableaux are specifically arranged semantic-tableaux in which certain formulae are repeated. The specific arrangement and repetition is required in order that closed tableaux may be mechanically ~~an~~ transformed into Fitch-style natural deduction proofs. (On deductive tableaux and their conversion into natural deduction proofs, see Borth [ ] and [ ] and the papers of Beth ~~[illegible]~~ referred to therein; and on Fitch-style natural deduction proofs for E, P, R and their fragments, see Anderson [ ] and Anderson & Belnap [ ]).

In ~~[illegible]~~ presenting the rules for deductive tableaux the rules and parts of Borth [ ], for classical sentential logic, are adapted. In the rules that follow Δ is the class of all wff in the left column at the stage when the rule in question is applied, and Σ the class of all wff in the right column. Δ', Δ'' are ~~all~~ subclasses of Δ, and Σ, Σ' are subclasses of Σ; each of these classes may be null. Λ is the null class of wff. The rules are given the form of transformation rules. The table~~s~~ on the left of the symbol '$\longrightarrow$' is the table~~s~~ to which the rule listed is applied & the table to the right of '$\longrightarrow$' is the resulting table after application of the rule. On the far right in each case the ensuing natural deduction is displayed. Subsequently natural deduction rules are provided which ensure that the resulting natural deduction is valid providing the dotted vertical lines can be filled in correctly. The deductive tableaux rules stated are those for E and fragments. Qualifications needed for rules for P are stated ~~in~~, where required, in square brackets.

Tableau closure:

$$\begin{array}{c|c} L & R \\ \hline \Delta' & \Sigma' \\ A & A \\ \Delta'' & \end{array} \quad \longrightarrow \quad \begin{array}{c|c} L & R \\ \hline \Delta' & \Sigma' \\ A & A \\ \Delta'' & \\ \hline\hline \end{array}$$

Repetition (Rep):

$$\begin{array}{ll} \Delta' & \\ A & \\ \Delta'' & \\ A & \text{rep} \\ \Sigma' & \end{array}$$

$\rightarrow r$:

$$\begin{array}{c|c} L & R \\ \hline \Delta & \Sigma' \\ & A \rightarrow B_\alpha \\ & \Lambda \end{array} \quad \longrightarrow \quad \begin{array}{c|c} L & R \\ \hline \Delta & \Sigma' \\ & A \rightarrow B_\alpha \\ \cdots\cdots & \cdots\cdots \\ A_\beta & B_{\alpha+\beta} \end{array}$$

(β new)

[For P: β new and $\max(\beta) \geq \max(\alpha)$]

Implication ~~elimination~~ introduction ($\rightarrow I$)

$$\begin{array}{ll} \Delta & \\ \quad A_\beta & \text{Hyp} \\ \quad \vdots & \\ \quad B_{\alpha+\beta} & \\ A \rightarrow B_\alpha & \rightarrow I \\ \Sigma' & \end{array}$$

$\rightarrow l$:

$$\begin{array}{c|c} L & R \\ \hline \Delta' & \Sigma \\ A \rightarrow B_\alpha & \\ \Delta'' & \end{array} \quad \longrightarrow \quad \begin{array}{cc|cc} & L & R & \\ \hline & \Delta' & \Sigma & \\ & A \rightarrow B_\alpha & & \\ \hline 1 & 2 & 1 & 2 \\ & B_{\alpha+\beta} & A_\beta & \end{array}$$

(any β)

Implication elimination ($\rightarrow E$)

$$\begin{array}{ll} \Delta' & \\ A \rightarrow B_\alpha & \\ \Delta'' & \\ \vdots & \\ A_\beta & \\ B_{\alpha+\beta} & \rightarrow E \\ \vdots & \\ \Sigma & \end{array}$$

[For P: for any β with $\max(\beta) \geq \max(\alpha)$]
The particular form of the result of the transformation is selected to fit neatly with the convention for linearizing tableaux that is chosen.

$\rightarrow$ transmission:

$$\begin{array}{c|c} L & R \\ \hline \Delta' & \Sigma' \\ A \rightarrow B_\alpha & \\ \Delta'' & \\ \cdots\cdots & \cdots\cdots \\ \Delta''' & \Sigma'' \end{array} \quad \longrightarrow \quad \begin{array}{c|c} L & R \\ \hline \Delta' & \Sigma' \\ A \rightarrow B_\alpha & \\ \Delta'' & \\ \Delta''' & \Sigma'' \end{array}$$

Reiteration (Reit)

$$\begin{array}{ll} \Delta' & \\ A \rightarrow B_\alpha & \\ \Delta'' & \\ \quad \Delta''' & \\ \quad A \rightarrow B_\alpha & \text{Reit} \\ \quad \Sigma'' & \end{array}$$

&r :

$$\begin{array}{c|c} L & R \\ \hline \Delta & \Sigma' \\ & A\&B_\alpha \\ & \Delta \end{array} \quad \longrightarrow \quad \begin{array}{c|c} L & R \\ \hline \Delta & \Sigma' \\ & A\&B_\alpha \\ \hline \begin{array}{c|c} 1 & 2 \\ & \end{array} & \begin{array}{c|c} 1 & 2 \\ A_\alpha & B_\alpha \end{array} \end{array}$$

(&I)
Conjunction introduction)

$$\begin{array}{l} \Delta \\ \vdots \\ A_\alpha \\ \vdots \\ B_\alpha \\ A\&B_\alpha \ \&I \\ \vdots \\ \Sigma' \end{array}$$

&l :

$$\begin{array}{c|c} L & R \\ \hline \Delta' & \Sigma \\ A\&B_\alpha & \\ \Delta'' & \end{array} \quad \longrightarrow \quad \begin{array}{c|c} L & R \\ \hline \Delta' & \Sigma \\ A\&B_\alpha & \\ \Delta'' & \\ A_\alpha & \\ B_\alpha & \end{array}$$

Conjunction Elimination (&E):

$$\begin{array}{l} \Delta' \\ A\&B_\alpha \\ \Delta'' \\ A_\alpha \ \&E \\ B_\alpha \ \&E \\ \vdots \\ \Sigma \end{array}$$

∨r :

$$\begin{array}{c|c} L & R \\ \hline \Delta & \Sigma' \\ & A\vee B_\alpha \\ & \Delta \end{array} \quad \longrightarrow \quad \begin{array}{c|c} L & R \\ \hline \Delta & \Sigma' \\ & A\vee B_\alpha \\ \hline \begin{array}{c||c} 1 & 2 \\ & \end{array} & \begin{array}{c||c} 1 & 2 \\ A_\alpha & B_\alpha \end{array} \end{array}$$

Disjunction Introduction (∨I)

$$\begin{array}{l} \Delta \\ \vdots \\ A_\alpha \\ A\vee B_\alpha \\ \Sigma' \end{array} \quad \text{OR} \quad \begin{array}{l} \Delta \\ \vdots \\ B_\alpha \\ A\vee B_\alpha \\ \Sigma' \end{array}$$

The double vertical lines indicate that only one of the sub tableaux need be closed for the tableau to be closed. This form is chosen so that deductive tableaux rules are always applied on the right first.

$\vee l$:

$$\begin{array}{c|c} L & R \\ \hline \Delta' & \Sigma' \\ A \vee B_\alpha & C_{\alpha+\beta} \\ \Delta'' & \end{array} \longrightarrow \begin{array}{c|c|c|c} \multicolumn{2}{c|}{L} & \multicolumn{2}{c}{R} \\ \hline \multicolumn{2}{c|}{\Delta'} & \multicolumn{2}{c}{\Sigma'} \\ \multicolumn{2}{c|}{A \vee B_\alpha} & \multicolumn{2}{c}{C_{\alpha+\beta}} \\ \multicolumn{2}{c|}{\Delta'} & \multicolumn{2}{c}{} \\ \hline 1 & 2 & 1 & 2 \\ A_\alpha & B_\alpha & C_{\alpha+\beta} & C_{\alpha+\beta} \end{array}$$

Disjunction Elimination ($\vee E$

Δ'
$A \vee B_\alpha$
Δ''
A_α Hy
$\vdots$
$C_{\alpha+\beta}$
B_α Hy
$\vdots$
$C_{\alpha+\beta}$
$C_{\alpha+\beta}$ $\vee E$
Σ'

$\sim r$:

$$\begin{array}{c|c} L & R \\ \hline \Delta & \Sigma' \\ & \sim A_\alpha \\ & \Delta \end{array} \longrightarrow \begin{array}{cc|c} & L & R \\ \hline & \Delta & \Sigma' \\ & & \sim A_\alpha \\ P(\alpha+\gamma) & A_\gamma & \cancel{P(\alpha+\gamma)} \\ & & \sim A_\alpha \end{array}$$

Negation Introduction ($\sim I$

Δ
$P(\alpha+\gamma)$ A_γ
$\vdots$
$\cancel{P(\alpha+\gamma)}$
$\sim A_\alpha$
$\sim A_\alpha$
Σ'

$\sim\sim r$:

$$\begin{array}{c|c} L & R \\ \hline \Delta & \Sigma' \\ & \sim\sim A_\alpha \\ & \Delta \end{array} \longrightarrow \begin{array}{c|c} L & R \\ \hline \Delta & \Sigma' \\ & A_\alpha \\ & \sim\sim A_\alpha \end{array}$$

Double Negation Elimination ($\sim\sim E$):

Δ
$\vdots$
$\sim\sim A_\alpha$
A_α $\sim\sim E$
Σ'

$\sim l$:

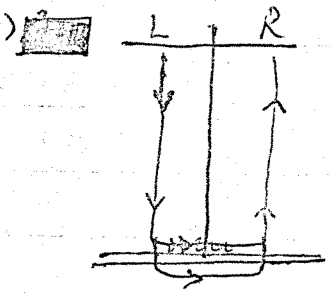
$$\begin{array}{cc|c} & L & R \\ \hline P(\beta+\gamma) & \Delta' & \Sigma' \\ & \sim B_\beta & \cancel{P(\beta+\gamma)} \\ & \Delta'' & \end{array} \longrightarrow \begin{array}{cc|c} & L & R \\ \hline P(\beta+\gamma) & \Delta' & \Sigma' \\ & B_\beta & \cancel{P(\beta+\gamma)} \\ & \Delta'' & B_\gamma \end{array}$$

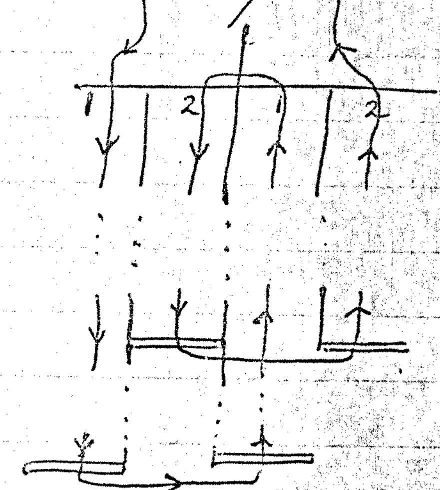
Negation Elimination ($\sim E$):

$P(\beta+\gamma)$ Δ'
$\sim B_\beta$
Δ''
$\vdots$
B_γ
Σ'

Once again the right rules are corresponding.

The general method of linearization of closed deductive tableaux is best indicated diagrammatically:

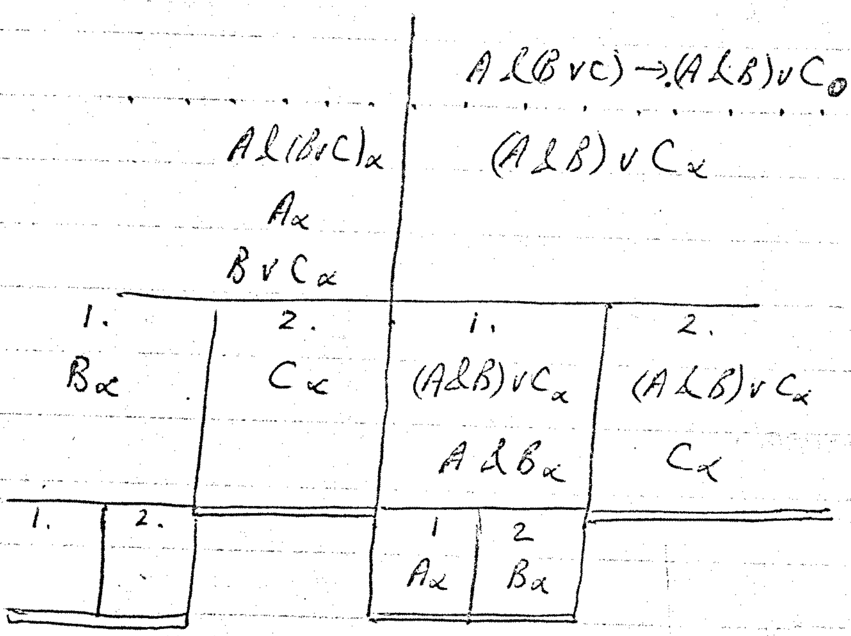
(1) 

(2) 

A closed tableaux is ~~[illegible]~~ transformed to a natural deduction sequence by ~~[illegible]~~ rewriting the formulae in the tableaux in a vertical arrangement in the order in which they appear along the arrow in this linearization diagram, and by inserting in the resulting vertical arrangement hypotheses introduction & removal signs and marginal notes as to natural deduction rules applied. Both of these latter features are listed systematically along with the rules given above for each connective.

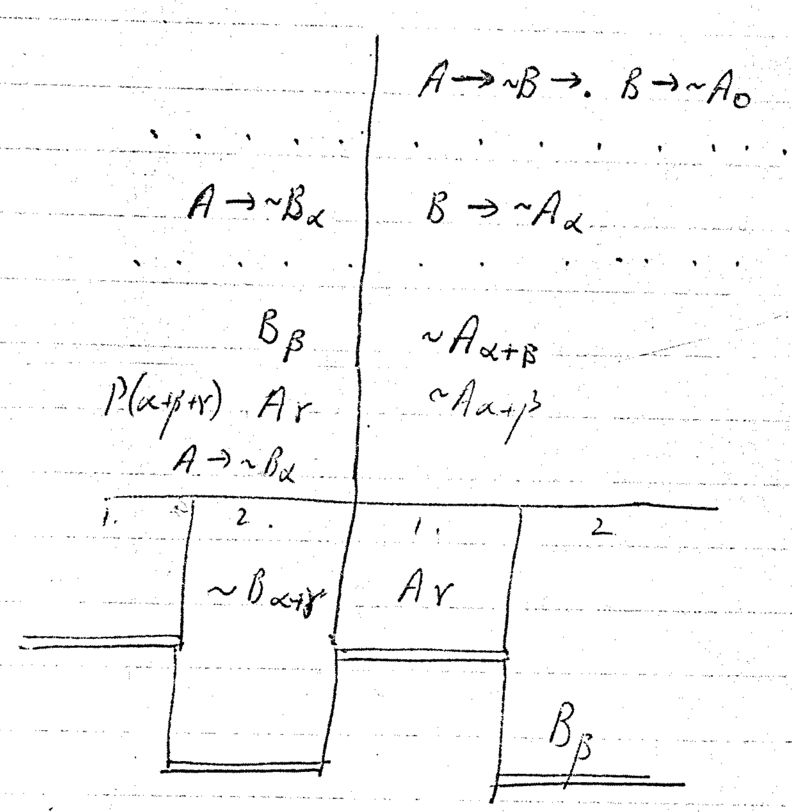
To illustrate two important examples of closed deductive tableaux and their transformations in natural deduction form are given

(1) Distribution (E11).



$A \& (B \vee C)_\alpha$	Hyp
A_α	$\&E$
$(B \vee C)_\alpha$	$\&E$
A_α	rep
B_α	Hyp
A_α	reit
B_α	rep
$(A \& B)_\alpha$	$\&I$
$(A \& B) \vee C_\alpha$	$\vee I$
C_α	Hyp
$(A \& B) \vee C_\alpha$	$\vee I$
$(A \& B) \vee C_\alpha$	$\vee E$
$A \& (B \vee C) \rightarrow . (A \& B) \vee C$	$\rightarrow I$

(2) Contraposition (E13).



$A \rightarrow \sim B_\alpha$	Hyp
B_β	Hyp
$P(\alpha+\beta+\gamma)$ A_γ	Hyp
$A \rightarrow \sim B_\alpha$	reit
A_γ	rep
$\sim B_{\alpha+\gamma}$	$\rightarrow E$
B_β	rep
$\sim A_{\alpha+\beta}$	
$\sim A_{\alpha+\beta}$	$\sim I$
$B \rightarrow \sim A_\alpha$	$\rightarrow I$
$A \rightarrow \sim B \rightarrow . B \rightarrow \sim A_0$	$\rightarrow I$

For R-deductive tableaux, the →-transmission rule is deleted and the implication rule →r is amended to the following:

$\rightarrow r$:

$$\begin{array}{c|l} L & R \\ \hline \Delta & \Sigma' \\ & A \to B_\alpha \\ & \Delta \end{array} \quad \longrightarrow \quad \begin{array}{c|l} L & R \\ \hline \Delta & \Sigma' \\ & A \to B_\alpha \\ A_\beta & B_{\alpha+\beta} \end{array}$$

(β new)

Implication Introduction:

$$\begin{array}{ll} \Delta & \\ \hline A_\beta & \text{Hyp} \\ \vdots & \\ B_{\alpha+\beta} & \\ \hline A \to B_\alpha & \to I \\ \Sigma' & \end{array}$$

The remaining rules are just as for E-deductive tableaux.

Theorem. If the L-s-construction for A is closed then the L-deductive tableaux construction for A is closed.

The Fitch-style natural deduction systems, E^*, P^* and R^*, introduced differ from those of Anderson & Belnap, in particular in that two sorts of hypotheses are admitted. (However the systems of Anderson & Belnap are taken for granted as background knowledge; see especially [ ]). In stating the rules those for E^* are taken as central; and differences and qualifications needed for P^* and R^* are noted where needed. Such standard features of natural deduction systems as vertical arrangement and subproof arrangement are taken for granted.

(i) <u>Structural Rules</u>.

<u>New World Hypotheses</u> (<u>N. Hyp</u>). A step B_α may be introduced as the new world hypothesis of a new subproof, where such new hypothesis B receives a new subscript α from N. The introduction of such ~~an~~ subscripted hypothesis is marked by the sign '⌐ . . . ¬' written above it, and 'N hyp' written to its right; and the hypothesis is eliminated with the paired sign '∟ . . . ⌟'.

<u>Ordinary Hypotheses</u> (<u>O. Hyp</u>). A step B_α may be introduced as the ordinary hypothesis of a new subproof in the application of an extensional logical rule. The introduction of such an hypothesis is marked by the sign '—¬' and its removal by the coupled sign '—⌟'.

<u>Repetition</u>. B_α may be repeated within a proof or subproof.

<u>Reiteration</u>. B_α may be reiterated, retaining its subscript, ;

(i) ordinary hypothetical subproofs, with no restriction;

(ii) new world hypothetical subproofs, provided B has the form $C \to D$.

NC) In the case of R^* B_α may be reiterated into new world hypothetical subproofs whatever its form. Hence the distinction between N hypotheses and O hypotheses largely vanishes in R and by restepping the implication rules N hypotheses can be eliminated altogether from R^*.

(ii) Logical Rules. These rules have already been exhibited schematically. They are, to summarise, as follows:

(iia) Implicational Rules.

$\to$I. From a proof of $B_{\alpha+\beta}$ on ~~[illegible]~~ N hypothesis A_β to infer $A \to B_\alpha$

For P^* it is required that $max(\beta) \geq max(\alpha)$. In the case of R^*, where N hypotheses are eliminated, the rule is modified to:

$\to$I(R). From a proof of $B_{\alpha+\beta}$ on hypotheses A_β ~~β is a new label from N~~ to infer $A \to B_\alpha$, provided β is a new label from N except in the case of $\vee$E and ~I below.

$\to$E. From A_β and $A \to B_\alpha$ to infer $B_{\alpha+\beta}$

For P^* it is required that $max(\beta) \geq max(\alpha)$)

(iib) Extensional Rules.

&I. From A_α and B_α to infer $(A\&B)_\alpha$

&E. From $(A\&B)_\alpha$ to infer both A_α and B_α.

$\vee$I. From A_α to infer $(A \vee B)_\alpha$. From B_α to infer $(A \vee B)_\alpha$

$\vee$E. From $(A \vee B)_\alpha$ and a proof of $C_{\alpha+\beta}$ on O hypothesis A_α and a proof of $C_{\alpha+\beta}$ on O hypothesis B_α to infer $C_{\alpha+\beta}$.

~~~I. From a proof of ~A_α on O hypothesis A_{α^*} to infer ~A_α~~

~~~E. From $(A \to {\sim}B)_\beta$ and B_α to infer ~$A_{\alpha+\beta}$~~

? ~~For P^* it is required that $\alpha \geq \beta$~~

~~E. From ~~A_β to infer A_β.

~I. From a proof of ~A_α on O hypothesis A_γ where $P(\alpha+\gamma)$ to infer ~A_α

~E From ~A_β where $P(\beta+\gamma)$ to infer A_γ

Theorem. If the deductive L-tableau for A is closed then A is a theorem of L^*.

Proof: Transform the deductive L-tableau into vertical form. Then no gaps remain since the L-tableau is closed ~~[illegible]~~, so a proof in L^* results.

Theorem (Anderson completeness): If A is a theorem of L^*, then A is a theorem of L. This may be proved either by the methods of Anderson [ ], or using deduction theorems. Note the the same sort of proof could be applied directly to deductive tableaux.

So far for ordinary systems only.

§4. <u>Independent Gentzen formulations of the positive systems</u>

A GENTZEN FORM OF R^+ R^+_A

<u>Axiom rules</u>: $C_\alpha \Vdash C_\alpha$

In the following formulation Γ, Θ etc. are sets of subscripted wff. A singular formulation is given.

<u>Structural rules</u> in antecedent

Weakening (Thinning):
$$\frac{\Gamma \Vdash D_\delta}{C_\alpha, \Gamma \Vdash D_\delta}$$

Contraction:
$$\frac{C_\alpha, C_\alpha, \Gamma \Vdash D_\delta}{C_\alpha, \Gamma \Vdash D_\delta}$$

Interchange:
$$\frac{\Delta, C_\alpha, D_\beta, \Theta \Vdash E_\delta}{\Delta, D_\beta, C_\alpha, \Theta \Vdash E_\delta}$$

<u>Logical rules</u>:

in succedent   in antecedent

$\underline{\to}$
$$\frac{A_\alpha, \Gamma \Vdash B_\beta}{\Gamma \Vdash A \to B_{\beta-\alpha}}, \qquad \frac{\Delta \Vdash A_\alpha \quad B_{\alpha\cup\beta}, \Gamma \Vdash D_\delta}{A \to B_\beta, \Delta, \Gamma \Vdash D_\delta}$$

<u>provided</u> $\alpha \neq 0$, $\alpha \subseteq \beta$, $\alpha \notin \delta$ for $C_\delta \in \Gamma$

$\underline{\&}$
$$\frac{\Gamma \Vdash A_\alpha \quad \Gamma \Vdash B_\alpha}{\Gamma \Vdash (A \& B)_\alpha} \qquad \frac{A_\alpha, \Gamma \Vdash D_\delta}{(A \& B)_\alpha, \Gamma \Vdash D_\delta} \qquad \frac{B_\alpha, \Gamma \Vdash D_\delta}{(A \& B)_\alpha, \Gamma \Vdash D_\delta}$$

$\underline{\vee}$
$$\frac{\Gamma \Vdash A_\alpha}{\Gamma \Vdash (A \vee B)_\alpha} \qquad \frac{\Gamma \Vdash B_\alpha}{\Gamma \Vdash (A \vee B)_\alpha} \qquad \frac{A_\alpha, \Gamma \Vdash D_\delta \quad B_\alpha, \Gamma \Vdash D_\delta}{(A \vee B)_\alpha, \Gamma \Vdash D_\delta}$$

~~2 forms of $\Vdash\to$:~~
$$\frac{\Gamma \Vdash A_\alpha \quad B_{\alpha\cup\beta}, \Gamma \Vdash D_\delta}{A \to B_\beta, \Gamma \Vdash D_\delta}$$

<u>Cut</u>:
$$\frac{\Delta \Vdash C_\delta \qquad C_\delta, \Gamma \Vdash D_\gamma}{\Delta, \Gamma \Vdash D_\gamma}$$

A CUT-FREE REFORMULATION. R^+_*

The following changes are made to the first formulation.

<u>Weakening</u>:
$$\frac{\Gamma \Vdash D_\delta}{C_\alpha, \Gamma \Vdash D_\delta}\text{, provided } \alpha \subseteq \delta$$

$\Vdash\to$
$$\frac{A_\alpha, \Gamma \Vdash B_\beta}{\Gamma \Vdash A\to B_{\beta-\alpha}}$$
provided $\alpha \neq 0$, $\alpha \subseteq \beta$, α disj δ for $C_\delta \in \Gamma$

$\to\Vdash$
$$\frac{\Gamma \Vdash A_\alpha \qquad B_{\alpha+\beta}, \Gamma \Vdash D_\delta}{A\to B_\beta, \Gamma \Vdash D_\delta}$$

<u>Cut</u> is omitted.

<u>Lemma</u>: If $\Gamma \Vdash D_\delta$ and $C_\alpha \in \Gamma$ then $\alpha \subseteq \delta$.

Proof is by induction over the rules. The only case that is not immediate is $\to\Vdash$. Suppose $C_\gamma \in \Gamma$. Then $\gamma \subseteq \beta$. But also $\alpha \subseteq \beta$ and α disj γ; hence $\gamma \subseteq \beta - \alpha$.

<u>Elimination theorem for R^+_*</u>.

If $\Delta \Vdash C_\delta$ and $C_\delta, \Gamma \Vdash D_\gamma$ then $\Delta, \Gamma \Vdash D_\gamma$.

Proof: Cut may be replaced by the following rule, Mix:

$$\frac{\Delta \Vdash M_\delta \qquad \Sigma \Vdash D_\gamma}{\Delta, \Sigma_{M_\delta} \Vdash D_\gamma}\ (Mix),$$

where $M_\delta \in \Sigma$ and sequence Σ_{M_δ} is obtained from Σ by suppressing all occurrences of M_δ.

Cut follows from Mix, and conversely, thus:

$$\cfrac{\cfrac{\Delta \Vdash C_\delta \qquad C_\delta, \Gamma \Vdash D_\gamma}{\Delta, \Gamma_{C_\delta} \Vdash D_\gamma}\ Mix}{\Delta, \Gamma \Vdash D_\gamma}\ ;$$

$$\cfrac{\Delta \Vdash M_\delta \qquad \cfrac{\Sigma \Vdash D_\gamma}{M_\delta, \Sigma_{M_\delta} \Vdash D_\gamma}}{\Delta, \Sigma_{M_\delta} \Vdash D_\gamma}$$

by weakening, since, by hypothesis, $\delta \subseteq \gamma$.

Proof that all uses of Mix ~~...~~ may be ~~replaced~~ eliminated follows Kleene [ ] p. 454 ff. Kleene's definition of rank is modified: M is replaced by M_δ in the definitions, since M_δ is now the mix formula.

The core structure of the double induction is the same as in Moore, but some cases treated by Moore no longer occur in a simpler formulation. In some remaining cases subscript inclusion has to be established. The mix to be eliminated is written:

$$\frac{\Pi \Vdash M_\delta \qquad \Sigma \Vdash D_\gamma}{\Pi, \Sigma_{M_\delta} \Vdash D_\gamma} \quad \text{or briefly} \quad \frac{S_1 \qquad S_2}{S_3},$$

where $M_{\delta_1} \in \Sigma$.

A. Preliminary cases.

Case 1a. $M_\delta \in \Pi$. If $C_\alpha \in \Pi$ then, in view of S_1, $\alpha \leq \delta$, [since $M_\delta \in \Sigma$, $\delta \leq \gamma$; hence $\alpha \leq \delta$] hence S_3 comes from S_2 by weakening.

Case 2a does not occur, but case 2b does:—

$$\frac{\Pi \Vdash M_\delta \qquad \dfrac{a \Vdash D_\gamma}{M_\delta, a \Vdash D_\gamma}\,T}{\Pi, a_{M_\delta} \Vdash D_\gamma}\,\text{Mix} \qquad \text{with } M_\delta \notin a$$

Since $\Pi \Vdash M_\delta$, if $C_\alpha \in \Pi$, $\alpha \leq \gamma$. Thus the proof figure may be modified to eliminate Mix, thus

$$\frac{a \Vdash D_\gamma}{\Pi, a_{M_\delta} \Vdash D_\gamma}\,T \qquad (a_{M_\delta} = a).$$

B. Further cases. These cases differ from those in Moore only [in] the matter of showing that relevant conditions are satisfied. Main examples:—

B1, where rank is 2.

Case 3.

$$\frac{\dfrac{A_\alpha, \Pi \Vdash B_{\alpha\gamma\beta}}{\Pi \Vdash A \to B_\beta}\,(\text{with conditions}) \qquad \dfrac{\Gamma \Vdash A_\alpha \qquad B_{\alpha\gamma\beta}, \Gamma \Vdash D_\delta}{A \to B_\beta, \Gamma \Vdash D_\delta}}{\Pi, \Gamma \Vdash D_\delta}\,\text{Mix}$$

Since $\alpha \leq \beta + \alpha \leq \delta$ the figure may be amended to (after change of subscript pattern)

$$\frac{\dfrac{\Gamma \Vdash A_\alpha \qquad A_\alpha, \Pi \Vdash B_{\alpha\gamma\beta}}{\Gamma, \Pi_{A_\alpha} \Vdash B_{\alpha\gamma\beta}}\,\text{Mix} \qquad B_{\alpha\gamma\beta}, \Gamma \Vdash D_\delta}{\dfrac{\Gamma, \Pi_{A_\alpha}, \Gamma_{B_{\alpha\gamma\beta}} \Vdash D_\delta}{\Pi, \Gamma \Vdash D_\delta}}\,\text{Mix}$$

Further detail to handle (see: Sec 3.)

B2, where rank exceeds 2.

Then the left rank ≥ 2, so M_δ occurs in the succedent of at least one of the premisses for the inference of S_1.

Case 4a. S_1 is by an antecedent structural rule S.

Case 4. (a) (b) similar.

$$\dfrac{\dfrac{\Pi \Vdash A_\alpha \qquad \Pi \Vdash B_\alpha}{\Pi \Vdash (A \& B)_\alpha} \qquad \dfrac{A_\alpha, \Gamma \Vdash D_\delta}{(A \& B)_\alpha, \Gamma \Vdash D_\delta}}{\Pi, \Gamma \Vdash D_\delta}\ \text{Mix}$$

Hence $\alpha \leq \delta$

Alter to:

$$\dfrac{\dfrac{\Pi \Vdash A_\alpha \qquad A_\alpha, \Gamma \Vdash D_\delta}{\Pi, \Gamma_{A_\alpha} \Vdash D_\delta}\ \text{Mix}}{\Pi, \Gamma \Vdash D_\delta}$$

using $\alpha \leq \delta$

Case 5 (a)

$$\dfrac{\dfrac{\Pi \Vdash A_\alpha}{\Pi \Vdash (A \vee B)_\alpha} \qquad \otimes \dfrac{\Gamma, A_\alpha \Vdash D_\delta \qquad \Gamma, B_\alpha \Vdash D_\delta}{\Gamma, A \vee B_\alpha \Vdash D_\delta}}{\Pi, \Gamma \Vdash D_\delta}\ \text{Mix}$$

$\otimes$ + conditions.

Alter to:

$$\dfrac{\dfrac{\Pi \Vdash A_\alpha \qquad \Gamma, A_\alpha \Vdash D_\delta}{\Pi, \Gamma_{A_\alpha} \Vdash D_\delta}\ \text{Mix}}{\Pi, \Gamma \Vdash D_\delta}$$

Hence $\alpha \leq \delta$, so weakening is permissible.

Thus all conditions are met automatically.

Case 3. General form

$$\dfrac{\dfrac{A_\gamma, \Pi \Vdash B_{\gamma+\beta}}{\Pi \Vdash A \to B_\beta} \qquad \dfrac{\Gamma \Vdash A_\alpha \qquad B_{\alpha+\beta}, \Gamma \Vdash D_\delta}{A \to B_\beta, \Gamma \Vdash D_\delta}}{\Pi, \Gamma \Vdash D_\delta}\ \text{Mix}$$

By hypothesis there is a proof without mix of $A_\gamma, \Pi \Vdash B_{\gamma+\beta}$, where ξ disj γ for every $C_\xi \in \Pi$. If $C_\alpha \in \Pi$ for some C choose a new distinct subscript η and change α to η throughout the proof of $A_\gamma, \Pi \Vdash B_{\gamma+\beta}$. As for Case 3.5 is there the new figure is a proof. The procedure eliminates all occurrences of α from Π's subscripts. Finally change γ to α throughout the proof figure. Then the same figure as before, only relettered, provides a proof of $A_\alpha, \Pi \Vdash B_{\alpha+\beta}$, satisfying the conditions for $\Vdash\to$.

Then the figure on the left is replaced by the figure on the right

$$\dfrac{\dfrac{\Pi_1 \Vdash M_\delta}{\Pi \Vdash M_\delta}\,S \qquad \Sigma \Vdash D_\gamma}{\Pi, \Sigma_{M_\delta} \Vdash D_\gamma}\,Mix \qquad\qquad \dfrac{\dfrac{\Pi_1 \Vdash M_\delta \qquad \Sigma \Vdash D_\gamma}{\Pi_1, \Sigma_{M_\delta} \Vdash D_\gamma}\,Mix}{\Pi, \Sigma_{M_\delta} \Vdash D_\gamma}$$

In this case the rule S is weakening by C_α, note that $\alpha \le \delta \le \gamma$, since $M_\delta \in \Sigma$. The new figure reduces the rank of the mix by one.

Case IIa. S_1 is by a one premise logical rule L, either $\Vdash \to$, $\& \Vdash$, or $\Vdash \vee$. Since the formulation is ~~single~~ singular and the left rank ≥ 2 only the following case can occur with $\Lambda_\alpha = A_\alpha$ or B_α. All similar cases on the right.

$$\dfrac{\dfrac{\Lambda_\alpha, \Gamma \Vdash M_\delta}{(A\&B)_\alpha, \Gamma \Vdash M_\delta}\,\&\Vdash \qquad \Sigma \Vdash D_\gamma}{A\&B_\alpha, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\,Mix \qquad\qquad \dfrac{\dfrac{\Lambda_\alpha, \Gamma \Vdash M_\delta \qquad \Sigma \Vdash D_\gamma}{\Lambda_\alpha, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\,Mix}{(A\&B)_\alpha, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\,\&\Vdash$$

Case IIb. S_1 is by a two-premise logical rule L, either $\to\Vdash$, or $\vee\Vdash$, since $\Vdash\&$ is impossible.

$\vee\Vdash$:

$$\dfrac{\dfrac{A_\alpha, \Gamma \Vdash M_\delta \qquad B_\alpha, \Gamma \Vdash M_\delta}{(A\vee B)_\alpha, \Gamma \Vdash M_\delta}\,L \qquad \Sigma \Vdash D_\gamma}{(A\vee B)_\alpha, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\,Mix$$

Alter to the following figure which reduces the rank of the mix.

$$\dfrac{\dfrac{A_\alpha, \Gamma \Vdash M_\delta \qquad \Sigma \Vdash D_\gamma}{A_\alpha, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\,Mix \qquad \dfrac{B_\alpha, \Gamma \Vdash M_\delta \qquad \Sigma \Vdash D_\gamma}{B_\alpha, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\,Mix}{(A\vee B)_\alpha, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\,L$$

$\to\Vdash$:

$$\dfrac{\dfrac{\overset{\Gamma}{\cancel{\Delta}} \Vdash A_\alpha \qquad B_{\alpha+\beta}, \Gamma \Vdash M_\delta}{A\to B_\beta, \cancel{\Delta}, \Gamma \Vdash M_\delta}\,L \qquad \Sigma \Vdash D_\gamma}{A\to B_\beta, \cancel{\Delta}, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\,Mix$$

Alter to:

$$\dfrac{\dfrac{\overset{\Gamma}{\cancel{\Delta}} \Vdash A_\alpha \qquad \dfrac{B_{\alpha+\beta}, \Gamma \Vdash M_\delta \qquad \Sigma \Vdash D_\gamma}{B_{\alpha+\beta}, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\,Mix}{A\to B_\beta, \cancel{\Delta}\Gamma, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\,L}{A\to B_\beta, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\,C$$

B2.2: the right rank ≥ 2, so M_δ occurs in the antecedent of at least one of the premises for the inference. Cases 9b and 10b. S_2 is by an antecedent structural rule S. ~~with (Ms ... Dγ)~~ The figure is amended as on the right.

$$\dfrac{\Pi \Vdash M_\delta \qquad \dfrac{\Sigma_1 \Vdash D\gamma}{\Sigma \Vdash D\gamma}\,S}{\Pi, \Sigma_{M_\delta} \Vdash D\gamma}\,Mix \qquad\qquad \dfrac{\dfrac{\Pi \Vdash M_\delta \qquad \Sigma_1 \Vdash D\gamma}{\Pi, \Sigma_{1M_\delta} \Vdash D\gamma}\,Mix}{\Pi, \Sigma_{M_\delta} \Vdash D\gamma}\,S$$

In case M_δ is an interchanged or contracted formula in application of S, Σ_{1M_δ} is Σ_{M_δ} and the last S step is unnecessary. In case some formula $C\alpha$, not M_δ, is introduced by application of S then $\alpha \leq \gamma$ so S can be applied after Mix.

Case 11b. S_2 is by a one premiss logical rule L. The rule has the form

$$\dfrac{\Lambda_1, \Gamma \Vdash \Omega, \Lambda_2}{\Xi_1, \Gamma \Vdash \Omega, \Xi_2}\,L$$

where each of Λ_1, Λ_2 is either a side formula or empty and one of Ξ_1, Ξ_2 is the principal formula while the other is empty, and at least one of Ω and Λ_2 and of Ω and Ξ_2 is empty.

Subcase 1: Ξ_1 is not M_δ, so $M_\delta \in \Gamma$.

$$\dfrac{\Pi \Vdash M_\delta \qquad \dfrac{\Lambda_1, \Gamma \Vdash \Omega, \Lambda_2}{\Xi_1, \Gamma \Vdash \Omega, \Xi_2}\,L}{\Pi, \Xi_1, \Gamma_{M_\delta} \Vdash \Omega, \Xi_2}\,Mix$$

The altered proof figure is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi \Vdash M_\delta \qquad \Lambda_1, \Gamma \Vdash \Omega, \Lambda_2}{\Pi, \Lambda_1, \Gamma_{M_\delta} \Vdash \Omega, \Lambda_2}\,Mix}{\Lambda_1, \Pi, \Gamma_{M_\delta} \Vdash \Omega, \Lambda_2}\,I}{\Xi_1, \Pi, \Gamma_{M_\delta} \Vdash \Omega, \Xi_2}\,L}{\Pi, \Xi_1, \Gamma_{M_\delta} \Vdash \Omega, \Xi_2}\,I$$

The new mix is of rank one less than the original. It remains to show in case L is $\to\Vdash$ that for each $C\xi \in \Pi$ ~~[illegible]~~ ξ disj α, where $\Lambda_1 = A\alpha$. By [illegible] S_1, $\xi \leq \alpha$ and by the original premiss for the application of L, ξ disj α since $M_\delta \in \Gamma$. Hence ξ disj α.

<u>Subcase 2</u>: Ξ_1 is M_δ. Then Ξ_2 is empty, Ω is $D\gamma$, and Λ_2 is empty. Also Λ_1 is not M_δ, so $M_\delta \in \Gamma$. (Thus L can only be $\& \Vdash$, but the more general case is [illegible] because later we come to extensions of R^+).

$$\frac{\Pi \Vdash M_\delta \qquad \dfrac{\Lambda_1, \Gamma \Vdash D\gamma}{M_\delta, \Gamma \Vdash D\gamma}\,L}{\Pi, \Gamma_{M_\delta} \Vdash D\gamma}\,Mix$$

Amend to:

$$\frac{\Pi \Vdash M_\delta \qquad \dfrac{\dfrac{\dfrac{\Pi \Vdash M_\delta \qquad \Lambda_1, \Gamma \Vdash D\gamma}{\Pi, \Lambda_1, \Gamma_{M_\delta} \Vdash D\gamma}\,Mix}{\Lambda_1, \Pi, \Gamma_{M_\delta} \Vdash D\gamma}\,I}{M_\delta, \Pi, \Gamma_{M_\delta} \Vdash D\gamma}\,L}{\dfrac{\Pi, \Pi, \Gamma_{M_\delta} \Vdash D\gamma}{\Pi, \Gamma_{M_\delta} \Vdash D\gamma}\,C}\,Mix$$

<u>Case 12b</u>. S_2 is by a two premiss logical rule L. The rule has the form

$$\frac{\Lambda_{11}, \Gamma \Vdash \Omega, \Lambda_{12} \qquad \Lambda_{21}, \Gamma \Vdash \Omega, \Lambda_{22}}{\Xi_1, \Gamma \Vdash \Omega, \Xi_2}\,L$$

<u>Subcase 1</u>. Ξ_1 is not M_δ, so $M_\delta \in \Gamma$.

$$\frac{\Pi \Vdash M_\delta \qquad \dfrac{\Lambda_{11}, \Gamma \Vdash \Omega, \Lambda_{12} \qquad \Lambda_{21}, \Gamma \Vdash \Omega, \Lambda}{\Xi_1, \Gamma \Vdash \Omega, \Xi_2}}{\Pi, \Xi_1, \Gamma_{M_\delta} \Vdash \Omega, \Xi_2}\,Mix$$

The amended proof figure is:

$$\frac{\dfrac{\dfrac{\Pi \Vdash M_\delta \qquad \Lambda_{11}, \Gamma \Vdash \Omega, \Lambda_{12}}{\Pi, \Lambda_{11}, \Gamma_{M_\delta} \Vdash \Omega, \Lambda_{12}}\,Mix}{\Lambda_{11}, \Pi, \Gamma_{M_\delta} \Vdash \Omega, \Lambda_{12}}\,L \qquad \dfrac{\dfrac{\Pi \Vdash M_\delta \qquad \Lambda_{21}, \Gamma \Vdash \Omega, \Lambda_{22}}{\Pi, \Lambda_{21}, \Gamma_{M_\delta} \Vdash \Omega, \Lambda_{22}}}{\Lambda_{21}, \Pi, \Gamma_{M_\delta} \Vdash \Omega, \Lambda_{22}}}{\dfrac{\Xi_1, \Pi, \Gamma_{M_\delta} \Vdash \Omega, \Lambda_{22}}{\Pi, \Xi_1, \Gamma_{M_\delta} \Vdash \Omega, \Xi_2}\,I}\,L$$

Subcase 2. Ξ_1 in M_δ. The case reduces to

$$\dfrac{\Pi \Vdash M_\delta \qquad \dfrac{\Lambda_{11}, \Gamma \Vdash D_\gamma \qquad \Lambda_{21}, \Gamma \Vdash D_\gamma}{M_\delta, \Gamma \Vdash D_\gamma}\,L}{\Pi, \Gamma_{M_\delta} \Vdash D_\gamma}\,Mix$$

$M_\delta \in \Gamma$ (The rule can only be $\Vdash \vee$).

The amended proof figure is:

$$\dfrac{\Pi \Vdash M_\delta \qquad \dfrac{\dfrac{\dfrac{\Pi \Vdash M_\delta \quad \Lambda_{11}, \Gamma \Vdash D_\gamma}{\Pi, \Lambda_{11}, \Gamma_{M_\delta} \Vdash D_\gamma}\,Mix}{\Lambda_{11}, \Pi, \Gamma_{M_\delta} \Vdash D_\gamma} \qquad \dfrac{\dfrac{\Pi \Vdash M_\delta \quad \Lambda_{21}, \Gamma \Vdash D_\gamma}{\Pi, \Lambda_{21}, \Gamma_{M_\delta} \Vdash D_\gamma}}{\Lambda_{21}, \Pi, \Gamma_{M_\delta} \Vdash D_\gamma}}{M_\delta, \Pi, \Gamma_{M_\delta} \Vdash D_\gamma}\,I}{\dfrac{\Pi, \Pi, \Gamma_{M_\delta} \Vdash D_\gamma}{\Pi, \Gamma_{M_\delta} \Vdash D_\gamma}\,C}\,Mix$$

Corollaries: (1) R^+_* proofs without Cut have the subformula property.

(2) The separation theorem holds.

(3) The decidability theorem holds.

Weakening is available in a form which does not affect Meyer's cognation requirement. (In fact in $_1R^+_*$ weakening is available with qualification).

The Elimination theorem for $_1R^+_*$ (without Cut).

Proof is as for R^+_*, except that in B2.2 it is assumed both that the right rank exceeds 1 and that the left rank $= 1$. In this way the restrictions on $\Vdash\rightarrow$ needed in case 11b are guaranteed by the form of S1.

<u>Equivalence Theorem</u> for R^+ systems.

$\vdash_{R^+} A$ iff $\Vdash A_o$ in R^+_*
iff $\Vdash A_o$ in R^+_*

Proof: One half amounts to a direct demonstration of the axioms of R^+; for rules [illegible] follows [illegible] Cut and adjunction follows from $\Vdash$-I.
For the converse, the sequent $\Gamma \Vdash A_S$ of the Gentzen system is interpreted as $\Gamma \vdash_R A_S$, i.e. as an R^+-proof of A_S from hypotheses Γ. Then the axioms always hold, and in the case of each rule if the premisses hold then the conclusion holds, using the deduction theorems already established, & [illegible] conditions. Hence if $\Vdash A_o$ in R^+_* then there is an R^+-proof of A from null hypotheses, so $\vdash_R A$.

~~As indicated, [illegible]~~

A CUT-FREE FORMULATION $\Box R^+_x$

The following rules are added to R^+_x

$\Box\Vdash$
$$\frac{A_\alpha, \Gamma \Vdash D_\delta}{\Box A_\alpha, \Gamma \Vdash D_\delta}$$

$\Vdash\Box$
$$\frac{\Box\Gamma \Vdash D_\delta}{\Box\Gamma \Vdash \Box D_\delta}$$

$\Box\Gamma$ is the sequence of subscripted wffs formed by prefixing $\Box$ to each wff in sequence Γ.

Lemma If $\Gamma \Vdash D_\delta$ and $C_\alpha \in \Gamma$, then $\alpha \leq \delta$.

Elimination theorem for $\Box R^+_x$

There are the following new cases:

B1, where rank is 2.

Case i.
$$\frac{\dfrac{\Box\Pi \Vdash C_\gamma}{\Box\Pi \Vdash \Box C_\gamma} \qquad \dfrac{C_\gamma, \Gamma \Vdash D_\delta}{\Box C_\gamma, \Gamma \Vdash D_\delta}}{\Box\Pi ; \Gamma \Vdash D_\delta}\ \text{Mix}$$

Amend the figure to:
$$\frac{\Box\Pi \Vdash C_\gamma \qquad C_\gamma, \Gamma \Vdash D_\delta}{\Box\Pi, \Gamma \Vdash D_\delta}\ \text{Mix}$$

B2, where rank exceeds 2,

Case IIa. Only the following new case can occur.
$$\frac{\dfrac{A_\alpha, \Gamma \Vdash M_\delta}{\Box A_\alpha, \Gamma \Vdash M_\delta}\ \Box\Vdash \qquad \Sigma \Vdash D_\gamma}{\Box A_\alpha, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\ \text{Mix}$$

Amended to:
$$\frac{\dfrac{A_\alpha, \Gamma \Vdash M_\delta \qquad \Sigma \Vdash D_\gamma}{A_\alpha, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\ \text{Mix}}{\Box A_\alpha, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\ \Box\Vdash$$

Case IIb. Already treated generally: except one case: see p10

Corollaries: Separation & decidability theorems for $\Box R^+_x$, and hence for $\Box R^+$.

Case IIb subcase 1. $M_S \in \Box\Gamma$; so M_S is $\Box N_S$

The proof figure to be amended is:

$$\frac{\Pi \Vdash \Box N_S \qquad \dfrac{\Box\Gamma \Vdash D\gamma}{\Box\Gamma \Vdash \Box D\gamma}\,{\Vdash\Box}}{\Pi\,,\ \Box\Gamma_{\Box N_S} \Vdash \Box D\gamma}\,\text{Mix}$$

Case a. The left rank is 1. Then $\Box N_S$ must have been introduced by $\Vdash\Box$. Thus Π is of the form $\Box\Theta$. The proof figure is amended as follows:

$$\frac{\dfrac{\Box\Theta \Vdash \Box N_S \qquad \Box\Gamma \Vdash D\gamma}{\Box\Theta\,,\ \Box\Gamma_{\Box N_S} \Vdash D\gamma}\,\text{Mix}}{\Box\Theta\,,\ \Box\Gamma_{\Box N_S} \Vdash \Box D\gamma}\,{\Vdash\Box}$$

Case b. The left rank exceeds 1, so $\geqslant 2$. The case is already treated under B2.1

GENTZEN FORMS OF E^+

Add to the forms for R^+ the further proviso on $\Vdash\rightarrow$: provided every member of Γ is a subscripted entailment.

The elimination theorem holds (for case B2.2, assume also that left rank $= 1$).

CUT FREE FORMULATIONS OF PARTS OF P.

(I) P_I^*.

$\boxed{\text{Formulation 1}}$. Axiom $C_\alpha \Vdash C_\alpha$

Structural rules: Contraction and Interchange (as for R_+^t)

Weakening
$$\frac{\Gamma \Vdash D_\delta}{C_\alpha, \Gamma \Vdash D_\delta} \quad ; \text{provided } \alpha \subseteq \delta$$

Logical Rules:

$\to\Vdash$
$$\frac{\Gamma \Vdash A_\alpha \qquad B_{\alpha\cup\beta}, \Gamma \Vdash D_\delta}{A \to B_\beta, \Gamma \Vdash D_\delta}$$
provided $\max(\alpha) \geq \max(\beta)$

$\Vdash\to$
$$\frac{A_\alpha, \Gamma \Vdash B_\beta}{\Gamma \Vdash A \to B_{\beta-\alpha}}$$
provided $\alpha \neq 0$, $\alpha \subseteq \beta$, and for $C_\delta \in \Gamma$ α disj δ and $\max(\alpha) > \max(\delta)$.

Lemma: If $\Gamma \Vdash D_\delta$ and $C_\alpha \in \Gamma$ then

(i) $\alpha \subseteq \delta$

(ii) $\max(\alpha) \leq \max(\delta)$.

New details in the elimination theorem

(1) For $\to$. In cases 12a and 12b the restrictions in the original figures carry over to the amended figures. In case 12b where 1, the further restriction on $\to\Vdash$ is derived thus. Lemma for $C_\xi \in \Pi$, $\max(\xi) \leq \max(\delta)$ by the premiss S_1. But $M_\delta \in \Gamma$ so $\max(\alpha) > \max(\delta)$; hence $\max(\alpha) > \max(\xi)$.

$\boxed{\text{Formulation 2}}$ As for above but

(a) proviso on weakening removed

(b) $\Vdash\to$ replaced by following rule

$\Vdash\to$
$$\frac{A_\alpha, \Gamma \Vdash B_\beta}{\Gamma \Vdash A \to B_{\beta-\alpha}},$$
provided $\alpha \neq 0$, $\max(\alpha) \in \beta$, and $\max(\alpha) > \max(\delta)$ for $C_\delta \in \Gamma$.

Since the elimination theorem holds, P_I is [illegible] by the methods of Kleene.

(II) P_{Id}^*

Add to the formulations the subscripted rules for &.
Everything holds.

III P_+^*

Add to P_{Id}^* the subscripted rules for v.

Then everything holds BUT the equivalence theorem. It breaks down because I have not been able to prove primeness, i.e. (here)

(?) if $\Gamma, A_\beta \Vdash_p C_\delta$ and $\Gamma, B_\beta \Vdash_p C_\delta$
then $\Gamma, (A \vee B)_\beta \Vdash_p C_\delta$

There is a Gentzen formulation using primeness in the form I have managed to establish, but then the proof of the elimination theorem breaks down.

Confidential (1)

COVERING NOTE.

There are several gaps in the argument, and no doubt many invisible errors — I should be grateful if you would point out all those you think I have failed to see. The chief gaps and deficiencies are these:

1) A proof of the following primeness theorem is still outstanding in the case of E and P:
if $\Gamma, A_\alpha \vdash C_\beta$ and $\Gamma, B_\alpha \vdash C_\beta$
then $\Gamma, (A \vee B)_\alpha \vdash C_\beta$.

2) A separation theorem for the positive logics of R, Ch, E and P is still lacking. I've put very little work into looking for one. The lack of one is symptomatic of the next.

3) The lack of a satisfactory treatment of negation. Two reasons:

i) an inadequate logic of negation

ii) the implication evaluation function is not quite right. There are clearly lots of variations on the implication rule with the right sort of features:
I started out eg with the following functions:

$h(A \supset B, H_1) = T$ iff
for every H_2, H_3 if $H_1 R(H_2, H_3)$ and $h(A, H_2) = T$
then $h(B, H_3) = T$.

$h(\sim A, H) = T$ iff $h(A, H^*) = F$

Then the condition for contraposition is the not unpleasing: if $H R(H_1, H_2)$ then $H R(H_2^*, H_1^*)$.
But the conditions for $\Leftrightarrow$ implication theorems get quite complicated. It should be possible to [illegible] a suitable simplification. An improved implication rule might enable a solution of 1) too. [illegible] the trouble case: [illegible] $\supset F$

4) The later parts — §6 on — are sketchy and even temporarily deficient, but I would hope the deficiencies (gaps) can be repaired when earlier problems ((1) & (3)) have been solved.

5) The simple rule for negation on the right: if ~Ax is on the right of H put Ax on the left appears to work fine, but I haven't been able to show its adequacy. But its rules for negation on the right would involve Disjunctive Syllogism, & in any case it isn't adequate for that them Contraposition.

6) I'm still unhappy about disjunction behaviour in {→, &, f} formulation of R. There's more to this than has met my eye.

7) All of my "proofs" that simplified models will work for R have broken down. I now think a proof will result using the methods of the sketchy §8: at least it seems to follow that models for R^+ may be simplified as in §2.3.

8) The basic idea of §8, which I have only in rough form is that deductive tableaux rewritten, from bottom to top, provide a Gentzen cut-free proof method. This completeness follows using an interpretation theorem for the Gentzen system. You'll see how the positive Gentzen system ~~is~~ looks in §9