Semantic Analysis of Entailment and Relevant Implication: I

Author: Richard Routley (1970/1)

Transcribed by Nicholas Ferenz

Contents

Semantical analyses are provided for several intensional logics, in particular for (substantial parts of) the systems R of relevant implication, $\Box$R of relevant implication with necessity, P of ticket entailment, and E of entailment, and what is the same theory as E the system Π of rigorous implication. The analyses provided are used to provide semantical completeness results and decidability results for the main systems discussed, and are applied to settle some of the open questions concerning E and R and their fragments (on these questions see Anderson [**?**]).

The analyses extend the set-up analysis of the first-degree theory of entailment provided in [**?**]. (The discussion in [**?**] is presupposed in the remainder of this introductory section). The rules for set-up membership for conjunctive, disjunctive, and negated formulae are essentially the rules already defended in [**?**]; viz.

$(A\&B)$ is in set-up H_α iff A is in H_α and B is in H_α

$(A \vee B)$ is in set-up H_α iff A is in H_α or B is in H_α

$\sim A$ is in set-up H_α iff A is not in H^*_α,

with complementary set-up H^*_α of H_α explained as in [**?**]. The chief innovation is a more sophisticated rule for the introduction of entailmental formaulae, of the form $A \to B$, which enables the design of set-ups which falsify entailmental principles, and in particular of set-ups which falsify the law of identity $A \to A$ for any given A. This is done by evaluating higher degree entailments not over a single (possible) situation as in strict implication, but over a pair of (modus-ponens-) connected situations. Thus the special form of the implication rule is as follows:—

> $A \to B$ is in H_α iff for every pair of set-ups H_β and H_γ which are R-related to H_α if A is in H_β then materially B is in H_γ; in short, if $R(H_\alpha, H_\beta, H_\gamma)$ and A is in H_β then B is in H_γ. Canonically relation R is the following: $R(H_\alpha, H_\beta, H_\gamma)$ iff for every wff B and C, if $B \to C$ is in H_α and B is in H_β, then C is in H_γ.

But the general implication rule requires special conditions for practically every pure implicational thesis; so while it is a fine tool for independence proofs and for systems with weak pure entailment parts, it considerably complicates first attempts to prove completeness. To take advantage of known results, e.g. in system E, the implication rule is recast as follows: $A \to B$ is in H_α iff for every set-up H_β which is R-related to H_α, if A is in H_β then B is in $H_{\alpha+\beta}$, where $H_{\alpha+\beta}$ is a certain compounded set-up constructed from H_β taking account of H_α. In fact, the connections may now be made using Anderson's rule of entailment elimination: If $A \in H_\alpha$ and $A \to B \in H_\beta$ then $B \in H_{\alpha+\beta}$ where $\alpha + \beta$ is the set of lattice union of α and β.

For analysis of E, P, and R (and for typing) it is convenient to transform H_α into the pair (α, H) and to consider α and H as independent units. Then for all the systems mentioned the R-relation of H_α to H_β, now replaced by the relation of (α, H_1) to (β, H_2), can be analyzed broken down into two independent relations, of H_1RH_2 and of $\alpha Z\beta$. Relation R is the now familiar alternativeness relation of modal logic; and in the case of system E it is required, as for S4, that R is reflexive and transitive. In the case of systems like E and R, which, unlike P countenance implicative suppression or implicative commutation principles, the ordering relation Z does not figure, since $\alpha Z\beta$ for every α and β; accordingly the implication rule can be simplified in these cases to:

> $A \to B$ is in (α, H) iff for every H_1 and every β, if H_1RH_2 and A is in (β, H_1) then B is in $(\alpha + \beta, H_2)$.

For system R where relation R is an equivalence relation and where the hereditariness condition:

> if A is in (α, H_1) and H_1RH_2, then A is in (α, H_2)

is satisfied, a further simplification can be made: relation R and its field can be omitted altogether. This for system R the implication rule reduced to:

> $A \to B$ is in α iff for every β, if A is in β then B is in $\alpha + \beta$.

The rule for E can be recovered from this rule for R by combining it with the $\Box$ necessity rule, for "$\Box$"; viz.

> $\Box A$ is in (α, H) iff, for every H_1, if HRH_1 then, materially, A is in (α, H_1).

It will follow then from the semantics that E is effectively an S4-modalization of R.

The strict implication rule is a special case of the entailment rule for E; the strict implication rule results from equating α with β for every α and β. Thus the semantics include conditions for normal modal logics as special cases.

In the presentation a characteristic function h is used to indicate whether or not a given wff is in or holds in a given situation, i.e. $h(A, (\alpha, H)) = T$, or $= F$, according as A is in, of is not in, (α, H). Finally, $h(A, (\alpha, H))$ is shortened to $h(A, \alpha, H)$.

The paper is heavily indebted to the work of Anderson and Belnap and Meyer and Dunn and coworkers (and I hope debt will increase). The paper presupposes some of their work, and it also presupposes semantical analysis of modal logics, especially the work of Kripke.

The methods of the paper may be applied to provide semantics for a number of other systems related to these studied.

1 The Axiomatic Systems

[PDF p. 6][1] The postulates of system E are (in favoured Anderson & Belnap form) as follows:

E1. $((A \to A) \to B) \to B$
E2. $A \to B \to .B \to C \to .A \to C$
E3. $(A \to .A \to B) \to A \to B$
E4. $A\&B \to A$
E5. $A\&B \to B$
E6. $(A \to B)\&(A \to C) \to .A \to (B\&C)$
E7. $NA\&NB \to N(A\&B)$
E8. $A \to A \vee B$
E9. $B \to A \vee B$
E10. $(A \to C)\&(B \to C) \to .(A \vee B) \to C$
E11. $A\&(B \vee C) \to (A\&B) \vee C$
E12. $A \to\sim A \to . \sim A$
E13. $A \to\sim B \to .B \to\sim A$
E14. $\sim\sim A \to A$

Modus Ponens (MP): From A and $A \to B$ to infer B.
Adjunction (Adj): From A and B to infer $A\&B$.
The connectives "&" (symbolizing conjunction) "$\sim$" (negation) and "$\to$" (implication or entailment) are taken as primitive; "$\vee$" (disjunction) is either taken as primitive, or defined in the full system; $A \vee B =_{Df} \sim (\sim A\& \sim B)$ and "N" (necessity) is defined: $NA =_{Df} (A \to A) \to A$. The pure implicational fragment E_I, of E, has as postulates E1–E3 and MP; the implication-negation fragment, E_I^-, the postulates E1–E3 and E12–E14 with MP; the implication-conjunction fragment, $\mathrm{E}_\&$, the postulates E1–E7 with MP and Adj; the positive fragment, E^+, E1–E11 with MP and Adj.

The postulates of system R and those of E together with the scheme

E0. $A \to .(A \to A) \to A$

or one of its equivalents. Further each fragment of R adds E0 to the corresponding fragments of E; e.g. R^+ is E^++ E0. Scheme E7 is however redundant whenever it occurs in R systems; and scheme E12 may be proved using E3 (or vice versa) in R systems (see [?]).

System R_f (of Meyer [?]) takes the propositional constant f as primitive in place of N, and replaces negation axioms E12–E14 of R by the single axiom: $A \to f \to f \to A$.

System $\Box$R, of relevant implication with S4-necessity, results upon adding to R the new primitive '$\Box$' and the intended S4 principles (see e.g. Meyer [?])

$\Box$1. $\Box A \to A$
$\Box$2. $\Box(A \to B) \to .\Box A \to \Box B$
$\Box$3. $\Box A\&\Box B \to \Box(A\&B)$
$\Box$4. $\Box A \to \Box\Box A$

[1] For convenience, sections begin with a reference to the page number of the PDF of the scanned manuscript. In the manuscript, some pages are not numbered, and the numbering of the remaining pages restarts at certain points. The reference to the page numbers of the PDF can be used by the reader to locate the beginning of sections by counting from the first page of the manuscript.

Necessitation (Nec): From A to infer $\Box A$.
Entailment is defined thus in $\Box$R: $A \Rightarrow B =_{Df} \Box(A \rightarrow B)$. The $\Box$R translation of a wff $\Box A$ of E is the wff A' which results on replacing each occurrence of '$\rightarrow$' in A by '$\Rightarrow$' and each occurrence of 'N' by '$\Box$'.

$\Box$R may be reaxiomatized so as to avoid the rule of necessitation by doubling up on the axioms as follows: for each axiom Ax of the given system the new axiom $\Box Ax$ is added. For example, in the reaxiomatization, both $\Box A \rightarrow A$ and $\Box(\Box A \rightarrow A)$ are taken as axioms. In the reaxiomatized system the rule of necessitation is a derivable rule provable by induction over proofs.

System P (of [?]) differs from E in just these respects. In place of E1 the scheme E1$'$. $A \rightarrow A$ is adopted; E7 is deleted; and the permuted form E2$'$, $A \rightarrow B \rightarrow .C \rightarrow A \rightarrow .C \rightarrow B$, of E2 is added. The pure implication fragment P_I of P has as postulates E1$'$, E2, E2$'$, and E3 with MP; the implication-negation fragment P_I^- has the postulates of P_I together with E12–E14; the implication-conjunction fragment $\mathrm{P}_\&$ the postulates of P_I together with E4–E6 and Adj; and the positive fragment P^+ the postulates of P_I together with E4–E6, E8–E11 and Adj.

The system EΛ (of [?]) adds to E a propositional constant Λ satisfying these postulates:

Λ1. $A \rightarrow \Lambda \rightarrow \sim A$

Λ2. $\sim (A \rightarrow A) \rightarrow \Lambda$

EΛ, which is a conservative extension of E, corresponds to Ackermann's system Π' (of [?]) as E corresponds to Ackermann's system Π, i.e. they have the same class of theorems.

Several other systems are singled out for attention. First, S5-modalizations of the focal systems. E5 (P5) adds to E (P) the S5 principle

E15. $\sim NA \rightarrow N \sim NA$

$\Box$R5 adds to $\Box$5 the postulate $\sim \Box A \rightarrow \Box \sim \Box A$, and, in reaxiomatized form, the postulate $\sim \Box A \Rightarrow \Box \sim \Box A$. Second, extensions of the focal systems by a special S5 type principle to the effect that some logically false proposition entails its necessary falsity. Thus $\Box \mathrm{R}_f 5$ adds to $\Box \mathrm{R}_f$ the postulate $f \rightarrow \Box f$ (and $f \Rightarrow \Box f$), and EΛ5 adds to EΛ the postulate $\Lambda \rightarrow N\Lambda$.

Third, non-transitive analogues of E and P. Here E2 and P2 resemble S2 in the way E resembles S3; they weaken the Exported Syllogism principles E2 and E2$'$ to the imported form: $(A \rightarrow B)\&(B \rightarrow C) \rightarrow .A \rightarrow C$. Naturally, compensation for the loss of an over-powerful proofs principle such as exported syllogism has to be made elsewhere. Thus P2, formulated with primitive connective set $\{\rightarrow, \sim, \&\}$, reforges P as follows

P1. $A \rightarrow A$

P2. $A \rightarrow B\&B \rightarrow C \rightarrow .A \rightarrow C$

P3. $A \rightarrow (B \rightarrow C) \rightarrow A\&B \rightarrow C$

P4. $A\&B \rightarrow A$

P5. $(A \rightarrow B)\&(C \rightarrow C) \rightarrow .A\&C \rightarrow D\&B$

P6. $A \rightarrow A\&A$

P11–P14 are the same as E11–E14: to the rules of P is added the further rule of substitutability of entailments: from $C(A)$ and $(A \leftrightarrow B)$, i.e. $A \to B \& B \to A$, to infer $C(B)$. The sole pure entailment axiom P1 is of course derivable from P4 and P6.

Forth, extensions of P$^+$ and E$^+$ by different negation principles. Of special interest are the systems PP (P proper) and EP obtained from P and E respectively by weakening E12; for once impossible situations are admitted as semantically valuable, the reductio principle E12 appears as an unnecessary and undesirable restriction. Moreover, in the case of P leads to anomalies; e.g. P has as a theorem $((A \vee \sim A) \to B) \to B$ though rejecting the theorem $((A \to A) \to B) \to B$ characteristic of E; yet the grounds for objecting to the second of these are also grounds for objecting to the first.[2]

[2] There is a note here. The first half reads "add $A \to B \to \sim (A \& \sim B)$ &", and I have been unable to figure out the last half.

2 The Semantical Systems

2.0 Positive Models

[PDF p. 10] An E$^+$-model $\mathfrak{M}$ is a structure $\mathfrak{M} = \langle G, K, R, 0, N, h\rangle$, where K is a set; $G \in K$; R is a reflexive and transitive relation on K; N is a set of sets including the null set 0 and closed under the set union operation $+$, and h is a 2-place holding function such that for every atomic wff p and every $H \in K$ and $\alpha \in N$, $h(p, \alpha, H) = T$ or $= F$.

The holding function h is extended to all wff of E$^+$ as follows:—

$h(A\&B, \alpha, H) = T$ iff $h(A, \alpha, H) = T = h(B, \alpha, H)$

$h(A \vee B, \alpha, H) = T$ iff $h(A, \alpha, H) = T$ or $h(B, \alpha, H) = T$

$h(A \to B, \alpha, H) = T$ iff for every $H' \in K$ and $\beta \in N$ if HRH' and $h(A, \beta, H') = T$ then materially $h(B, \alpha + \beta, H') = T$.

A wff B is true in E$^+$-model $\mathfrak{M}$ iff $h(B, 0, G) = T$, false in $\mathfrak{M}$ iff $h(B, 0, G) = F$, E$^+$-valid iff B is true in every E$^+$-model. E$^+$-model $\mathfrak{M}$ falsifies B iff $h(B, 0, G) = F$; $\mathfrak{M}$ satisfies Γ iff for every wff $B \in \Gamma$, $h(B, 0, G) = T$.

An R$^+$-model $\mathfrak{M}$ is an E$^+$-model such that:
(i) if $h(p, \alpha, H_1) = T$ and $H_1 R H_2$ then $h(p, \alpha, H_2) = T$, for every atomic p and every $H_1, H_1 \in K$ (the hereditariness requirement).

A $\Box$R$^+$-model $\mathfrak{M}$ is a structure $\mathfrak{M} = \langle G, K, R, 0, N, W, h\rangle$ where $\langle G, K, R, 0, N, h\rangle$ is an R$^+$-model and W is another reflexive and transitive relation on K.

A P$^+$-model is an E$^+$-model where the elements of sets of N are ordered. A convenient choice is to take N as a set of sets of positive integers (or ordinals). Then, as in Anderson & Belnap [?], for $\alpha \in N$,[3]

2.1 Forced Negation Models

An R-model $\mathfrak{M}$ is a structure $\mathfrak{M} = \langle G, K, R, 0, N, P, h\rangle$ where K is a set, $G \in K$, R is a reflexive and transitive relation on K, N is a set of sets including the null set 0 and closed under the set union operation $+$, P is a relation on elements of N and K such that
(i) if, for every $\beta \in N$ and $H \in K$ $H_1 R H$ and $P(\alpha + \beta, H)$ materially imply HRH_2 and $P(\gamma + \beta, H_2)$, then $(\alpha, H_1) = (\gamma, H_2)$ (the reduction requirement).
Finally h is a 2-place holding (or valuation) function such that for every atomic wff p and every $H \in K$ and every $\alpha \in N$, $h(p, \alpha, H) = T$ or $= F$, and such that
(ii) for every atomic wff p and every $H_1, H_2 \in K$ and every $\alpha \in N$, if $H_1 R H_2$ and

[3]This sentence appears to be missing the latter half.

$h(p, \alpha, H_1) = T$, then $h(p, \alpha, H_2) = T$ (the hereditariness requirement); and
(iii) for every atomic p and every $\alpha \in N$ and every $H_1 \in K$ if $h(p, \alpha, H_1) = F$ then for some $H_2 \in K$ and $\beta \in N$ $H_1 R H_2$ and $P(\alpha + \beta, H_2)$ (the falsity requirement).

The holding function h is extended from atomic wff to all wff of R as follows:—

$h(A \& b, \alpha, H) = T$ iff $h(A, \alpha, H) = T = h(B, \alpha, H)$

$h(A \rightarrow B, \alpha, H) = T$ iff, every every $H' \in K$ and every $\beta \in N$, if HRH' and $h(A, \beta, H') = T$ then, materially, $h(B, \alpha + \beta, H') = T$

$h(\sim A, \alpha, H) = T$ iff for every $H' \in K$ and every $\beta \in N$ if HRH' and $P(\alpha + \beta, H')$ then, materially, $h(A, \beta, H') = F$.

An R-model may be simplified. G may be defined: $G = H(H \in K)$; and R may be eliminated (in the way explained in [**?**]). If requirements (i) and (iii) and dropped a minimal logic version of R which does not validate E14 results.

Lemma 1. *For every wff A, if $H_1 R H_2$ and* $\mathrm{h}(A, \alpha, H_1) = T$ *then* $\mathrm{h}(A, \alpha, H_2) = T$.

Proof. Proof is by induction from the stipulated basis. There are 3 cases:
Ad &: If $h(B \& C, \alpha, H_1) = T$ and $H_1 R H_2$ then $h(B \& C, \alpha, H_2) = T$
Ad $\rightarrow$: by transitivity of R and definition of h
Ad $\sim$: by transitivity of R and definition of h □

Lemma 2. . *For every wff A and every $H_1 \in K$ if* $\mathrm{h}(A, \alpha, H_1) = F$ *then, for some $H_2 \in K$ and some $\beta \in N$, $H_1 R H_2$ and $P(\alpha + \beta, H_2)$.*

Proof. Proof is by induction from the stipulated basis.
Ad &: If $h(B \& C, \alpha, H_1) = F$ then either $h(B, \alpha, H_1) = F$ or $h(C, \alpha, H_1) = F$. In either case the derived rule follows by induction hypothesis.
Ad $\rightarrow$: If $h(B \rightarrow C, \alpha, H_1) = F$ then for some H_3 and some $\gamma \in N$, $H_1 R H_3$ and $h(B, \gamma, H_3) = T$ and $h(C, \alpha + \gamma, H_3) = F$. Since $h(C, \alpha + \gamma, H_3) = F$, by induction hypothesis, for some $H_2 \in K$ and some $\delta \in N$, $H_3 R H_2$ and $P(\alpha + \gamma + \delta, H_2)$. Thus, as R is transitive, for some H_2 and some $\beta = \gamma + \delta$, $H_1 R H_2$ and $P(\alpha + \beta, H_2)$.
Ad $\sim$: If $h(\sim B, \alpha, H_1) = F$, then, for some $H_2 \in K$ and some $\beta \in N$, $H_1 R H_2$ and $P(\alpha + \beta, H_2)$. □

It is simplest to use the β yielded by this lemma in applying the reduction requirement.

A wff B is true in R-model $\mathfrak{M}$ iff $h(B, 0, G) = T$; B is R-valid iff B is true in every R-model. R-model $\mathfrak{M}$ falsifies B iff $h(B, 0, G) = F$. $\mathfrak{M}$ satisfies Γ iff for every wff $B \in \Gamma$, $h(B, 0, G) = T$.

Lemma 3. *Where* $\vee$ *is defined:* $A \vee B =_{Df} \sim (\sim A \& \sim B)$,
(i) if $\mathrm{h}(A, \alpha, H) = T$ *or* $\mathrm{h}(B, \alpha, H) = T$ *then* $\mathrm{h}(A \vee B, \alpha, H) = T$
(ii) if $\mathrm{h}(A \vee B, \alpha, H) = T$ *then* $\mathrm{h}(A, \alpha, H) = T$ *or* $\mathrm{h}(B, \alpha, H) = T$ *provided that for some wff* C *and some* β, $\mathrm{h}(C, \alpha + \beta, H) = F$.

Proof. (i) If $h(\sim (\sim A \& \sim B), \alpha, H) = F$ then for some γ and some H_1, $P(\alpha + \gamma, H_1)$ and $h(\sim A, \gamma, H_1) = T = h(\sim B, \gamma, H_1)$. Hence since $P(\alpha + \gamma, H_1)$, $h(A, \alpha, H_1) = F = h(B, \alpha, H_1)$. Since HRH_1, by hereditariness, $h(A, \alpha, H) = F = h(B, \alpha, H)$.

(ii) If $h(\sim (\sim A \& \sim B), \alpha, H_1) = T$ then for every β and H, if $P(\alpha + \beta, H)$ and $H_1 R H$ then $h(\sim A \& \sim B, \beta, H) = F$, i.e.: either $h(\sim A, \beta, H) = F$ or $h(\sim B, \beta, H) = F$. Suppose further that $h(\sim A, \beta, H) = F$. Then for some γ and some H_2, $P(\beta + \gamma, H_2)$ and HRH_2 and $h(A, \gamma, H_2) = T$. By the reduction requirement then $(\gamma, H_2) = (\alpha, H_1)$: i.e. in this case $h(A, \alpha, H_1) = T$. Similarly, on the alternative assumption that $h(\sim B, \beta, H) = F$ $h(B, \alpha, H_1) = T$ follows. Thus, using the falsity requirement to guarantee that for some β and H, $H_1 R H$ and $P(\alpha + \beta, H)$, either $h(A, \alpha, H_1) = T$ or $h(B, \alpha, H_1) = T$. □

An $\underline{\mathrm{R}_f\text{-model}}$ is an R-model; only the extension of h differs as follows: the clause for negated wff is replaces by this clause for f:
$h(f, \alpha, H) = F$ iff $P(\alpha, H)$
Thus P may be eliminated in the case of R_f-models.

Lemma 4. *A wff* A *of* R *is R-valid iff its* R_f*-translation* A_f*, obtained by eliminating each part* $\sim B$ *using the definition* $\sim B =_{Df} B \to f$*, is* R_f*-valid.*

Proof. Suppose A_f is not R_f-valid. Then for some R-model $\mathfrak{M}$ $h'(A_f, 0, G) = F$ where h' is the R-extension of h it follows by induction over sub formulae of A_f that $h'(A, 0, G) = F$. The converse half is similar. □

An R-model $\mathfrak{M}$ <u>for wff</u> A is an R-model $\mathfrak{M}$ where h assigns truth vales only for atomic sub wff of A (and for f). Function h is extended as before for sub wff of A. Further in the case of disjunction h is extended as follows, for sub wff B and C:
if $h(B, \alpha, H) = T$ or $h(C, \alpha, H) = T$ then $h(B \vee C, \alpha, H) = T$
if $h(B \vee C, \alpha, H) = T$ and, for some sub wff D of A (or f) and some β, $h(D, \alpha + \beta, H) = F$, then $h(B, \alpha, H) = T$ or $h(C, \alpha, H) = T$.

Under this definition, a wff A of R is valid (c-valid) iff A is true in every R-model for A, i.e. $h(A, 0, G) = T$ for every R-model for A.

Theorem. *Every theorem of R is both valid and c-valid.*

A <u>□R-model</u> $\mathfrak{M}$ is a structure $\mathfrak{M} = \langle G, K, R, 0, N, P, W, h \rangle$ where $\langle G, K, R, 0, N, P, h \rangle$ is an R-model and W is a reflexive and transitive relation on K such that

(v) if H_1RH_2 then H_1WH_2

Hence if H_1RH_2 and H_2WH_3 then H_1WH_3. The holding function h is extended as for system R; and, in addition,

> $h(\Box A, \alpha, H) = T$ iff either for every H_1 such that HWH_1 $h(A, \alpha, H_1) = T$ or for every H_2 and β, if HRH_2 then not $P(\alpha + \beta, H_2)$.

Truth in $\mathfrak{M}$, $\Box$R-validity, $\Box$R-c-validity, etc, are defined along the same lines as before.

Lemma 5. *For every wff A, if H_1RH_2 and $\mathrm{h}(A, \alpha, H_1) = T$, then $\mathrm{h}(A, \alpha, H_2) = T$.*

Lemma 6. *For every wff A, if $\mathrm{h}(A, \alpha, H_1) = F$, then, for some H_2 and β, H_1RH_2 and $P(\alpha + \beta, H_2)$.*

The new induction step, for $\Box$, is immediate from the holding function for $\Box$, & helps explain its design.

A $\Box$R5-model $\mathfrak{M}$ is a $\Box\mathrm{R}_f$-model such that

(vi) if H_1WH_2 and $P(\alpha, H_2)$ then $P(\alpha, H_1)$.

In the case of $\Box$R4 the holding function h may be extended in the expected way for $\Box$; e.g.:

> $h(\Box A, \alpha, H) = T$ iff for every H_1 such that HWH_1, $h(A, \alpha, H_1) = T$

The lemmata shown both hold.

It follows from the $\Box$R5 modeling that necessary entailments [are][4] evaluated as follows:

> $h(A \to B, \alpha, H_1) = T$ iff for every H_2 and H_3 and β, if H_1WH_2 and H_2RH_3 and $h(A, \beta, H_3) = T$ then $h(B, \alpha + \beta, H_3) = T$.

In the case of $\Box$R-modeling the following alternative is added: or else for every H_4 and γ, if H_1RH_4 then not $P(\alpha + \gamma, H_4)$. In view of condition (v) and given quantification logic, the main clause can be simplified to the following:

> $h(A \to B, \alpha, H_1) = T$ iff, for every H_3 and β, if H_1WH_3 and $h(A, \beta, H_3) = T$ then $h(B, \alpha + \beta, H_3) = T$.

For, for some H_2, H_1WH_2 and H_2RH_3 iff H_1WH_3, by quantification logic.

An[5] EΛ5-model is a structure $\mathfrak{M} = \langle G, K, R, 0, N, P, h \rangle$ where K is a set, $G \in K$, R is a reflexive and transitive relation on K, N is a set of sets including the null set 0 closed under

[4] Cut off

[5] This page is marked for omission. I have indicated the contents of the page by the horizontal lines.

set union operation $+$, P is a relation on elements of K and N such that
(i) If, for every $\beta \in N$ and $H \in K$ H_1RH and $P(\alpha+\beta, H)$ materially imply HRH_2 and $P(\gamma+\beta, H_2)$, then $(\alpha, H_1) = (\gamma, H_2)$, for $\alpha, \gamma \in N$, $H_1, H_3 \in K$.
(ii) If $P(\alpha, H_2)$ and H_1RH_2 then $P(\alpha, H_1)$ for $\alpha \in N$ and $H_1, H_2 \in K$.
Finally h is a 2-place holding (or valuation) function such that for every atomic p and $H \in K$ and $\alpha \in N$, $h(p, \alpha, H) = T$ or $= F$, and such that
(iii) for every atomic p and $\alpha \in N$, $H_1 \in K$ if $h(p, \alpha, H_1) = F$ then, for some $H_2 \in K$ and $\beta \in N$, H_1RH_2 and $P(\alpha+\beta, H_2)$.

The holding function h is extended to wff of EΛ5 as follows:—

$h(A\&B, \alpha, H) = T$ iff $h(A, \alpha, H) = T = h(B, \alpha, H)$

$h(\Lambda, \alpha, H) = F$ iff $P(\alpha, H)$

$h(A \Rightarrow B, \alpha, H) = T$ iff for every $H' \in K$ and every $\beta \in N$, if HRH' and $h(A, \beta, H') = T$ then materially $h(B, \beta+\alpha, H') = T$.

$h(\sim A\alpha, H) = F$ iff for some $H_1 \in K$ and $\beta \in N$ HRH_1 and $P(\alpha+\beta, H_1)$ and $h(A, \beta, H_1) = T$.

A wff B is true in a EΛ5-model $\mathfrak{M}$ iff $h(B, 0, G) = T$; etc. Since the distinguishing postulate $\sim NA \rightarrow N \sim NA$ of E5 is EΛ5-valid, it is tempting to define E5-validity as EΛ5-validity of a Λ-free wff.

Lemma 7. *For every wff A, if* h$(A, \alpha, H) = F$ *then for some H_1 and some β, HRH_1 and $P(\alpha+\beta, H_1)$.*

Lemma 8. *For every wff B, if* h$(\sim B, \alpha, H) = T$[6]

$$max(\alpha) = \begin{cases} \text{the largest element of } \alpha, & \text{if } \alpha \neq 0 \\ \text{zero,} & \text{if } \alpha = 0 \end{cases}$$

In the case of P^+ the holding function for $\rightarrow$ is extended to the following:—

$h(A \rightarrow B, \alpha, H) = T$ iff for every $H' \in K$ and $\beta \in N$ if HRH' and $\max(\beta) \geq \max(\alpha)$ and $h(A, \beta, H') = T$ then, materially, $h(B, \alpha+\beta, H') = T$.

A wff B is true in P^+-model $\mathfrak{M}$ iff $h(B, 0, G) = T$; etc.

Modelings for systems E_I, R_I, P_I, $\mathrm{E}_{I\&}$, $\mathrm{R}_{I\&}$, and $\mathrm{P}_{I\&}$ are obtained from the modelings given by deleting clauses for inoperative connectives.

[6] Page is cut off here

Theorem. *If* $\vdash_L A$ *then* A *is* L*-valid for each of the positive systems and their fragments.*

An S4^T-model and indeed an S4-model, is and E$^+$-model where
$N = \{0\}$, i.e. $\alpha = 0$ for every $\alpha \in N$. It is this clear that by varying conditions on the relation R implicational analogues of normal modal systems can be got. For a characterization of entailment proper there is, as Lewis emphasized, a case for abandoning the transitivity requirement on R, and thereby cutting Exported Syllogism, E2, back to Conjunctive Syllogism: $A \to B \& B \to C \to .A \to C$.

2.2 Direct Negation Models

The models so far studied cause substantial problems with respect to the assessment of formulae where negation occurs essentially (and not simple on a substitution instance of a positive wff). To reduce the problems the initial models are supplanted by models which treat negation more directly.

An E-model $\mathfrak{M}$ is a structure $\mathfrak{M} = \langle G, K, R, 0, N, h \rangle$ where K is a set of elements, including $G = H_0$, such that for every $H_i \in K$ there is a unique element $J_i \in K$; and R is a transitive and reflexive relation on $M = \{H_i : H_i \in K\}$, 0 and N are as before; and h is, as before, two-valued holding function which assigns one of T or F to every atomic wff for every H_i and $J_i \in K$ and every $\alpha \in N$. But h also assigns one of T and F to every entailment for every $\alpha \in N$ and $J_i \in K$, i.e. entailments are assigned values arbitrarily at J-situations.

The symbols I, I_1, I_2, $\ldots$, I', $\ldots$ are used as general variables ranging over elements of K. h is extended from atomic wff to all wff of E thus:

$h(A \& B, \alpha, I) = T$ iff $h(A, \alpha, I) = h(B, \alpha, I) = T$

$h(A \vee B, \alpha, I) = T$ iff $h(A, \alpha, I) = T$ or $h(B, \alpha, I) = T$

$h(\sim A, \alpha, H_i) = T$ iff $h(A, \alpha, J_i) = F$

$h(\sim A, \alpha, J_i) = T$ iff $h(A, \alpha, H_i) = F$

if $h(A \to B, \alpha, H_i) = T$ then, for every $\beta \in N$ and $H_j \in K$, if $H_i R H_j$, then if $h(A, \beta, H_j) = T$ $h(B, \alpha + \beta, H_j) = T$ and if $h(A, \alpha + \beta, J_j) = T$ $h(B, \beta, J_j) = T$; further if $h(A \to B, \alpha, H_i) = T$ and $h(B, \alpha, H_i) = F$ then $h(A, \alpha, J_i) = F$. (The last condition is the reduction condition; the complication of the first condition is to take account of contraposition principles.)

If $h(A \to B, \alpha, H_i) = F$ then, for some $\beta \in N$ and some $H_j \in K$, $H_i R H_j$ and $h(A, \beta, H_j) = T$ and $h(B, \alpha + \beta, H_j) = F$, $h(A, \alpha + \gamma, J_k) = T$ and $h(B, \gamma, J_k) = F$

(N.B. A single quantification, for some $\beta \in N$, covering the whole consequent can be used in place of the separate quantifications for β and γ.)

A wff B is true in E-model $\mathfrak{M}$ iff $h(B, 0, G) = T$, i.e. $h(B, 0, H_0) = T$; etc.

An R-I-model $\mathfrak{M}$ is an E-model $\mathfrak{M}$ such that

(1) R is extended to $\overline{M}$, i.e. $K - M$ through the equivalence: J_1RJ_2 iff H_2RH_1, for every $J_1, J_2 \in K$;

(2) if $h(A, \alpha, I_1) = T$ and I_1RI_2 then $h(A, \alpha, I_2) = T$, for every initial case, i.e. (1) for every atomic wff for every $\alpha \in N$ and $I_1, I_2 \in K$, and (ii) for every entailment for every $\alpha \in N$ and $I_1, I_2 \in \overline{M}$ (the Hereditariness requirement).

Lemma 9. *Where $\mathfrak{M}$ is an R-I-model, if* $\mathrm{h}(A, \alpha, I_1) = T$ *and* I_1RI_2 *then* $\mathrm{h}(A, \alpha, I_2) = T$, *for every wff A, every $\alpha \in N$ and $I_1, I_2 \in K$.*

Proof is by induction over connectives in A.

A is R-I-valid iff A is true in every R-I-model.

A $\Box$R-I-model $\mathfrak{M}$ is a structure $\mathfrak{M} = \langle G, K, R, 0, N, W, h\rangle$ where $\langle G, K, R, 0, N, h$ is an R-I-model, and W is a reflexive and transitive relation on M such that if H_1RH_2 then H_1WH_2, and $h(\Box A, \alpha, J)$ is a further initial case, i.e. $\Box A$ is evaluated arbitrarily in (α, J) situations. The hereditariness lemma results.[7]

A is $\Box$R-I-valid iff A is true in every $\Box$R-I-model.

A P-model is simple an E-model where N is an ordered set; however entailment wff are evaluated differently in H-situations, i.e. the extension of h differs from that for E in the following:—

If $h(A \to B, \alpha, H_i) = T$ then, for every $\beta \in N$ and $H_j \in K$, if H_iRH_j and $\max(\beta) \geq \max(\alpha)$, then if $h(A, \beta, H_j) = T$ $h(B, \alpha + \beta, H_j) = T$ and if $h(A, \alpha + \beta, J_j) = T$ $h(B, \beta, J_j) = T$; further if $h(A \to B) = T$ then if $h(A, \alpha, J_i) = T$ $h(B, \alpha, H_i) = T$.

If $h(A \to B, \alpha, H_i) = F$ then for some $\beta \in N$ and $H_i \in K$ $\max(\beta) \geq \max(\alpha)$ and H_iRH_j and $h(A, \beta, H_j) = T$ and $h(B, \alpha + \beta, H_j) = F$ and also for some $\gamma \in N$ and $H_k \in K$ $\max(\gamma) \geq \max(\alpha)$ and H_iRH_k and $h(A, \alpha + \gamma, J_k) = T$ and $h(B, \gamma, H_k) = F$.

B is P-valid iff B is true in every P-model, in effect P-true in every E-model; etc.

In the case of the positive part, P^+, of P the entailment evaluation rule simplifies to the following:—

$h(A \to B) = T$ iff for every $\beta \in N$ and $H' \in K$ if HRH' and $\max(\alpha) \leq \max(\beta)$ and $h(A, \beta, H') = T$ then $h(B, \alpha + \beta, H') = T$.

[7] I have used the word "results" here as a placeholder, as I am unable to decipher the original.

2.3 Simplified Models for Systems based on R and R^+

Lemma 10. *Every non-valid wff of R [□R etc] has a connected R- [□R- etc] countermodel, i.e. every R-satisfiable wff has a connected R-model (etc).*

Proof. Proof is as in Kripke [?]. Define $K' = \{H \in K : GR_*H\}$[8] where R_* is the ancestral of R; R' is the restriction of R to K'; and for $H \in K'$, $h'(p, \alpha, H') = h(p, \alpha, H)$[9]. Then $\mathfrak{M}' = \langle G, K', R', 0, N, P, h' \rangle$ is a connected R-model; & it follows, by induction, that for every $H \in K'$, $h'(B, \alpha, H) = h(B, \alpha, H)$. □

Lemma 11. *(i) For every $H \in K$,* $\mathrm{h}(A, \alpha, H) = \mathrm{h}(\sim\sim A, \alpha, H)$

(ii) If H_1RH_2, $\mathrm{h}(\sim A, \alpha, H_1) = \mathrm{h}(\sim A, \alpha, H_2)$.

(iii) If H_1RH_2, then $\mathrm{h}(A, \alpha, H_1) = \mathrm{h}(A, \alpha, H_2)$

Proof of (i) uses falsity and reduction requirements, and proof of (ii) the transitivity of R.

A simplified R-model $\mathfrak{M}$ is a structure $\mathfrak{M} = \langle 0, N, P, h \rangle$ where 0 and N are as before, P is a property of elements of N and h is a holding function such that for every atomic wff p and every $\alpha \in N$, $h(p, \alpha) = T$ or $= F$. It is required:
(i) If, for every $\beta \in N$, $P(\alpha + \beta)$ materially implies $P(\gamma + \beta)$, then $\alpha = \gamma$ (the simplified reduction requirement).
(ii) For every atomic p and every $\alpha \in N$ if $h(p, \alpha) = F$ then for some $\beta \in N$ $P(\alpha + \beta)$ (the simplified falsity requirement).
The holding function is extended in the expected way upon after deletion of H, viz:

$$h(A\&B, \alpha) = T \text{ iff } h(A, \alpha) = T = h(B, \alpha)$$

$$h(A \to B, \alpha) = T \text{ iff, for every } \beta \in N, \text{ if } h(A, \beta) = T \text{ then, materially, } h(B, \alpha + \beta) = T$$

$$h(\sim A, \alpha) = T \text{ iff, for every } \beta \in N, \text{ if } P(\alpha + \beta) \text{ then, materially, } h(A, \beta) = F$$

A wff B is true in a simplified R-model iff $h(B, 0) = T$. B is R-s-valid iff true in every simplified R-model.[10]

A simplified R_f-model is a structure $\langle 0, N, h \rangle$: P is eliminated using f.

Lemma 12. *In R-models the reduction requirement can be simplified without affecting R- or R_f-validity to the following:*
if, for every β, $P(\alpha + \beta, H_1)$ implies H_1RH_2 and $P(\beta + \gamma, H_2)$ then $(\alpha, H_1) = (\gamma, H_2)$.

[8] Part is cut off. I merely guess that the clause is GR_*H.
[9] Again, this part was partially cut off.
[10] Last sentence cut off.

Proof. The only postulate that requires the reduction principle $\sim\sim A \to A$ remains valid using the simpler requirement, by direct verification. The converse presupposes completeness material of §4. If $A_\alpha \notin H_1$ and $A_\gamma \in H_2$ then $A \to f \to f_\alpha \notin H_1$. Hence, for some H and β[11] H_1RH and $A \to f_\beta \in H$, and $f_{\alpha+\beta} \notin H$. By the preceding lemma since H_1RH_2 $A \to f_\beta \in H_1$ and $f_{\alpha+\beta} \notin H_1$. Remaining details in establishing the simplified reduction requirement are as for the (presupposed) proof of the reduction requirement in §4. □

Theorem.

(i) If $\vdash_R B$ ($\vdash_{R_f} B$) then B is R-s-valid (R_f-s-valid).

(ii) B is R_f-s-valid iff B is R_f-valid.

Proof. Proof of (i) is by induction over proof of B. As to (ii) if B is R_f-valid then B is R_f-s-valid since R_f-s-models are R-models with $K = \{G\}$. Suppose, for the converse that B is not R_f-valid; then there is a connected R_f countermodel $\mathfrak{M}$ to B; Since $\mathfrak{M}$ is connected and R is transitive, by a lemma for every $H \in K$, $h(A, \alpha, H) = h(A, \alpha, G) = h(A, \alpha)$ say. The restriction of K to $\{G\}$ thus provided a countermodel also, and hence B is not R_f-s-valid.
Ad (a). Since M is connected, for every $H \in K$, GRH. Thus $P(\gamma + \beta, G)$ implies GRH & $P(\gamma + \beta, G)$ which implies GRH & $P(\gamma + \beta, H)$. Hence for every β, $P(\alpha + \beta, G)$ implies $P(\gamma + \beta, G)$ implies that for every β, $P(\alpha + \beta, G)$ implies GRH & $P(\gamma + \beta, H)$. Therefore, using the previous lemma, $(\alpha, G) = (\gamma, H)$. Since, however, GRH, for evaluation of wff, $(\alpha, G) = (\gamma, G)$. Next, if $h(p, \alpha, G) = F$ then for some β and H GRH ad $P(\alpha + \beta, H)$, i.e. $h(f, \alpha + \beta, H)$; hence for some β $h(f, \alpha + \beta, G)$, i.e. $P(\alpha + \beta, G)$.
Ad (b). By induction over connectives, each step which shows that A holds or fails to hold in (α, H) may be reflected in (α, G). (of the analogous proof in the decidability section). □

An S-model, for system S of classical two-valued logic, is a simplified R-model where $N = \{0\}$. [Thus model wise it seems that R is related to classical logic as E is to S4: but, though R includes S, E does not include S4].

A simplified $\Box\mathrm{R}_f$-model is a structure $\langle G, K, R, 0, N, h\rangle$ where $\langle G, 0, N, h\rangle$ is a simplified R_f-model with $\alpha = (\alpha, G)$, and K is a set with base G and R is a reflexive and transitive relation on K. Further:

$$h(\Box A, \alpha, H) = T \text{ iff for every } H' \in K \text{ if } HRH' \text{ then, materially, } h(A, \alpha, H') = T.$$[12]

[11] This transcription may be incorrect due to unreadability.
[12] There is a scratched out lemma below this.

3 Deduction Theorems and Primeness Theorems

[PDF p. 25] Where L is one of the systems E or R or their parts, and where, as before $\alpha, \alpha_1, \ldots, \beta, \gamma, \theta$, etc are sets (or lattice elements), 0 is the null new (or minimal element) and $\alpha + \beta$ is the set (or lattice) union of α and β, define:—

$A^1_{\alpha_1}, \ldots, A^n_{\alpha_n} \Vdash_L B_\beta$ is an L-proof of B_β from hypotheses $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}$ iff there is a sequence $C^1_{\gamma_1}, \ldots, C^m_{\gamma_m}$ with $C^m_{\gamma_m} = B_\beta$, where each elements of the sequence is either

(i) one of the hypotheses, or

(ii) D_0 where D is an axiom of L, or

(iii) obtained from predecessors in the sequence by application of the $\to E$ rule: from A_α and $(A \to D)_\beta$ to infer $D_{\alpha+\beta}$, or

(iv) obtained from predecessors in the sequence by application of the rule $\&I$: from A_α and D_α to infer $(A\&D)_\alpha$.

As before $\alpha, \alpha_1, \ldots, \alpha_n, \beta$ are sets, 0 is the null set and $\alpha + \beta$ is the set union of α and β.

$\nabla \Vdash_L B_\beta$ iff for some $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n} \in \nabla$ $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n} \Vdash_L B_\beta$; in this case B_β is L-provable from ∇. ∇_α is a set of α-subscripted wff.

For systems like P and P2 and their parts, it is necessary, once again, to use sets where elements are ordered. Sets of ordinals are a convenient choice. For these systems the rule $\to E$ is simplified by adding the proviso: provided $\max(\alpha) \not< \max(\beta)$, where, as before,

$$max(\alpha) = \begin{cases} \text{the largest element of } \alpha, & \text{if } \alpha \neq 0 \\ 0, & \text{if } \alpha = 0 \end{cases}$$

The first deduction theorems proved for E, R and P and their parts are given essentially in Anderson [?] and Anderson and Belnap [?].

Lemma 13. *If $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}, A_\theta \Vdash_E B_\beta$ and $\theta \neq 0$, $\theta \not\leq \beta$ and $\theta \not\leq \alpha_i$ for any i, $1 \leq i \leq n$ and each $A^i_{\alpha_i}$ is an entailment, i.e. of the form $(E_1 \to E_2)_{\alpha_i}$, for $1 \leq i \leq n$, then $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}, [A_\theta] \Vdash_E NB_\beta$ from which hypothesis A_θ may be deleted.*

Proof. Let the assumed proof sequence be represented

(α) $B^1_{\beta_1}, \ldots, B^m_{\beta_m}$ with $B^m_{\beta_m} = B_\beta$.

Form a new sequence

(β) $D^1_{\delta_1}, \ldots, D^p_{\delta_p}$ with $D^p_{\delta_p} = B_\beta$, obtained from (α) by deleting every $B^i_{\beta_i}$ [13] such that $\theta \leq \beta_i$. Then (β) guarantees.

$$A^1_{\alpha_1}, \ldots, A^n_{\alpha_n} \Vdash_E B_\beta.$$

For no wff with a subscript α including θ occurs essentially in a proof of B_β from hypotheses $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}$, for if it did it would follow that $\theta \leq \beta$. For $\rightarrow E$ and $\&I$ eliminate no subscripts, and $\theta \neq 0$, so no axioms used have a θ subscript. Now form a new sequence

(γ) $ND^1_{\delta_1}, \ldots, ND^p_{\delta_p}$

[In the case of the pure calculus of entailment the more general form, $((D^i \rightarrow C) \rightarrow C)_{\delta_i}$ for arbitrary C, can displace $ND^1_{\delta_i}$: see [**?**]] Proof of the adequacy of (γ) uses the same proof strategy as the deduction theorem which follows. There are these cases:
<u>Case 1:</u> $D^i_{\delta_i}$ is one of $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}$, say, $A^r_{\alpha_r}$. Then insert before $ND^i_{\delta_i}$ in (γ) the zero subscripted E-proof sequence of $\Box(A^r \rightarrow NA^r)_0$, using the fact that A^r is an entailment. $ND^i_{\delta_i}$ then results by $\rightarrow E$.
<u>Case 2:</u> $D^i_{\delta_i}$ is C_0 for some axiom C of system E.
<u>Case 3:</u> $D^i_{\delta_i}$ is inferred by $\rightarrow E$ from $D^j_{\delta_j}$ and $D^k_{\delta_k}$, with $j < i$, $k < i$. Then $D^j_{\delta_j}$ (say) is $(D^k \rightarrow D^i)_{\delta_j}$ and $\delta_i = \delta_j + \delta_k$. By induction hypothesis, $ND^k_{\delta_k}$ and $ND^j_{\delta_j}$ i.e. $N(D^k \rightarrow D^i)_{\delta_j}$ are available. Insert before $D^i_{\delta_i}$ a zero subscripted E-proof of $(N(D^k \rightarrow D^i) \rightarrow .ND^k \rightarrow ND^i)_0$; and $ND^i_{\delta_j+\delta_k}$ results by two applications of $\rightarrow E$.
<u>Case 4:</u> $D^i_{\delta_i}$ is implied by $\&I$ from $D^j_{\delta_j}$ and $D^k_{\delta_k}$, with $j < i$, $k < i$. Then $\delta_i = \delta_j = \delta_k$ and $D^i = D^j \& D^k$. By induction hypothesis $ND^j_{\delta_j}$ and $ND^k_{\delta_k}$ are available in (γ). Insert before $ND^i_{\delta_i}$ the axiom $ND^j \& ND^k \rightarrow N(D^j \& D^k)$, and $(ND^j \& ND^k)_{\delta_i}$.[14]

$\Box$

Lemma 14. *If* $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n} \Vdash_{\Box R} B_\beta$, *then* $\Box A^1_{\alpha_1}, \ldots, \Box A^n_{\alpha_n} \Vdash_{\Box R} \Box B_\beta$.

Proof. Let the given proof sequence be represented

$$C^1_{\delta_1}, \ldots, C^m_{\delta_m} = B_\beta$$

Form a new sequence

$$\Box C^1_{\delta_1}, \ldots, \Box C^m_{\delta_m};$$

then this sequence provides a proof of $\Box B_\beta$ from hypotheses $\Box A^1_{\alpha_1}, \ldots, \Box A^n_{\alpha_n}$. The cases are these at stage $C^i_{\delta_i}$:[15]
<u>Case 1:</u> $C^i_{\delta_i}$ is $A^j_{\alpha_j}$: Then $\Box C^i_{\delta_i}$ is $\Box A^j_{\alpha_j}$.

[13] Original text appears to read $B^i_{\alpha_i}$
[14] I have fixed a number of typographical errors in this paragraph.
[15] I have added the formatting and case numbers to the following to be consistent with the above proof.

Case 2: $C^i_{\delta_i}$ is D_0 where D is an axiom; then $\Box C^i_{\delta_i}$ is $\Box D_0$, which can be introduced in a $\Box$R proof from hypothesis.
Case 3: $C^i_{\delta_i}$ is obtained by rule $\to E$ from C^j_α and $(C^j \to C^i)_\beta$ with $\delta_i = \alpha + \beta$; then $\Box C^i_{\delta_i}$ is obtained from $\Box C^j_\alpha$ and $\Box(C^j \to C^i)_\beta$ which occur in the new sequence by the following inserted steps:[16]
$\Box(C^j \to C^i) \to .\Box C^j \to \Box C^i_0$, $\Box C^j \to \Box C^i_\beta$ and one of $\to E$.
Case 4: $C^i_{\delta_i}$ is obtained from $D^j_{\delta_j}$ and $C^k_{\delta_k}$ by $\&I$, then $\delta_i = \delta_j = \delta_k$ and $C^i = C^j \& C^k$: then $\Box(C^j \& C^k)_{\delta_i}$ is obtained from $\Box C^j_{\delta_i}$ and $\Box C^k_{\delta_i}$ by $\&I$, $\to E$ and the following inserted steps: $(\Box C^j \& \Box C^k)_{\delta_i}$, $(\Box C^j \& \Box C^k) \to \Box(C^j \& C^k)_0$.
Case 5:$C^i_{\delta_i}$ is $\Box D_0$ where D_0 is an axiom; then $\Box C^i_{\delta_i}$[17] is $\Box\Box D_0$ and is obtained by $\to E$ from the following inserted formulae, $\Box D_0$, $\Box D \to \Box\Box D_0$. $\Box$

Lemma 15. *If $\nabla \Vdash_E B_\beta$ and each wff in ∇ is an entailment then $\nabla \Vdash_E \boxminus B_\beta$, where $\boxminus B_\beta \leftrightarrow .B \to B \to B$.*

Proof is like the preceding lemma; it uses the following theorems of E:
$C \to C \to N(C \to D)$; $N(C \to D) \to .NC \to ND$; $NC \& ND \to N(C \& D)$.

Theorem (First Deduction Theorems for E and R and their Parts).
If $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}, A_\delta \Vdash_L B_\beta$ and $\delta \neq 0$, $\delta \subseteq \beta$ but $\delta \not\subseteq \alpha_i$ for any i, $1 \leq i \leq n$, then

$$A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}, \Vdash_L A \to B_{\beta - \delta}$$

where

(1) L is system R

(2) L is system E, and for each i, $1 \leq i \leq n$, A^i is an entailment, i.e. of the form $(D_1 \to D_2)$.

Proof. By assumption there is a sequence

$$B^1_{\beta_1}, \ldots, B^m_{\beta_m} \text{ with } B^m_{\beta_m} = B_\beta$$

which provides a proof of B_β from hypotheses $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}$.[18] Form a new sequence

$$B^1_{\beta_1}{}', \ldots, B^m_{\beta_m}{}'$$

[16] I have made corrections in the following wffs
[17] Corrected
[18] Corrected to $A^n_{\alpha_n}$: subscript in original possibly crossed out.

where

$$B^{i}_{\beta_i}{}' = \begin{cases} (A \to B^i)_{\beta_i - \delta}, & \text{if } \delta \subseteq \beta_i \\ B^{i}_{\beta_i}, & \text{if } \delta \not\subseteq \beta_i \end{cases}$$

Then since $\delta \subseteq \beta$, $B^{m}_{\beta_m}{}' = (A \to B)_{\beta-\delta}$. Following the proof strategy of Church [?, p. 88–89] it is shown how to make insertions in the new sequence so that is provides a proof from hypotheses $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}$ of its last element $B^{m}_{\beta_m}{}'$. Suppose the insertions have been completed just up to the $(i-1)^{th}$ stage. At the i^{th} stage there are these cases:

<u>Case 1a:</u> $B^i_{\beta_i}$ is A_δ. Then $B^{i}_{\beta_i}{}'$ is $(A \to A)_0$. Insert before $B^i_{\beta_i}$ a proof sequence of $(A \to A)_0$ using zero subscripted axioms and $\to E$.

<u>Case 1b:</u> $B^i_{\beta_i}$ is one of $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}$, say $A^r_{\alpha_r}$. Then $B^{i}_{\beta_i}{}'$ is also $A^r_{\alpha_r}$, since $\delta \not\subseteq \alpha_r$. Thus $B^{i}_{\beta_i}{}'$ occurs as one of the hypotheses (case (i) in an L-proof from hypotheses).

<u>Case 2:</u> $B^i_{\beta_i}$ is D_0 for some axiom D of L. Since $\delta \neq 0$, as a consequence of $\delta \not\subseteq \alpha_i$, $B^{i}_{\beta_i}{}'$ is also D_0. Thus $B^{i}_{\beta_i}{}'$ occurs as a zero subscripted axiom (case (ii) in an L-proof from hypotheses).

<u>Case 3:</u> $B^i_{\beta_i}$ is inferred by $\to E$ from $B^j_{\beta_j}$ and $B^k_{\beta_k}$, with $j < i$, $k < i$. Then $B^j_{\beta_j}$ (say) is $(B^k \to B^i)_{\beta_j}$ and $\beta_i = \beta_j + \beta_k$. There are 4 subcases:—

<u>Case 3a:</u> $\delta \subset \beta_j$ and $\delta \subset \beta_k$; so $\delta \subset \beta_i$. Then $B^{k}_{\beta_k}{}'$ is $(A \to B^k)_{\beta_k - \delta}$. $B^{j}_{\beta_j}{}'$ is $(A \to (B^k \to B^i))_{\beta_j - \delta}$, and $B^{i}_{\beta_i}{}'$ is $(A \to B^i)_{B_j + B_k - \delta}$. Insert before $B^{i}_{\beta_i}{}'$ a zero subscripted proof sequence of $(A \to (B^k \to B^i) \to .A \to B^k \to .A \to B^i)_0$; then insert $(A \to B^k \to .A \to B^i)_{\beta_j - \delta}$. $B^{i}_{\beta_i}{}'$ is inferred by $\to E$.

<u>Case 3b:</u> $\delta \subseteq \beta_k$ and $\delta \not\subseteq \beta_j$. Thus $B^{k}_{\beta_k}{}'$ is $(A \to B^k)_{\beta_k - \delta}$ but $B^{j}_{\beta_j}{}'$ if $(B^k \to B^i)_{\beta_j}$. Insert the axiom $(B^k \to B^i \to .A \to B^k \to .A \to B^i)_0$ and $(A \to B^k \to .A \to B^i)_{\beta_j}$ before $B^{i}_{\beta_i}{}'$. Then $B^{i}_{\beta_i}{}'$, i.e. $(A \to B^i)_{\beta_j + \beta_k - \delta}$, results by $\to E$.

<u>Case 3c:</u> $\delta \not\subseteq \beta_k$ and $\delta \not\subseteq \beta_j$. Thus $B^{k}_{\beta_k}{}'$ is $B^k_{\beta_k}$, $B^{j}_{\beta_j}{}'$ is $(B^k \to B^i)_{\beta_j}$, and $B^{i}_{\beta_i}{}'$, i.e. $B^i_{\beta_j + \beta_k}$, is inferred by $\to E$.

<u>Case 3d:</u> $\delta \subseteq \beta_j$ and $\delta \not\subseteq \beta_k$. Thus $B^{k}_{\beta_k}{}'$ is $B^k_{\beta_k}$, $B^{j}_{\beta_j}{}'$ is $(A \to (B^k \to B^i))_{\beta_j - \delta}$, and $B^{i}_{\beta_i}{}'$ is $(A \to B^i)_{\beta_j + \beta_k - \delta}$.

(1) L is system R. Insert before $B^{i}_{\beta_i}{}'$ a zero subscripted proof sequence of $(A \to (B^k \to B^i) \to .B^k \to .A \to B^i)_0$, and then insert $(B^k \to .A \to B^i)_{\beta_j - \delta}$. $B^{i}_{\beta_i}{}'$ then results from $\to E$.

(2) L is system E. By a lemma, since $\gamma \not\subseteq \beta_k$, there is an E-proof from hypotheses of $(NB^k)_{\beta_k}$. Insert this sequence, then insert the zero-subscripted proof sequence of $(A \to (B^k \to B^i) \to .NB^k \to .A \to B^i)_0$, and finally insert $(NB^k \to .A \to B^i)_{\beta_j = \delta}$. $B^i_{\beta_i}$ then results by $\to E$.

<u>Case 4:</u> $B^i_{\beta_i}$ is inferred by $\&I$ from $B^j_{\beta_j}$ and $B^k_{\beta_k}$ with $j < i$, $k < i$. Then $\beta_i = \beta_j = \beta_k$ and B^i is $(B^j \& B^k)$. There are 2 subcases:—

Case 4a: $\delta \subseteq \beta_i$. Then ${B^k_{\beta_k}}'$ is $(A \to B^k)_{\beta_k - \delta}$, ${B^j_{\beta_j}}'$ is $(A \to B^j)_{\beta_k - \delta}$ and ${B^i_{\beta_i}}'$ is $(A \to B^j \& B^k)_{\beta_k - \delta}$. Insert before ${B^i_{\beta_i}}'$ the axiom $((A \to B^j)\&(A \to B^k) \to .A \to (B^j \& B^k))_0$ and, what is inferred by $\&I$, $(A \to B^j)\&(A \to B^k)_{\beta_k - \delta}$. Then ${B^i_{\beta_i}}'$ is inferred by $\to E$.
Case 4b: $\delta \not\subseteq \beta_i$. Then ${B^k_{\beta_k}}'$ is $B^k_{\beta_k}$, ${B^j_{\beta_j}}'$ is $B^j_{\beta_k}$, and ${B^i_{\beta_i}}'$ is $(B^j \& B^k)_{\beta_k}$, which is inferred, as before, by $\&I$. □

This deduction theorem holds also for such extensions of E and R as EΛ, □R, □R5, etc. It is not, of course, the only deduction theorem for E and R. Alternative deduction theorems for R_I are given in [**?**] and [**?**][19], and an alternative deduction theorem for E_I is as follows: if $A_1, \ldots, A_n, A \vdash_{E_I} B$ and $A_1, \ldots, A_n$ are entailments and A is used in the proof then $A_1, \ldots, A_n \vdash_E A \to B$. In order to deal with disjunction in R then following deduction theorem is needed:

Theorem (A Second Deduction Theorem for R and E). *If $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}, A_\delta \Vdash_R B_\beta$ then either $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}, \Vdash_R A \to B_{\beta-\delta}$ with $\delta \subseteq \beta$ or $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}, \Vdash_R B_\beta$. Similarly for E when $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}$ are entailments.*

Proof. Let $B^1_{\beta_1}, \ldots, B^m_{\beta_m} = B_\beta$ be a proof of B_β from hypotheses $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}$. [It] is shown by induction for each $B^i_{\beta_i}$ that either (i) $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n} \Vdash_R A \to B^i_{\beta_i - \delta}$ and $\delta \subseteq \beta_i$ or (ii) $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n} \Vdash_R B^i_{\beta_i}$.
Case 1: $B^i_{\beta_i}$ is A_δ. Then $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n} \Vdash_R A \to B^i_{\beta_i}$ using $A \to A_0$.
Case 2: $B^i_{\beta_i}$ is a zero-subscripted axiom of one of $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}$. Then $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n} \Vdash B^i_{\beta_i}$.
Case 3: $B^i_{\beta_i}$ is inferred by $\to E$. The cases are as before. Note that $B^i_{\beta_i}$ results when and only when both the previous are of form (i). $\delta \subseteq \beta_i$ follows from $\delta \subseteq \beta_j$ or $\delta \subseteq \beta_k$.
Case 4: $B^i_{\beta_i}$ is inferred by $\&I$. □

Corollary 1. *(Primeness Theorem for R) If $\Gamma, A_\delta \Vdash_R B_\beta$ and $\Gamma, C_\delta \Vdash_R B_\beta$ then $\Gamma, (A \vee C)_\delta \Vdash_R B_\beta$.*

Proof. Given the premises, either $\Gamma \Vdash_R B_\beta$, and so $\Gamma, (A \vee C)_\delta \Vdash_R B_\beta$, or both $\Gamma \Vdash_R A \to B_{\beta-\delta}$ and $\Gamma \Vdash_R C \to B_{\beta-\delta}$ and $\delta \subseteq \beta$. Since $\vdash_R (A \to B)\&(C \to B) \to .A \vee C \to B$, $\Gamma \Vdash_R (A \vee C \to B)_{\beta-\delta}$ with $\delta \subseteq \beta$, whence $\Gamma, A \vee C_\delta \Vdash_R B_{(\beta-\delta)+\delta}$, i.e. $\Gamma, A \vee C_\delta \Vdash_R B_\beta$, since $\beta \subseteq \delta$. □

Corollary 2. *(A primeness result for E). As in corollary 1 but with Γ consisting only of entailments.*

Theorem (Alternative Form of the First Deduction Theorem for R and for E). *If $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}, A_\delta \Vdash_R B_\beta$ and $\delta \neq 0$, $\delta \subseteq \beta$ and δ disjoint from α_i for $1 \leq i \leq n$, then $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}, \Vdash_R A \to B_{\beta-\delta}$. Similarly for E where $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}$ are entailments.*

[19] This footnote is in the original: "The simple use-of-hypotheses account breaks down over conjunction."

Proof. Redefine

$$B^{i}_{\beta_i}{}' = \begin{cases} (A \to B^i)_{\beta_i - \delta}, & \text{if } \delta \text{ not disjoint } \beta_i \\ B^i_{\beta_i}, & \text{if } \delta \text{ disjoint } \beta_i \end{cases}$$

Then $B^m_{\beta_m}{}'$ is $(A \to B)_{\beta-\delta}$. The proof is as before. Note in case 3c if δ *disj* β_k and δ *disj* β_j then δ *disj* $(\beta_j + \beta_k)$; in 3b δ *disj* β_j but not *disj* β_k then δ not *disj* $(\beta_j + \beta_k)$; in 3a δ not *disj* β_k and not *disj* β_j then δ not *disj* $(\beta_j + \beta_k)$; in 3d that δ is not *disj* $(\beta_j + \beta_k)$. □

Theorem (First Deduction Theorem for P and its parts). *If* $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}, A_\delta \Vdash_L B_\beta$ *where* $\delta \neq 0$, $m = max(\delta) \in \beta$ *but* m *exceeds* $max(\alpha_i)$ *for each* i *in* $1 \leq i \leq n$, *then* $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}, \Vdash_L A \to B_{\beta-\delta}$

Proof. Using the assumed sequence $B^1_{\beta_1}, \ldots, B^m_{\beta_m} = B_\beta$, for a new sequence $B^1_{\beta_1}{}', \ldots, B^m_{\beta_m}{}'$ where

$$B^{i}_{\beta_i}{}' = \begin{cases} (A \to B^i)_{\beta_i - \delta}, & \text{if } \max(\delta) \in \beta_i \\ B^i_{\beta_i}, & \text{if } \max(\delta) \notin \beta_i \end{cases}$$

<u>Cases 1 and 2 and 4:</u> as before.
<u>Case 3:</u> $B^i_{\beta_i}$ is inferred by $\to E$ from $B^k_{\beta_k}$ and $B^j_{\beta_j} = (B^k \to B^i)_{\beta_j}$ with $j < i$, $k < i$, $\beta_i = \beta_j + \beta_k$, and $\max(\beta_k) \not< \max(\beta_j)$.
<u>Case 3a:</u> $m = \max(\delta) \in \beta_j$ and $m \in \beta_k$; so $m \in \beta_i$. Thus $B^k_{\beta_k}{}'$ is $(A \to B^k)_{\beta_k - \delta}$, $B^j_{\beta_j}{}'$ is $(A \to (B^k \to B^i))_{\beta_j - \delta}$, and $B^i_{\beta_i}{}'$ is $(A \to B^i)_{\beta_j + \beta_k - \delta}$.
<u>Case 3a(i):</u> $\max(\beta_k - \delta) \not< \max(\beta_j - \delta)$. Insert a zero-subscripted proof sequence of $(A \to (B^k \to B^i) \to .A \to B^k \to .A \to B^i)_0$, then insert $(A \to B^k \to .A \to B^i)_{\beta_j - \delta}$. In view of the ordering conditions $\to E$ may be applied to infer $B^i_{\beta_i}{}'$.
<u>Case 3a(ii):</u> $\max(\beta_k - \delta) < \max(\beta_j - \delta)$. Insert a proof sequence for $((A \to B^k) \to .A \to (B^k \to B^i) \to .A \to B^i)_0$, then insert $A \to (B^k \to B^i) \to .A \to B^i)_{\beta_k - \delta}$ (since $\max(\beta_k - \delta) \geq 0$).
<u>Case 3b:</u> $m \in \beta_k$ but $m \notin \beta_j$; so $m \in \beta_k + \beta_j$, and $B^k_{\beta_k}{}'$ is $(A \to B^k)_{\beta_k - \delta}$, $B^i_{\beta_i}{}'$ is $(A \to B^i)_{\beta_i - \delta}$ but $B^j_{\beta_j}{}'$ is $(B^k \to B^i)_{\beta_j}$.
<u>Case 3b(i):</u> $\max(\beta_j) \leq \max(\beta_k - \delta)$. Insert $(B^k \to B^i \to .A \to B^k \to .A \to B^i)_0$, $(A \to B^k \to .A \to B^i)_{\beta_j}$ before $B^i_{\beta_i}{}'$; and use $\to E$ twice.
<u>Case 3b(ii):</u> $\max(\beta_j) \not< \max(\beta_k - \delta)$. Insert $(A \to B^k \to .B^k \to B^i \to .A \to B^i)_0$, $(B^k \to B^i \to .A \to B^i)_{\beta_k - \delta}$ before $B^i_{\beta_i}{}'$; and use $\to E$ twice.
<u>Case 3c:</u> as before.
<u>Case 3d:</u> impossible. For since $m \in \beta_j$ the largest element that can occur belongs to β_j. As $m \notin \beta_k$, $\max(\beta_j) > \max(\beta_k)$, contradicting an assumption for case 3. □

Theorem (A Second Deduction Theorem for P and its parts). *If* $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}, A_\delta \Vdash_P B_\beta$ *and* $m = max(\delta)$ *and* $m > max(\alpha_i)$ *for every* α_i, $1 \le i \le n$, *then either* $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n} \Vdash_P A \to B_{\beta-\delta}$ *and* $m \in \beta$ *or* $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n} \Vdash_P B_\beta$ *and* $m \notin \beta$.

Proof is like the second deduction theorem for R.

This deduction theorem is not sharp enough to provide the basis for a disjunction rule for P. For that the following rule seems to be needed.

Theorem (Improved Second Deduction Theorem for P (conjecture only)). [20] *If* $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}, A_\delta \Vdash_P B_\beta$ *and* $max(\delta) \ge max(\alpha_i)$ *for every* α_i, $1 \le i \le n$, *then either* $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n} \Vdash_P A \to B_{\beta-\delta}$ *and* $m \in \beta$ *or* $A^1_{\alpha_1}, \ldots, A^n_{\alpha_n} \Vdash_P B_\beta$.

Corollary 3. *(A Primeness result for P)*

If $\Gamma, A_\beta \Vdash_P C_\delta$ *and* $\Gamma, B_\beta \Vdash_P C_\delta$ *then* $\Gamma, (A \vee B)_\beta \Vdash_P C_\delta$, *provided* $max(\beta) > max(\gamma)$ *for each* $D_\gamma \in \Gamma$.

Theorem (Qualified Primeness Theorem for P and E). *If* $\Gamma_\alpha, A_\alpha \Vdash C_\alpha$ *and* $\Gamma_\alpha, B_\alpha \Vdash C_\alpha$ *then* $\Gamma_\alpha, (A \vee B)_\alpha \Vdash C_\alpha$ *for every* α. Γ_α *is a set of wff all subscripted with* α.

Proof. (α'). If $\Gamma_\alpha, A_\alpha \Vdash C_\alpha$ then $\Gamma_\alpha, (A \vee B)_\alpha \Vdash (C \vee B)_\alpha$.

Let given sequence in (α') be

$$A^1_{\gamma_1}, \ldots, A^m_{\gamma_m} = C_\alpha$$

Then $\gamma_i = 0$ or α according as A^i is a theorem or is a consequence of at least one of the hypotheses. Form a new sequence:

$$A^1_{\gamma_1}{}', \ldots, A^m_{\gamma_m}{}'$$

where

$$A^i_{\gamma_i}{}' = \begin{cases} (A^i \vee B)_{\gamma_i}, & \text{if } \gamma_i = \alpha \\ A^i_0, & \text{otherwise} \end{cases}$$

There are these cases:—

<u>Case $\to E$:</u> $A^i_{\gamma_i}$ follows by $\to E$ from $A^k_{\gamma_k}$ and $(A^k \to A^i)_{\gamma_j} = A^j_{\gamma_j}$ and $\gamma_i = \gamma_j + \gamma_k$.

<u>Case 1:</u> $\gamma_j = \gamma_k = \alpha$. Then $\gamma_i = \alpha$, and by hypotheses have in new sequence $(A^k \vee B)_\alpha$ and $(A^k \to A^i \vee .B)_\alpha$. Then insert $(A^k \vee B \& A^k \to A^i \vee .B)_\alpha$ appropriate theorems leading to, in turn, to

$([A^k \& (A^k \to A^i)] \vee [A^k \& B] \vee [B \& A^k \to A^i] \vee [B \& B])_\alpha$

[20] Above "conjecture" is written "unlikely?".

$(A^i \vee B \vee B \vee B)_\alpha$
$(A^i \vee B)_\alpha$
Case 2: $\gamma_j = \gamma_k = 0$ Then result just as before by $\rightarrow E$.
Case 3: $\gamma_k = \alpha$ and $\gamma_j = 0$. Then $\gamma_i = \alpha$ and ${A^k_{\gamma_k}}'$ is $(A^k \vee B)_\alpha$, ${A^j_{\gamma_j}}'$ is $(A^k \rightarrow A^i)_0$, and ${A^i_{\gamma_i}}'$ is $(A^i \vee B)_\alpha$. Insert $(B \rightarrow B)_0$, $(A^k \rightarrow A^i \& .B \rightarrow B)_0$, $((A^k \rightarrow A^i)\&(B \rightarrow B) \rightarrow .(A^k \vee B) \rightarrow (A^i \vee B))_0$, whence $(A^k \vee B) \rightarrow (A^i \vee B)_0$ so $(A^i \vee B)_\alpha$.
Case 4: $\gamma_k = 0$ and $\gamma_j - \alpha$; so $\gamma_i = \alpha$. This case is impossible for P unless $\alpha = 0$, in which case the result follows as for case 1. For E, ${A^k_{\gamma_k}}'$ is A^k_0, ${A^j_{\gamma_j}}'$ is $((A^k \rightarrow A^i) \vee B)_\alpha$ and ${A^i_{\gamma_i}}'$ is $(A^i \vee B)_\alpha$. If $\alpha = 0$ then the result folows as for case 1; if $\alpha \neq 0$ then A^k must be a theorem. Hence $(A^k \rightarrow A^i) \rightarrow A^i$ is a theorem. So insert $((A^k \rightarrow A^i) \rightarrow A^i)_0$ [and] insert $(B \rightarrow B)_0$ then $(B \rightarrow B\&(A^k \rightarrow A^i) \rightarrow A^i)_0$, then [something unreadable], then $((A^k \rightarrow A^i) \vee B \rightarrow A^i \vee B)_0$. Result by $\rightarrow E$.
Base Hyp: $A^i_{\gamma_i} \in \Gamma_\alpha$ or $A^i_{\gamma_i}$ is A_α; then ${A^i_{\gamma_i}}'$ is $(A^i \vee B)_\alpha$. Insert $A^i \rightarrow (A^i \vee B)$.
Case Axiom: $A^i_{\gamma_i} = D_0$; then ${A^i_{\gamma_i}}' = D_0$ also.
Case &I: $A^i_{\gamma_i} = (A^j \& A^k)_{\gamma_i}$ follows by &I from $A^j_{\gamma_j}$ and $A^k_{\gamma_k}$; then $\gamma_j = \gamma_k = \gamma_i$.
Case 1: $\gamma_i = \alpha$. So ${A^k_{\gamma_k}}' = (A^k \vee B)_\alpha$, ${A^j_{\gamma_j}}' = (A^j \vee B)_\alpha$, and ${A^i_{\gamma_i}}' = (A^j \& A^k \vee B)_\alpha$. Apply &$I$ to get $((A^j \vee B)\&(A^k \vee B))_\alpha$, then insert appropriate theorems to get $(A^j \& A^k \vee B)_\alpha$.
Case 2: $\gamma_i = 0$; then ${A^i_{\gamma_i}}' = A^i_0$, ${A^j_{\gamma_j}}' = A^j_0$, and ${A^k_{\gamma_k}}' = A^k_0$.[21]

(b') If $\Gamma_\alpha, B_\alpha \Vdash C_\alpha$ then $\Gamma_\alpha, (C \vee B)_\alpha \Vdash (C \vee C)_\alpha$. The proof is similar to (b').[22]

(c') $\Gamma_\alpha, (C \vee C)_\alpha \Vdash C_\alpha$. For $\vdash C \vee C \rightarrow C$.

The theorem then follows on combining (a'), (b'), and (c'). □

[21] The rest cut off the page.We use & intro on the two theorems to get A^i, then we simply insert $A^i \rightarrow A^i \vee B$. The result follows by $\rightarrow E$.

[22] Likely to mean (a').

4 Completeness by Maximal Set Methods

[PDF p. 41] ∇ is <u>L-consistent w.r.t.</u> N iff, for some $\delta \in N$ and $D_\delta \in \nabla$, D_δ is not L-provable from ∇.
∇ is an <u>L-ok set w.r.t.</u> N (where N is a set closed under + and including 0) iff

(i) ∇ is L-consistent w.r.t. N

(ii) $A_0 \in \nabla$ for every axiom A of L

(iii) for every $\alpha \in N$, if $A_\alpha \in \nabla$ and $B_\alpha \in \nabla$ then $(A\&B)_\alpha \in \nabla$

(iv) for every $\alpha, \beta \in N$, if $B_\beta \in \nabla$ and $(B \to C)_\alpha \in \nabla$ then $C_{\alpha+\beta} \in \nabla$[23] [provided $\max(\beta) \not< \max(\alpha)$, in the case of P systems].

Lemma 16. *If ∇ is an L-ok set w.r.t. N then*

(i) for every theorem A of L, $A_0 \in \nabla$,

(ii) for $\alpha \in N$, $(A\&B)_\alpha \in \nabla$ iff $A_\alpha \in \nabla$ and $B_\alpha \in \nabla$,

(iii) for $\alpha \in N$, $A_\alpha \in \nabla$ iff $\nabla \Vdash_L A_\alpha$

An L-ok set ∇ w.r.t. N is <u>prime</u> iff for every $\alpha \in N$ if $(A \vee B)_\alpha \in \nabla$ then either $A_\alpha \in \nabla$ or $B_\alpha \in \nabla$. If ∇ is prime then $A \vee B_\alpha \in \nabla$ iff $A_\alpha \in \nabla$ or $B_\alpha \in \nabla$.

Lemma 17. *If $(B \to C)_\alpha \notin \nabla$ where ∇ is an E-ok set, and ∇' is a set whose elements comprise every subscripted entailment $(D_1 \to D_2)_{\delta_i}$ in ∇ and B_δ for any $\delta \neq \alpha$, $\not\leq \delta_i$ for $(D_1 \to D_2)_{\delta_i} \in \nabla$, and $\neq 0$, then $C_{\alpha+\delta}$ is not E-provable from ∇'.*

Proof. Suppose on the contrary, $\nabla' \Vdash_E C_{\alpha+\delta}$. Then for some entailments, $D^1_{\delta_1}, \ldots, D^n_{\delta_n} \in \nabla$, and therefore in ∇', $D^1_{\delta_1}, \ldots, D^n_{\delta_n} B_\delta \Vdash_E C_{\alpha+\delta}$. Since $\delta \not\subseteq \delta_i$ for $1 \leq i \leq n$, B_δ must occur among the hypotheses. The conditions for the subscripted deduction theorem are satisfied; thus $D^1_{\delta_1}, \ldots, D^n_{\delta_n} \Vdash_E (B \to C)_\alpha$. Since, however, $D^1_{\delta_1}, \ldots, D^n_{\delta_n} \in \nabla$ and ∇ is E-ok, $(B \to C)_\alpha \in \nabla$,[24] contradiction the hypothesis. □

Lemma 18. *If $(B \to C)_\alpha \notin \nabla$ where ∇ is an R-ok set [$\Box$R-ok set], and ∇' is a set whose elements comprise those of ∇ and B_δ for any $\delta \not\subseteq \alpha$, $\not\subseteq \beta$ for $D_\beta \in \nabla$, and $\neq 0$, then $C_{\alpha+\delta}$ is not R-provable [$\Box$R-provable] from ∇'.*

A suitable δ can always be got, e.g. by deriving a new δ.

[23] Corrected N to ∇.
[24] Corrected $\notin \nabla$ to $\in \nabla$ to form said contradiction.

Lemma 19. *If $(B \to C)_\alpha \notin \nabla$ where ∇ is a P-ok set [P2-ok set], and ∇' is a set whose elements comprise every subscripted entailment $(D_1 \to D_2)_{\delta_i}$ in ∇ and B_γ where max(γ) is greater than all elements of α and of δ_i for $(D_1 \to D_2)_{\delta_i} \in \nabla$, then $C_{\alpha+\gamma}$ is not P-provable [P2-provable] from ∇'.*

Lemma 20. *If C_δ is not L-provable from ∇ then there is an L-ok extension ∇^+ of ∇ w.r.t. any countable set N which includes all subscripts of ∇ such that $C_\delta \notin \nabla'$.*

Proof. Enumerate N and numerate the wff of L, and then enumerate the wff of L with respect to the subscripts of N. Let the resulting enumeration of subscripted wff be represented:

$$D^0, D^1, \ldots, D^j, \ldots$$

Define

$$\begin{aligned} \nabla^0 &= \nabla \\ \nabla^{j+1} &= \nabla^j \text{ if } \nabla^j, D^j \Vdash_L C_\delta, \text{ and} \\ &= \nabla^j \cup \{D^j\} \text{ otherwise} \\ \nabla^+ &= \bigcup_{j<\omega} \nabla^j \end{aligned}$$

(i) C_δ is not L-provable from ∇^+. Proof is by induction over j from the given basis. The induction step is a consequence of the construction.

(ii) $C_\delta \notin \nabla^+$, by (i).

(iii) $D_0 \in \nabla^+$ where D is an axiom of system L.

(iv) ∇^+ is closed under $\to E$. Suppose, otherwise, that $B_\gamma \in \nabla^+$, $(B \to D)_\beta \in \nabla^+$ but $D_{\delta+\beta} \notin \nabla^+$ (and for P: max(γ) $\not<$ max(β)). Then $\nabla^+, D_{\gamma+\beta} \Vdash_L C_\delta$. But then since $\nabla^+ \Vdash_L B_\gamma$ and $\nabla^+ \Vdash_L (B \to D)_\delta$, $\nabla^+ \Vdash_L D_{\gamma+\delta}$, whence $\nabla^+ \Vdash C_\delta$, contradicting (i).

(v) ∇^+ is closed under $\&I$. Suppose $B_\gamma \in \nabla^+$, $D_\gamma \in \nabla^+$ but $(B\&D)_\gamma \notin \nabla^+$. Then $\nabla^+, (B\&D)_\gamma \Vdash_L C_\delta$, and (much as in (iv)) $\nabla^+ \Vdash_L C_\delta$, contradicting (i).

□

Lemma 21. *If, further, L is system R then ∇^+ is prime.*

Proof. Suppose $A \vee B_\alpha \in \nabla^+$, $A_\alpha \notin \nabla^+$ and $B_\alpha \notin \nabla^+$. Then for some p, $\nabla^p, A_\alpha \Vdash_R C_\delta$, $\nabla^p, B_\alpha \Vdash_R C_\delta$, but not $\nabla^p, (A \vee B)_\alpha \Vdash_R C_\delta$. By the second deduction theorem either $\nabla^p \Vdash A \to C_{\delta-\alpha}$ and $\nabla^p \Vdash_R C \to C_{\delta-\alpha}$ and $\alpha \subseteq \delta$ or else $\nabla^p \Vdash_R C_\delta$. Since the second is impossible, $\nabla^p \Vdash_R (A \to C)\&(B \to C)_{\delta-\alpha}$. Hence, since $\vdash_R (A \to C)\&(B \to C) \to .A \vee B \to C$, $\nabla^p \Vdash_R A \vee B \to C_{\delta-\alpha}$, and i.e. $\nabla^p, A \vee B_\alpha \Vdash_R C_\delta$. □

Lemma 22. *If no wff in non-null set Σ is L-provable from ∇ then there is an L-ok extension ∇^+ of ∇ w.r.t. any set N which includes all subscripts of ∇ such that no wff in Σ belongs to ∇^+.*

Proof is like the similar lemma where $\Sigma = \{C_\delta\}$ except the ∇^+ is redefined, as follows:

$$\begin{aligned} \nabla^0 &= \nabla \\ \nabla^{j+1} &= \nabla^j \text{ if } \nabla^j, D^j \Vdash_L D_\delta \text{ for } \textit{some } C_\delta \in \Sigma \\ &= \nabla^j \cup \{D^j\} \text{ otherwise} \\ \nabla^+ &= \bigcup_{j<\omega} \nabla^j \end{aligned}$$

Then no wff $C_\delta \in \Sigma$ is L-provable from ∇^+, and ∇^+ is L-ok.

Lemma 23. *If $\Box B_\alpha \notin \nabla$ where ∇ is a $\Box$R-ok set [$\Box$R4-ok set] then there is a set ∇', whose elements comprise each C_δ such that $\Box C_\delta \in \nabla$, such that B_α is not $\Box$R-provable [$\Box$R4-provable] from ∇'.*[25]

Proof. Suppose $C^1_{\delta_1}, \ldots, C^n_{\delta_n} \Vdash_{\Box\mathrm{R}} B_\alpha$. Then by a lemma $\Box C^1_{\delta_1}, \ldots, \Box C^n_{\delta_n} \Vdash_{\Box\mathrm{R}} \Box B_\alpha$. Hence, since ∇ is $\Box$R-ok, $\Box B_\alpha \in \nabla$. $\Box$

Lemma 24. *If C_δ is not $\Box$R-provable from ∇ then there is a $\Box$R-ok extension ∇^+ of ∇ w.r.t. countable set N which includes all subscripts of ∇ and that $C_\delta \notin \nabla^+$.*

Lemma 25. *If $\sim A_\alpha \in \nabla$ where ∇ is E-ok but $\sim (A \to A)_{\alpha+\beta} \notin \nabla$ (or $\Lambda_{\alpha+\beta} \notin \nabla$), and ∇' is any set containing every entailment in ∇ then A_β is not E-provable from ∇'.*

Proof. Suppose $\nabla' \Vdash_E A_\beta$. Then by a lemma since each member of ∇' is an entailment $\nabla' \Vdash_E A \to A \to A_\beta$; so $\nabla' \Vdash_E \sim A \to \sim (A \to A)_\beta$, and $\nabla', \sim A_\alpha \Vdash_E \sim (A \to A)_{\alpha+\beta}$. Hence since ∇ is E-ok $\sim (A \to A)_{\alpha+\beta} \in \nabla$, contradicting the hypothesis. (In the case of $\Lambda_{\alpha+\beta}$, use the principle $\sim (A \to A) \to \Lambda$.) $\Box$

Lemma 26. *If $(B \to C)_\alpha \notin \nabla$ where ∇ is an R-ok [$\Box$R-ok] set, then there is an R-ok [$\Box$R-ok] set Σ which includes ∇ such that $B_{\{k\}} \in \Sigma$ but $C_{\alpha_{\{k\}}} \notin \Sigma$ for some $\{k\}$.*

Proof combines previous lemmata.

Lemma 27. *If $(B \to C)_\alpha \notin \nabla$ where ∇ is an E-ok set then*

(i) there is an E-ok set Σ such that for some $\{k\}$ $B_{\{k\}} \in \Sigma$ and $(D_1 \to D_2)_\delta \in \Sigma$ if $(D_1 \to D_2)_\delta \in \nabla$ but $C_{\alpha+\{k\}} \notin \Sigma$.[26]

[25] Correction: added the prime to the last occurence of ∇ in the lemma.
[26] The rest of this lemma was marked for omission by Routley.

(ii) There is an E-ok set Ⓗ *which contains every subscripted entailment in* ∇ *and does not contain any subscripted negation not in* ∇ *(i.e. if* $\sim D_\delta \notin \nabla$ *then* $\sim D_\delta \notin$ Ⓗ*) such that for some* $\{k\}$ $B_{\{k\}} \in$ Ⓗ *but* $C_{\alpha+\{k\}} \notin$ Ⓗ.

Proof. Proof of (ii):[27] given ∇, there is a set ∇', whose elements comprise every subscripted entailment in ∇ and $B_{\{k\}}$ for suitable $\{k\}$, such that $C_{\alpha+\{k\}}$ is not E-provable from ∇'. Suppose, for some $\sim D_\delta$ not in ∇, $\nabla', B_{\{k\}} \Vdash_E \sim D_\delta$. Then $D_{\{k\}}$ must be used in the proof since $\sim D_\delta$ is not E-provable from ∇'; hence $k \in \delta$. Now choose any k such that for each $\sim D_\delta$ not in ∇ with $B \rightarrow \sim D_{\delta-\alpha} \in \nabla$ for some α, $k \notin \delta$. (Any new k will satisfy these conditions.)

Let Σ be the set consisting of $C_{\alpha+\{k\}}$ and every subscripted negation $\sim D_\delta$ not in ∇. Then Σ is not null and no element of Σ is E-provable from ∇'. Hence, by a lemma ∇' has an E-ok extension, Ⓗ say, such that no element of Σ belongs to Ⓗ. □

Theorem (Completeness Theorems for $\mathbf{R}_f$ and R and $\mathbf{R}^+$).

(i) If A_0 *is not* R_f*-provable from* Γ_0 *then there is an R-model* $\mathfrak{M} = \langle G, K, R, 0, N, P, h\rangle$ *with* K *and* N *denumerable which satisfies* Γ *and falsifies* A*. Hence every* R_f*-consistent set is satisfiable in a denumerable model.*

(ii) If A *is* R_f*-valid then* $\vdash_{R_f} A$.

(iii) If A *is R-valid then* $\vdash_R A$.

(iv) If A *is* R^+*-valid then* $\vdash_{R^+} A$.

Proof. (i). By a lemma there is an R_f-ok set, G say, w.r.t. $\{0\}$, which entails Γ_0 but excludes A_0. Define a canonical R-model $\mathfrak{M}$, with base G, as follows:—
K and N are defined by a joint inductive definition:

(i) $G \in K$ and $0 \in N$

(ii) if for $H_1 \in K$ and $\beta \in N$, $(B \rightarrow C)_\beta \notin H_1$ then by a lemma there is a new (singleton) subscript set γ and an R_f-ok set H_2, which extends H_1, such that $B_\gamma \in H_2$ and $C_{\beta+\gamma} \notin H_2$; fact $H_2 \in K$ and $\gamma \in N$. (A convenient choice of γ is as the set consisting of the first positive integer k not already in N.)

(iii) K is the set consisting of G and its successors.

(iv) N is the closure under set union of the sets assigned to it.

[27] This proof was marked for omission.

It follows, using set theory, that both K and N are denumerable. The remaining components of the canonical model are defined thus:

H_1RH_2 iff for every $\beta \in N$ and every wff C, if $C_\beta \in H_1$ then $C_\beta \in H_2$, i.e. iff $H_1 \subseteq H_2$;
$P(\alpha, H)$ iff $F_\alpha \notin H$, for $\alpha \in N$ and $H \in K$;
$h(p, \alpha, H) = T$ iff $p_\alpha \in H$, for every atomic wff p, every $\alpha \in N$ and every $H \in K$.

$$(*)\ h(A, \alpha, H) = T \text{ iff } A_\alpha \in H, \text{ for every } \alpha \in N \text{ and } H \in K.$$

Proof is by induction from the specified base.
Ad f:

$$\begin{aligned} h(f, \alpha, H) = T &\text{ iff } \sim P(\alpha, H) \\ &\text{ iff } f_\alpha \in H \end{aligned}$$

Ad &:

$$\begin{aligned} h(A\&B, \alpha, H) = T &\text{ iff } h(A, \alpha, H) = T = h(B, \alpha, H) \\ &\text{ iff } A_\alpha \in H \text{ and } B_\alpha \in H, \text{ by the induction hypothesis} \\ &\text{ iff } A\&B_\alpha \in H, \text{ since } H \text{ is } \mathrm{R}_f\text{-ok.} \end{aligned}$$

Ad $\rightarrow$:

(1) If $B \rightarrow C_\alpha \in H$ and HRH' then $B \rightarrow C_\alpha \in H'$; so if $B_\beta \in H'$ then, since H' is R_f-ok, $C_{\alpha+\beta} \in H'$, for any $\alpha, \beta \in N$. Thus if $B \rightarrow C_\alpha \in H$, then $h(B \rightarrow C, \alpha, H) = T$, using the induction hypothesis and applying quantification logic.

(2) If $B \rightarrow C_\alpha \notin H$ for $\alpha \in N$, then by the construction there is an $H' \in K$ and $\beta \in N$ such that HRH' and $B_\beta \in H'$ and $C_{\alpha+\beta} \in H'$. Hence, using the induction hypothesis, $h(B \rightarrow C, \alpha, H) = T$.

(+) $\mathfrak{M}$ is an R-model.

It is immediate that $G \in K$, $0 \in N$ and that N is a set of sets closed under union. Moreover since R is an inclusion relation, R is reflexive and transitive and the hereditariness requirements is satisfied. As to the falsity requirement, suppose $A_\alpha \notin H$. Then $(A \rightarrow f \rightarrow f)_\alpha \notin H$, so for some $\beta \in N$ and $H_1 \in K$ HRH_1 and $F_{\alpha+\beta} \notin H_1$; thus for some β and H_1 HRH_1 and $P(\alpha + \beta, H_1)$. Finally as to the reduction requirement, suppose $(\alpha, H_1) = (\gamma, H_2)$, then for some wff A, $A_\alpha \notin H_1$ and $A_\gamma \in H_2$ say (the other case is similar). Accordingly $(A \rightarrow f \rightarrow f)_\alpha \notin H_1$, whence for some $\beta \in N$ and $H \in K$, H_1RH and $A \rightarrow F_\beta \in H$ and

$f_{\alpha+\beta} \notin H$. Since $A \to F_\beta \in H$ and $A_\gamma \in H_2$ either $\sim HRH_2$ or $f_{\beta+\gamma} \in H_2$. Thus it is false for every β and every H that if H_1RH and $P(\alpha+\beta, H)$ then HRH_2 and $P(\beta+\gamma, H_2)$.

Applying $(*)$ since $A_0 \notin G$, $h(A, 0, G) = F$ and for $B_0 \in \Gamma$, $B_0 \in G$, so $h(B, 0, G) = T$. Hence, since by $(+)$ the canonical model $\mathfrak{M}$ is an R-model, $\mathfrak{M}$ satisfies Γ and falsifies A.

(ii). If A is not a theorem of R_f then A_0 is not R_f-provable from the null set of hypotheses Λ_0. Hence by (i) there is an R-model $\mathfrak{M}$ which falsifies A; so A is not R_f-valid.

(iii). If wff A of R is R-valid then (see §2) A is R_f-valid, so by (i), (ii) A is a theorem of $\mathrm{R}-f$. Hence, since A is a wff of R and R is a conservative extension of R_f (see Meyer [?]), A is a theorem of R.

(iv). As for (ii) but all statements and conditions concerning f and P are deleted. □

A direct proof of the completeness of R may be got as follows:

Theorem (Completeness and Skolem-Löwenheim Theorems for R).

(i) If A_0 is not R-provable from Γ_0 then there is an R-model $\mathfrak{M} = \langle G, K, R, 0, N, P, h\rangle$ with K and N denumerable which satisfies Γ and falsifies A.

(ii) If A is R-valid then $\vdash_R A$.

Proof. Proof of (i) varies the proof of the preceeding theorem at these points.

Since the primitive set $\{\to, \sim, \&\}$ replaces the primitive set $\{\to, f, \&\}$ of R_f, f is not a wff of R. P is redefined thus

$P(\alpha, H)$ iff for every wff C, $\sim (C \to C)_\alpha \notin H$

In terms of R, $f = \sim (p)(p \to p)$.

The induction step for $\sim$ in $(*)$ is proved as follows:—

(1) Suppose $\sim A_\alpha \in H$ and HRH_1 and $A_\beta \in H_1$. Then since $\vdash_R \sim A \to .A \to \sim (A \to A)$, $A \to \sim (A \to A) \in H$. Since HRH_1 and $A_\beta \in H_1$, $\sim (A \to A)_{\alpha+\beta} \in H_1$; hence $\sim P(\alpha+\beta, H_1)$. Hence $\sim A_\alpha \in H \supset .HRH_1 \& P(\alpha+\beta, H_1) \supset .A_\beta \notin H_1$, whence by the induction hypothesis and quantification logic, $\sim A_\alpha \in H \supset .h(\sim A, \alpha, H) = T$.

(2) Suppose $\sim A_\alpha \notin H$; then $A \to \sim (D \to D)_\alpha \notin H$ for arbitrary D. Hence for some H_1 and β, HRH_1 and $A_\beta \in H_1$ and $\sim (D \to D)_{\alpha+\beta} \notin H_1$, i.e. $P(\alpha+\beta, H_1)$. Hence $h(\sim A, \alpha, H) = F$. [The argument requires that for some H_1 and β for *every* D — not

for every D there is some H_1 and β — so its validity may be questioned. I think the substitution of $A \rightarrow\sim (p)(p \rightarrow p) \rightarrow . \sim A$ for $A \rightarrow\sim (D \rightarrow D) \rightarrow\sim A$ makes it plain that the argument is satisfactory. For the skeptical the semantics may be complicated by adding to $\mathfrak{M}$ a class X of individuals; by replacing $P(\alpha, H)$ by $P(\alpha, H, C)$ where $C \in X$, and by complicating appropriately the conditions of P. In the canonical model $\mathfrak{M}$, X is defined as the class of all wff and $P(\alpha, H, C)$ iff $\sim (C \rightarrow C)_\alpha \notin H$.]

Falsity and reduction requirements are established as follows: Suppose $A_\alpha \notin H$, then $\sim\sim A_\alpha \notin H$, so for some $\beta \in N$ and some H_1, HRH_1 and $P(\alpha+\beta, H_1)$, as required. Suppose that $(\alpha, H_1) \neq (\gamma, H_3)$. Then $A_\alpha \notin H_1$ and $A_\gamma \in H_3$ say. Since $A_\alpha \notin H_1$, $\sim\sim A_\alpha \notin H_1$, so by (2), for some H and β, H_1RH and $P(\alpha+\beta, H)$ but $\sim A_\beta \in H_3$. By (1) then for H_3 such that HRH_3, H_1RH and $P(\beta+\gamma, H_3)$, $A_\gamma \in H_3$, contradicting the supposition.[28] In turn, for some H and β, H_1RH and $P(\alpha+\beta, H_1)$ but it is not the case that both HRH_3 and $P(\beta+\gamma, H_3)$, as required. □

Corollary 4. *R_f is a conservative extension of R.*

A normalized R-model $\mathfrak{M}$ is an R-model $\mathfrak{M}$ such that $P(0, G)$.

Corollary 5. *$\vdash_R A$ ($\vdash_{R_f} A$) iff A is true in all normalized R-models.*

Proof. One half is immediate, by specialization. For the other half suppose A is not a theorem of R_f (or R). Then A_0 is not R_f-provable from Λ_0. But also $\sim\vdash_{R_f}$ ($\sim\vdash_R\sim (D \rightarrow D)$ for any D); hence f_0 is not $\mathrm{R}-f$ provable from Λ_0. Now let G be an R_f-ok set including Λ_0 which excludes both A_0 and f_0. The remainder of the completeness is just as before: A canonical R-model $\mathfrak{M}$ with base G is constructed. Moreover, $\mathfrak{M}$ is normalized; since $f_0 \notin G$, $P(0, G)$. □

Corollary 6. *(Meyer–Dunn Theorem for R) Material detachment is admissible for R, i.e. if $\vdash_R A$ and $\vdash_R\sim (A\& \sim B)$ then $\vdash_R B$.*

Proof. Suppose A and $\sim (A\& \sim B)$ are theorems of R but B is not. Then there is a normalized R-model $\mathfrak{M}$ such that $h(A, 0, G) = T = h(\sim (A\& \sim B), 0, G)$ but $h(B, 0, G) \neq T$. Since $h(\sim (A\& \sim B), 0, G)$ by an earlier lemma either $h(\sim A, 0, G) = T$ or $h(B, 0, G) = T$. As the second case is impossible, $h(\sim A, 0, G) = T$. Since, however, $P(0, G)$, $P(0+0, G)$, and $h(A, 0, G) \neq T$, which is impossible. □

[incomplete: breaks down for +ve parts]

[28] Perhaps Routley means $A_\gamma \notin H_3$, which would contradict the supposition.

Theorem (Separation Theorems for R formulated with $\{\rightarrow, \sim, \&\}$ and $\mathbf{R}_f$ formulated with $\{\rightarrow, f, \&\}$).

If A is a theorem of R, of R_f, and L is a fragment of R ($\{\rightarrow\}, \{\rightarrow, \sim\}, \{\rightarrow \&\}$ fragments), or of R_f ($\{\rightarrow\}, \{\rightarrow, f\}, \{\rightarrow \&\}$ fragments), then A is a theorem of L iff A is a wff of L.

Proof. Suppose, for the non-trivial half, that A is a wff of L and a theorem of R. Then A is R-valid, and, since a wff of L, also L-valid. Hence, by the relevant part of the completeness theorem, A is a theorem of L, provided L is a negative fragment of R. □

Theorem (Completeness Theorem for $\mathbf{R}^+$ and Separation Theorems for R formulated with $\{\rightarrow, \sim, \&\}$ and $\mathbf{R}_f$ formulated with $\{\rightarrow, f, \&\}$).

If L is a negation fragment of R, or R_f, ($\{\rightarrow\}, \{\rightarrow\sim\}, \{\rightarrow, f\}, \{\rightarrow \&\}, \{\rightarrow, \vee\}$) then:

(i) If A is a theorem of L, A is L-c-valid.

(ii) If A is a theorem of R, then A is a theorem of L iff A is a wff of L.

(iii) If A is a theorem of R^+ then where L is a fragment of R^+ A is a theorem of L iff A is a wff of L.

Proof. Proof of (i): By a lemma there is a prime L-ok set G which entails Λ_0 w.r.t but excludes A_0 (delete the requirements which do not apply). Define a canonical model $\mathfrak{M}$ with base G as before, except for the following points:— $B \rightarrow C_\beta$ is only considered in case $(B \rightarrow C)$ is a subformula of A; and when a new set $H \in K$ is introduced it is required that they set be a prime L-ok set — such a set is guaranteed by lemmata. $h(p, \alpha, H) = T$ iff $p_\alpha \in H$ for every atomic component p of A and for f, and for every $\alpha \in N$ and $H \in K$. $(*)$ is proved for subformulae of A. The new step for disjunction follows using primeness.

Proof of (ii) as before. □

It differs from the completeness proof that the qualification on the disjunction holding function can be lifted; for as such stage of the construction there is a suitable wff $C_{\alpha+\beta}$ (or $f_{\alpha+\beta}$) not in H.

Corollary 7.

1. *Church's theory of weak implication, R_I, is complete.*

2. *R_I is the pure implicational part of R.*

As to 2. If A is a theorem of R and a wff of R_I then A is a theorem of R_I^- by the preceding theorems. But if then follows, using a Gentzen formulation of R_I^- (got from the Kripke formulation of E_I^- in Belnap & Wallace [**?**] by dropping the restriction to entailments on the left of $\Vdash\rightarrow$; see also Meyer [**?**]).

Theorem (Completeness Theorem and Skolem-Löwenheim Theorem for □R).

(i) If A_0 is not □R-provable from Γ_0 then there is a □R-model $\mathfrak{M} = \langle G, K, R, 0, N, P, W, h\rangle$ with K and N countable which satisfies Γ and falsifies A.

(ii) If A is □R-valid then $\vdash_{\Box R} A$.

(iii) $\vdash_{\Box R}$ iff A is true in every normalized □R-model.

Proof. Proof is like that for R, but K is enlarged as follows:—
If for $H_1 \in K$ and $\beta \in N$, $\Box A_\beta \notin H_1$ then by a lemma there is a □R-ok set H_2 which contains every wff B_γ such that $\Box B_\gamma \in H_1$ such that $A_\beta \notin H_2$. □R-ok sets are of course used in place of R-ok sets. Further:
$H_1 W H_2$ iff for every $\alpha \in N$ and every wff B, if $\Box B_\alpha \in H_1$ then, materially, $B_\alpha \in H_2$.

(∗) $h(A, \alpha, H) = T$ iff $A_\alpha \in H$, for every $\alpha \in N$ and $H \in K$.
Ad □: If $\Box A_\alpha \in H$ then $h(\Box A, \alpha, H) = T$ by the definition of h and W and by quantification logic. Conveniently if $\Box A_\alpha \notin H$ then, by construction, for some H_1, HWH_1 and $A_\alpha \notin H_1$, i.e. by the induction hypotheses, $h(A, \alpha, H_1) = F$. Furthermore if $\Box A_\alpha \notin H$ then since H is □R-ok $\Box A \to f \to f_\alpha \notin H$ (or $\sim\sim \Box A_\alpha \notin H$); hence for some β and some H_2, HRH_2 and $f_{\alpha+\beta} \notin H_2$ i.e. $P(\alpha + \beta, H_2) = T$.[29]

(+) $\mathfrak{M}$ is a □R-model.
For W is reflexive, and since $\vdash_{\Box R} \Box A \to \Box\Box A$, transitive. Since R is an inclusion relation, $H_1 R H_2$ and $H_2 W H_3$ imply $H_1 W H_3$.
The remainder of the proof is like that for R_f (or R). □

Similar results can be proved for □R5. In particular using $f \to \Box f$, if $H_1 W H_2$ & $f_\alpha \notin H_2$ then $f_\alpha \notin H_1$, as required.

The admissibility of material detachment follows, as before, for both □R and □R5. In the case of □R however there is one further case because of the presence of P in the evaluation function for □.

Theorem (Separation theorems for □R formulated with $\{\to, \Box, f, \&\}, \{\to, \Box, \sim, \&\}, \{\to, \Box, f, \&\vee\}, \{\to, \Box, \sim, \&, \vee\}$).
Disjunction is only considered in the fragments $\{\to, \Box, \&, \vee\}$ and $\{\to, \&, \vee\}$; otherwise all proper fragments are considered.
If A is a theorem of □R and L is one of the chosen fragments of □R then A is a theorem of L iff A is a wff of L.

[29] This is certainly a typo.

Proof.

Case 1: L is a fragment including negation or falsity. Then the proof is as usual.

Case 2: L is a fragment not including negation or falsity.

□

[30]

Theorem (Completeness and Skolem-Löwenheim Theorems for EΛ5).

(i) If A_0 is not EΛ5-provable from Γ_0 then there is an EΛ5-model $\mathfrak{M} = \langle G, K, R, 0, N, P, \mathrm{h}\rangle$ with K and N denumerable which satisfies Γ and falsifies A.

(ii) If A is EΛ5-valid then $\vdash_{E\Lambda 5} A$.

Proof. Proof of (i) varies the corresponding result for R_f at these points:—
In the construction of K each new set H_2, which is introduced in order to falsify the subscripted wff $(B \to C)_\beta$ which is not in H_1, is related to H_1 as follows: If $(D_1 \to D_2)_\delta \in H_1$ then $(D_1 \to D_2) \in H_2$. Correspondingly R is defined thus:

> $H_1 R H_2$ iff for every $\beta \in N$ and every wff $(D_1 \to D_2)$ if $(D_1 \to D_2)_\delta \in H_1$ then $(D_1 \to D_2)_\delta \in H_2$. Furthermore $P(\alpha, H)$ iff $\Lambda_\alpha \notin H$.

$(*)$ $h(A, \alpha, H) = T$ iff $A_\alpha \in H$ for $\alpha \in N$ and $H \in K$.

<u>Ad $\sim$</u> (1) Suppose $\sim A_\alpha \notin H$. Then $A \to \Lambda_\alpha \notin H$. Hence for some H_2 and β, HRH_1 and $A_\beta \in H_1$ and $\Lambda_{\alpha+\beta} \notin H_1$, i.e. $P(\alpha+\beta, H_1)$. Hence $h(\sim A, \alpha, H) \neq T$

(2) Suppose $\sim A_\alpha \in H$ and HRH_1 and $P(\alpha+\beta, H_1)$. Then $\Lambda_{\alpha+\beta} \notin H_1$, and, since HRH_1, $N\Lambda_{\alpha+\beta} \notin H$. For if $NA_\gamma \in H$ then $(A \to A) \to A_\gamma \in H$; so if HRH_1 then if $A \to A_0 \in H_1$, as it does, $A_\gamma \in H_1$. Finally then $\Lambda_{\alpha+\beta} \in H$, since $\vdash_{E\Lambda 5} \Lambda \to N\Lambda$. The conditions are simplified to apply a lemma which asserts that A_β is not EΛ5-provable from any set ∇_1 comprising every entailment in H. By the construction of K the only $H \in K$ are obtained by applying a simple extension lemma. Hence for any $H_1 \supseteq \nabla_1$ in K, $A_\beta \notin H_1$. In sum, $\sim A_\alpha \in H$ implies $h(\sim A, \alpha, H) = T$.

$(+)$ $\mathfrak{M}$ is an EΛ5-model.
Since R is an inclusion of entailments relation it is reflexive and transitive. That $P(\alpha, H_2)$ and $H_1 R H_2$ imply $P(\alpha, H_1)$ follows as in (2) where using $\Lambda \to N\Lambda$ and that $NA_\beta \in H_1$ and $H_1 R H_2$ imply $A_\beta \in H_2$. Falsity and reduction requirements follow, using the theorem $\sim\sim A \to A$; just as in the case of system R. □

A <u>normalized</u> EΛ5-model is an EΛ5-model such that $P(0, G)$.

Corollary 8. *$\vdash_{E\Lambda 5} A$ iff A is true in every normalized EΛ5-model.*

Corollary 9. *Material detachment is admissible for EΛ5.*

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[30]The following theorem, proof, definition, and corollaries were marked for omission by Routley. I have indicated this section by horizontal lines.

[Primeness presupposed: also presupposed in superlat[31] theorem]

Theorem (Completeness and Skolem-Löwenheim Theorems for E and E+).

(i) If A_0 is not E-provable from Γ_0 then there is an E-model $\mathfrak{M} = \langle G, K, R, 0, N, \mathrm{h}\rangle$ with K and N denumerable which satisfies Γ and falsifies A. Similarly with E^+ for E.

(ii) If A is E-valid then $\vdash_E A$.

Proof. Proof of (i) follows the same lines are earlier proofs. $G = H_0$ is an E-ok extension of Γ_0 w.r.t. $\{0\}$ which excludes A_0. Then M and N are defined jointly, thus:

(i) $G \in M$ and $\{0\} \in N$.

(ii) if for $H_1 \in M$ and $\beta \in N$ $(B \to C)_\beta \notin H_1$ then by a lemma there is a new (singleton) subscript γ and an E-ok set H_2 such that $B_\gamma \in H_2$, $C_{\beta+\gamma} \notin H_2$ and such that if $(D_1 \to D_2)_\alpha \in H_1$ then $(D_1 \to D_2)_\alpha \in H_1$: put $H_2 \in M$ and $\gamma \in N$.

(iii) M is the new consisting of G and its successions under (ii).

(iv) N is the closure under set union of elements assigned to it.

Further:
if $H_i \in M$ then $J_i \in \overline{M}$, where for every $\beta \in N$ and every wff A, $A_\beta \in J_i$ iff $\sim A_\beta \notin H_i$.
$K = M \cup \overline{M}$.
$H_1 R H_2$ iff for every $\beta \in N$ and every wff $(D_1 \to D_2)$ if $(D_1 \to D_2)_\beta \in H_1$ then $(D_1 \to D_2)_\beta \in H_2$.
$h(p, \alpha, I) = T$ iff $p_\alpha \in I$ for every atomic wff p, every $\alpha \in N$ and $I \in K$; and $h((D_1 \to D_2), \alpha, J) = T$ iff $(D_1 \to D_2)_\alpha \in J$ for every wff $(D_1 \to D_2)$, every $\alpha \in N$, and $J \in \overline{M}$.

(∗) $h(A, \alpha, I) = T$ iff $A_\alpha \in I$ for every wff A, every $\alpha \in N$ and $I \in K$.
Proof is by induction from the specified initial cases.
Ad &: $h(B\&C, \alpha, H_i) = T$ iff $h(B, \alpha, H_i) = T = h(C, \alpha, H_i)$ is proved as before using induction hypothesis and properties of E and sets.

$$\begin{aligned}
h(B\&C, \alpha, J_i) = T &\text{ iff } h(B, \alpha, J_i) = T = h(C, \alpha, J_i)\\
&\text{ iff } B_\alpha \in J_i \text{ and } C_\alpha \in J_i\\
&\text{ iff } \sim B_\alpha \notin H_i \text{ and } \sim C_\alpha \notin H_i\\
&\text{ iff } \sim B_\alpha \vee \sim C_\alpha \notin H_i \text{ by primeness}\\
&\text{ iff } \sim (B\&C)_\alpha \notin H_i \text{ since } \vdash_E \sim B \vee \sim C \leftrightarrow \sim (B\&C)\\
&\text{ iff } B\&C \in J_i
\end{aligned}$$

[31] Unable to read what I have put here as "superlat".

<u>Ad $\vee$:</u> Similar in principle to the & case.
<u>Ad $\sim$:</u>

$$\begin{aligned} h(\sim A, \alpha, H_i) = T &\text{ iff } h(A, \alpha, J_i) \neq T \\ &\text{iff } A_\alpha \notin J_i \\ &\text{iff } \sim A_\alpha \in H_i \end{aligned}$$

$$\begin{aligned} h(\sim A, \alpha, J_i) &\text{ iff } A_\alpha \in H_i \\ &\text{iff } \sim\sim A_\alpha \in H_i \text{ by } \vdash_E \sim\sim A \leftrightarrow A \\ &\text{iff } \sim A_\alpha \notin J_i \end{aligned}$$

<u>Ad $\rightarrow$:</u> $h(B \rightarrow C, \alpha, J) = T$ iff $B \rightarrow C_\alpha \in J$, by stipulation.

(1) If $B \rightarrow C_\alpha \in H$, then, if HRH' and $B_\beta \in H'$[32] then $C_{\alpha+\beta} \in H'$ – is proved as before. Also if $B \rightarrow C_\alpha \in H$ then $\sim C \rightarrow\sim B_\alpha \in H$ since $\vdash_E B \rightarrow C \rightarrow . \sim C \rightarrow\sim B$. So similarly if HRH' then $\sim C_\beta \in H'$ materially implies $\sim B_{\alpha+\beta} \in H'$, i.e. $B_{\alpha+\beta} \in J'$ materially implies $C_\beta \in J'$. Finally since $\vdash_E B \rightarrow C \rightarrow\sim B \vee C$, if $B \rightarrow C_\alpha \in H$ then $\sim B \vee C_\alpha \in H$, so $\sim B_\alpha \in H$ of $C_\alpha \in H$, whence $B_\alpha \notin J$ or $C_\alpha \in J$ and $h(B, \alpha, J) \neq T$ of $h(C, \alpha, H) = T$.

(2) If $C \rightarrow C_\alpha \notin H$ then by construction for some $H' \in K$ and $\beta \in N$ $B_\beta \in H'$, $C_{\alpha+\beta} \notin H'$ and HRH'. Also if $B \rightarrow C_\alpha \notin H$ then $\sim C \rightarrow\sim B_\alpha \notin H$; this, by the construction, for some $H'' \in K$ and $\gamma \in N$, HRH'', $\sim C_\gamma \in H''$ and $\sim B_{\alpha+\gamma} \notin H''$, i.e. $B_{\alpha+\gamma} \in J''$ and $C_\gamma \notin J''$.

Much as before.

(+) $\mathfrak{M}$ is an E-model.

□

Theorem (Completeness and Skolem-Löwenheim Theorems for R using R-I-models).

Statement and proof are like the preceding result; but note:— Step (ii) in the construction of M is carried out as for R^+. H_1RH_2 iff, for every $\beta \in N$ and every wff C, if $C_\beta \in H_1$ then $C_\beta \in H_2$. J_1RJ_2 iff H_1RH_2. The model is an R-I-model, since $C_\beta \in H_1$ and H_1RH_2 materially imply $C_\beta \in H_2$ in virtue of the definition of R. As for the J-case, if $C_\beta \in J_2$ and J_1RJ_2 then $\sim C_\beta \notin H_2$ and H_1RH_2, so $\sim C_\beta \notin H_1$, i.e. $C_\beta \in J_1$.

[32] Correction

Theorem (Completeness and Skolem-Löwenheim Theorems for □R using □R-I-models).

Use □R-ok sets in place of the R-ok sets of the preceding result, and extend M by the following step: if $\Box A_\beta \notin H_1$ for $\beta \in N$ and $H_1 \in M$ then there is a □R-ok set H_2, which contains every wff B_γ such that $\Box B_\gamma \in H_1$ such that $A_\beta \notin H_2$:[33] put H_2 in M.

Theorem (Translation Theorem 1). *A is a theorem of E iff its □R-translation A^+ is a theorem of □R.*

Proof. One half, if $\vdash_E A$ then $\vdash_{\Box R} A$ follows by induction over the E-proof of A. As to the other half, suppose $\sim\vdash_E A$; then there is an E-model $\mathfrak{M} = \langle G, K, W, 0, N, h\rangle$ such that $h(A, O, G) = F$. Form a new model $\mathfrak{M}_1 = \langle G, K, Id, 0, N, W, h\rangle$ where Id is the identity relation on K and remaining elements are as before. Then $\mathfrak{M}_1$ is a □R-I-model which falsifies A^+; hence $\sim\vdash_{\Box R} A$.

□

Theorem (Translation Theorem 2). *A is a theorem of E^+ iff its □R-translation A^+ is a theorem of $\Box R^+$.*

[Primeness assumed]

Theorem (Completeness and Skolem-Löwenheim Theorems for P and P$^+$).

Statement and proof is like that for E and E$^+$ except at the following points:
At point (ii) in the construction of M it required that $m = \max(\gamma)$ exceeds every element of β and of α for $(D_1 \to D_2)_\alpha \in H_1$.
Ad →: If $B \to C_\alpha \in H$ and HRH' and $\max(\beta) \geq \max(\alpha)$ and $B_\beta \in H'$[34] then since $B \to C_\alpha \in H'$ $C_{\alpha+\beta} \in H'$ by the $\to E$ rule for P since H' is P-ok. The remainder is much like before but taking account of maximization requirements.

Theorem (Meyer-Dunn theorem for E and P). *(γ) is admissible: i.e. if $\vdash_L A$ and $\vdash_L \sim A \vee B$ then $\vdash_L B$, where L is P or E.*

Proof. Suppose otherwise that in some L-model $h(A, O, G) = T = h(\sim A \vee B, 0, G)$ and $h(B, 0, G) \neq T$. Since $h \sim A, 0, G) = T$ or $h(B, 0, G) = T$, $h(\sim A, 0, G) = T$; i.e. $h(A, 0, J_0) = F$. But $h(A \to A, 0, G) = T$, so that if $h(A, 0, J_0) = F$ then $h(A, 0, G) = F$ by the reduction condition. Hence $h(A, 0, G) = F$, contradicting $h(A, 0, G) = T$. □

[33] Correction.
[34] Correction.

Because of the unfortunate way negation and disjunction features are used in showing that the implication relation is correct in the canonical model, a separation theorem is not an immediate corollary of completeness theorems. However, some partial results may be obtained by building on independently _____[35] results.

Theorem (Separation theorem from $\{\rightarrow, \sim\}$ part, $\mathbf{E}_I^-$, of E). *If A is a theorem of E then A is a theorem of E_I^- iff A is a wff of E_I^-.*

Proof. Suppose A is a theorem of E and a wff of E_I^-. Then its $\Box$R-translation A^+ is a theorem of $\Box$R; but A^+ is a wff whose only connectives are $\rightarrow, \sim$ and $\Box$. Hence, by the separation theorem of $\Box$R, A^+ is a theorem of the $\{\rightarrow, \sim, \Box\}$ fragment of $\Box$R, i.e. of $\Box\mathrm{R}_I^-$. Then, however, by a result of Meyer [**?**], A is a theorem of E_I^-. $\Box$

Theorem (Separation theorem for the pure entailment part, $\mathbf{E}_I$, of E). *If A is a theorem of E then A is a theorem of E_I iff A is a wff of E_I.*

Proof. By the previous theorem if A is a theorem of E and a wff of E_I then A is a theorem of E_I^-. But it follows using the Belnap-Wallace Gentzen formulations of E_I^- (in [**?**]) that if A is a wff of E_I and a theorem of E_I^-, A is a theorem of E_I. $\Box$

Theorem (Separation theorem for $\mathbf{E}^+$ and $\mathbf{P}^+$). *If A is a theorem of E^+ (P^+) and L is one of these fragments of E (P)[36] — $\{\rightarrow\}, \{\rightarrow, \&\}$— then A is a theorem of L iff A is a wff of L.*

[35] Word unreadable.

[36] Correction.

5 Decidability

5.1

[PDF p. 63] An equivalence class method is used to show that the systems studied have the finite model property (for further details see [?] and [?]).

Where Ψ is a set of wff closed under subformulae, define $(\alpha_1, I_1) \equiv_\Psi (\alpha_2, I_2)$ iff, for every wff $B \in \Psi$, $h(B, \alpha_1, I_1) = T$ iff $h(B, \alpha_2, I_2)$. Then $\equiv_\Psi$ is an equivalence relation which partitions situations (α, I) into equivalence classes; and there are finitely many equivalence classes when Ψ is finite. Next $(\hat{I})_\Psi =_{Df} \{I' : (\cup\alpha \in N).(\alpha, I) \equiv)_\Psi(\alpha, I)\}$ and $(\hat{\alpha})_\Psi =_{Df} \{\alpha' : (\cup I \in K).(\alpha', I) \equiv_\Psi (\alpha, I)\}$. Then relative to a given Ψ $\hat{K} = \{\hat{I} : I \in K\}$, $\hat{H} = \{\hat{\alpha} : \alpha \in N\}$. Also $\hat{h}(A, \hat{\alpha}, \hat{I}) = T$ iff $h(A, \alpha, I) = T$ and $A \in \Psi$, for every initial case (and thus for every atomic wff A). In the case of system R, $\hat{H}_1\hat{R}\hat{H}_2$ iff for every $B \in \Psi$ and every $\alpha \in N$, if $h(B, \alpha, H_1) = T$ then, materially, $h(B, \alpha, H_2) = T$ and $\hat{J}_1\hat{R}\hat{J}_2$ iff $\hat{H}_2\hat{R}\hat{H}_1$. For $\Box$R, $\hat{H}_1\hat{W}\hat{H}_2$ iff for every $B \in \Psi$ and $\alpha \in N$ if $h(\Box B, \alpha, H_1) = T$ then, materially, $h(B, \alpha, H_2) = T$. For E and P, $\hat{H}_1\hat{R}\hat{H}_2$ iff for every wff $B, C \in \Psi$ and every $\alpha \in N$ if $h(B \to C, \alpha, H_1) = T$ then, materially, $h(B \to C, \alpha, H_2) = T$. This specification defined a <u>filtration</u> $\hat{\mathfrak{M}} = \langle \hat{G}, \hat{K}, \hat{R}, \hat{0}, \hat{N}, [\hat{W}], \hat{h} \rangle$ <u>of</u> L-model $\mathfrak{M}$ <u>through</u> Ψ, written $\hat{\mathfrak{M}} = \mathfrak{M}/\Psi$.

Lemma 28. *Where $\mathfrak{M}$ is an L-model (for L = R, $\Box$R, E, P or their parts) then*

(i) If H_1RH_2 then $\hat{H}_1\hat{R}\hat{H}_2$

(ii) If H_1WH_2 then $\hat{H}_1\hat{W}\hat{H}_2$

(iii) $\hat{R}$ is reflexive and transitive

(iv) $\hat{W}$ is reflexive and transitive

(v) where L is R or $\Box$R and $A \in \Psi$, if $\hat{H}_1\hat{R}\hat{H}_2$ and $\hat{\mathrm{h}}(A, \hat{\alpha}, \hat{H}_1) = T$, then $\hat{\mathrm{h}}(A, \hat{\alpha}, \hat{H}_2) = T$, for every $\hat{\alpha} \in \hat{N}$ and every $\hat{H}_1, \hat{H}_2 \in \hat{K}$[37]

Hence $\hat{\mathfrak{M}}$ is an L-model.

Lemma 29. *For every wff $A \in \Psi$, for every $I \in K$, $\hat{\mathrm{h}}(A, \hat{\alpha}, \hat{I}) = T$ iff $\mathrm{h}(A, \alpha, I) = T$.*

Proof is by induction on the number of connectives in A. The basis for initial cases is immediate, & the induction steps for "&", "$\vee$" and "$\sim$" are straightforward. The step for $\to$ is based on the fact that $h(A \to B, \alpha, H) = T$ iff for every β and H' if HRH' and $h(A, \beta, H') = T$ [and $\max(\beta) \geq \max(\alpha)$] then $h(B, \alpha + \beta, H') = T$ and similarly for $\hat{R}$. The case for J situations is an initial case. If $\hat{h}(B \to C, \hat{\alpha}, \hat{H}) = T$ then $h(B \to C, \alpha, H) = T$, since

[37] Last part of sentence cut off. I have merely guessed the last part.

HRH' implies $\hat{H}\hat{R}\hat{H}'$. Conversely, suppose $\hat{h}(B \to C, \hat{\alpha}, \hat{H}) \neq T$ and $\hat{h}(C, \alpha \hat{+} \beta, \hat{H}') = F$, whence $\hat{H}\hat{R}\hat{H}'$ and $h(B, \beta, H') = T$ and $h(C, \alpha + \beta, H') = F$ by the induction hypothesis. Hence, using the definition of $\hat{R}$, $h(B \to C, \alpha, H) \neq T$.

Theorem (Decidability Theorems).

(i) If wff A is false in L-model $\mathfrak{M}$ *then, where* Ⓗ *is the subformula closure of A, A is false in L-model* $\mathfrak{M}$/Ⓗ*;*

(ii) L has the finite model property, and accordingly is decidable; and therefore

(iii) E, P, and R and their isolable fragments are decidable.

(iv) E^+, P^+, *and* R^+ *and their fragments are decidable.*

Proof. Proof of (i). Applying previous lemmata $\hat{\mathfrak{M}} = \mathfrak{M}$/Ⓗ is an L-model, and $\hat{h}(A, \hat{0}, \hat{G}) = F$. Further K and N are finite since there are only finitely many equivalence classes, (α, I) when Ⓗ is finite. □

[How convincing!]

5.3 Decidability using Simplified $\mathbf{R}_f$

Where Ψ is a set of formulae closed under subformulae and including f, define $\alpha_1 \equiv_\Psi \alpha_2$ iff for every wff $B \in \Psi$, $h(B, \alpha_1) = h(B, \alpha_2)$; $(\hat{\alpha})_\Psi = \{\alpha' : \alpha' \equiv_\Psi \alpha\}$.

A filtration $\hat{\mathfrak{M}} = \mathfrak{M}/\Psi = \langle \hat{N}, \hat{0}, \hat{P}, \hat{h} \rangle$ of $\mathfrak{M}$ through Ψ is defined as follows (relative to a given Ψ):

$$\hat{N} = \{\hat{\alpha} : \alpha \in N\}; \hat{\alpha} + \hat{\beta} = \widehat{\alpha + \beta};$$
$$\hat{P}(\hat{\alpha}) \text{ iff } h(f, \alpha) = F \text{ since } f \in \Psi \text{ always;}$$
$$\hat{h}(p, \hat{\alpha}) = T \text{ iff } h(p, \alpha) = T \& p \in \Psi$$

Lemma 30. *Where $\mathfrak{M}$ is a simplified R-model, $\hat{\mathfrak{M}}$ is also.*

Lemma 31. *For every wff $A \in \Psi$, $\hat{\mathrm{h}}(A, \hat{\alpha}) = \mathrm{h}(A, \alpha)$*

Proof. Proof is by induction from the following dual basis:

$$\hat{h}(p, \hat{\alpha}) = T \text{ iff } h(p, \alpha) = T \ \& \ p \in \Psi \text{ iff } h(p, \alpha) = T$$

$$\hat{h}(f, \hat{\alpha}) = T \text{ iff } \sim \hat{P}(\hat{\alpha}) \text{ iff } h(f, \alpha) = T$$

<u>&</u> step is immediate;
<u>$\to$</u>

$$\begin{aligned} \hat{h}(A \to B, \hat{\alpha}) = T \ & \text{iff, for every } \hat{\beta} \in \hat{N}, \hat{h}(A, \hat{\beta}) = T \supset \hat{h}(B, \alpha \hat{+} \beta) = T \\ & \text{iff for every } \beta \in N, h(A, \beta) = T \supset h(B, \alpha + \beta) = T \\ & \quad \text{by the induction hypothesis} \\ & \text{iff } h(A \to B, \alpha) = T \end{aligned}$$

□

Theorem (Decidability for $\mathbf{R}_f$ and R).

6 Semantic Tableaux for the Systems

[PDF p. 66] A tableaux construction <u>for</u> a wff A (i.e. A_0) is begun by putting A_0 in the right column of the two columns of the main tableaux G of the construction. (The exposition presupposes the work of Kripke; see especially [?, p. 72 ft]). The construction is continued, in the case of wff of E (in form □R) and its fragments, by applying the following rules for any tableaux H and any subscript α:—

$\&l$ if $(A\&B)_\alpha$ is on the left of H, put both A_α and B_α on the left of H.

$\&r$ if $(A\&B)_\alpha$ is on the right of H, put either A_α on the right of H of B_α on the right of H. In such a case the tableaux is replaced by alternative cases (in a way well explained in [?]).

$\vee l$ if $(A \vee B)_\alpha$ is on the left of H, put either A_α on the left of H or B_α on the left of H.

$\vee r$ if $(A \vee B)_\alpha$ is on the right of H, put both A_α and B_α on the right of H.

$\rightarrow l$ If $(A \rightarrow B)_\alpha$ is on the left of H, for every tableaux H′ such that HRH′, put either A_β on the right of H' or $B_{\alpha+\beta}$ on the left of H′, for every subscript β in N.[38]

$\rightarrow r$ If $(A \rightarrow B)_\alpha$ is on the right of H begin a new tableaux H′, with A_β for <u>new</u> subscript $\beta \in N$ on the left of H' and $B_{\alpha+\beta}$ on the right of H′, such that HRH′.

N.B. these negation rules are not adequate for negation in combination with disjunction: try $B \rightarrow C \rightarrow . \sim B \vee C$.

$\sim l$ If $\sim A_\alpha$ is on the left of H, put A_α on the right of H for every γ in N such that $P(\alpha+\gamma, H)$.

$\sim r$ If $\sim A_\alpha$ is on the right of H, put A_γ, with new subscript $\gamma \in N$, on the left of H, and set $P(\alpha+\gamma, H)$ to the left of A_γ.

$\sim\sim r$ If A_α is on the right of H, put $\sim\sim A_\alpha$ on the right of H.

For an E-construction, tableaux relation R is assumed to have the same properties as modeling relation R, i.e. to be reflexive and transitive, P to have the same properties as P, and subscript operation + to satisfy the same conditions as its modeling correlate. Subscript set N is of course determined by the construction, beginning with element 0 and being enlarged through application of $\rightarrow r$ and $\sim r$; N is closed under operation +. For constructions for

[38]There is system R, the modeling relation R and tableaux relation R. Tableaux relation R and system R and written without italics. Context will be sufficient to discriminate between uses.

fragments of E, the inapplicable rules are simply deleted. Negation-free constructions have the subformula property. Negation operations are illustrated in the following example:[39,40,41]

$$
\begin{array}{lr|ll}
 & & \sim A \to\sim B \to .B \to A_0 & \\
 & & \text{G R} & \\
 & \sim A \to\sim B_\alpha & B \to A_\alpha & \text{By} \to r \\
 & & H_1 \text{ R} & \\
 & B_\beta & A_{\alpha+\beta} & \text{by} \to r \\
 & A & \sim\sim A_{\alpha+\beta} & \text{by} \sim\sim r \\
\mathrm{P}(\gamma+\alpha+\beta) & \sim A_\gamma & & \text{by} \sim r \\
 & \begin{array}{r|c} 1. & 2. \\ & \sim B_{\alpha+\gamma} \end{array} & \begin{array}{c|l} 1. & 2. \\ \sim A_\gamma & \end{array} & \text{by} \to l
\end{array}
$$

Beth's way of setting out alternatives and showing closure have been adopted in the example, but for more complicated examples it is useful to combine Beth's method with Kripke's method of recopying alternatives (see [**?**, p. 74 ft]).

For P-constructions, and constructions for systems which eliminate implicative suppression, the implication rules are amended by replacing "such that HRH′" in each case by "such that HRH′ and $\max(\beta) \geq \max(\alpha)$". In the case of wff of R, tableaux constructions corresponding to simplified models are easier. In this case the main tableaux G (together with its alternatives) is the only tableaux. The rules for conjunction, disjunction, and negation remain as before except that "H" is replaces throughout by "the tableaux" and deleted from $\mathrm{P}(\gamma, H)$. The implication rules are as follows:

$\to l$ If $(A \to B)_\alpha$ is on the left of the tableaux, put either A_β on the right of the tableaux or B_{α_β} on the left of the tableaux, for every β in N.

$\to r$ If $(A \to B)_\alpha$ is on the right of the tableaux, choose a new subscript β, put A_β on the left of the tableaux and $B_{\alpha+\beta}$ on the right of the tableaux, and put β in N.

A tableaux is closed iff some subscripted wff A_β appears on both sides of the tableaux; a set of tableaux is closed iff some tableaux in it is closed; a system of tableaux is closed

[39]Placement of G, H_1, ..., and the placement of the R slightly modified to fit the tabular environment in LaTeX.

[40]Last line of the following tableaux is cut off.

[41]I have inserted a picture of the original tableaux in the Transcriber's Note at the end of this document.

iff each of its alternative sets is closed; and a subscripted construction is closed iff at some stage of the construction a closed system of alternative sets appears. To facilitate closure it is required that rules are not applied to subscripted wff occurring in a closed tableaux and are not applied in case their result is repetitive, i.e. only reflects an application of the rules that has already been made (perhaps with relettered notation). The presence of A_α on the right of a tableaux and A_β on the left does not ensure closure unless $\alpha = \beta$.

It is advantageous to reformulate the rules so that the constructions may be based on a tree relation S instead of relation R: constructions and formulations based on S are called "s-constructions" and "s-formulations". These reformulations do not, of course, apply to constructions for wff of systems like R where they omit relation R altogether. Consider then an E-s-construction. The construction is begun as before; all the rules are as before except for $\rightarrow r$ where "S" replaces "R" and $\rightarrow l$ which is altered to:

$\rightarrow l$ If $(A \rightarrow B)_\alpha$ is on the left of the H, put either A_β on the right of H or $B_{\alpha+\beta}$ on the left of H, for each subscript β in N, and put $(A \rightarrow B)_\alpha$ on the left of H′ for any H′ such that HSH′.

The $\rightarrow l$ rule for s-constructions for other systems where R is both reflexive and transitive is similarly formulated. The statement as to what is meant by "alternative sets" also has to be reformulated with "S" in place of "R" (for a lucid statement see Kripke [**?**, p. 121]). For systems like $\sqsubset$T, ET, and D where relation R is reflexive but not transitive the s-formulation is a similar modification of the original formulation except for the rule $\rightarrow l$ which is altered to:

$\rightarrow l$ If $(A \rightarrow B)_\alpha$ is on the left of the H, put either A_β on the right of H or $B_{\alpha+\beta}$ on the left of H and of any tableaux H′ such that HSH′, for each subscript β in N.

Lemma 32. *The L-s-construction for A is closed iff the L-construction for A is closed, for each system L for which both constructions have been introduced.*

Proof consists of showing that one construction can be transformed into the other, and vice versa.

Note how we split $\rightarrow l$ into two parts— as for intuitionistic logic.

Theorem. *The L-construction for A is closed iff A is L-valid; for each semantical system L introduced.*

Proof reduces to two lemmas, and in each of these the arguments of Kripke ([**?**, p. 76–79]) are adopted.

Lemma 33. *If the L-construction for A is closed the A is L-valid.*

Proof. Suppose the L-construction is closed by A is not L-valid. Then there is an L-model $\mathfrak{M} = \langle G, K, R, N, h \rangle$ such that $h(A, \{\ \}, G) = F$. Also for each n, at the n^{th} stage of the closed L-construction, there is an alternative set $\mathfrak{D}$ of the construction and a mapping ϑ, mapping tableaux of $\mathfrak{D}$ into elements of K such that
$(*)$ If H is a tableaux of $\mathfrak{D}$, $H = \vartheta(\mathrm{H})$ and B_α is any wff occurring on the left (right) of H then $(B, \alpha, H) = T\ (F)$. Furthermore, if H_1 and H_2 are in $\mathfrak{D}$ and $H_1 = \vartheta(\mathrm{H}_1)$ and $H_2 = \vartheta(\mathrm{H}_2)$ then $\mathrm{H}_1\mathrm{R}\mathrm{H}_2$ implies $H_1 R H_2$.
Proof is by induction on n. For $n = 1$ there is only one tableaux G with $A_{\{\ \}}$ on the right; but $h(A, \{\ \}, G) = F$, as required. For the induction step suppose that in realizing the $(n+1)$ stage one of the rules is applied to some tableaux H of $\mathfrak{D}$, and that $(*)$ holds for all stages up to and including the n^{th}.
(incomplete in detail) □

Generally each semantical system yields a corresponding tableaux system.

7 Deductive Tableaux and Natural Deduction and an Alternative Route to Completeness

[PDF p. 72] Deductive tableaux are specifically arranged semantics-tableaux in which certain formulae are repeated. The specific arrangement and repetition is required in order that closed tableaux may be mechanically transformed into Fitch-style natural deduction proofs. (On deductive tableaux and their conversion into natural deduction proofs see Barth [**?**] and [**?**] and the papers of Beth referred to therein; and on Fitch-style natural deduction proofs for E, P, R and their fragments, see Anderson [**?**] and Anderson and Belnap [**?**]).

In presenting the rules for natural deduction tableaux the rules and format of Barth [**?**], for classical sentential logic, are adopted. In the rules that follow Δ is the class of all wff in the right column, Δ', Δ'' are subclasses of Δ, and Σ, Σ' are subclasses of Σ; each of these classes may be null. Λ is the null class of wff. The rules are given the form of transformation rules. The table on the left of the symbol "$\longrightarrow$" is the resulting table after application of the rule. On the far right in each case the ensuing natural deduction is displayed. Subsequently natural deduction rules are provided which ensure that the resulting natural deduction is valid provided the dotted vertical lines can be filled in correctly. The deductive tableaux rules stated are those for E and fragments. Qualifications needed for rules for P are stated, where required, in square brackets.

<u>Tableaux Closure:</u>

$$\begin{array}{r|l} \text{L} & \text{R} \\ \hline \Delta' & \Sigma' \\ \text{A} & \text{A} \\ \Delta'' & \end{array} \quad \longrightarrow \quad \begin{array}{r|l} \text{L} & \text{R} \\ \hline \Delta' & \Sigma' \\ \text{A} & \text{A} \\ \Delta'' & \\ \hline\hline \end{array}$$

<u>Repetition (Rep)</u>

$$\begin{array}{l} \Delta' \\ A \\ \Delta'' \\ A \text{ (Rep)} \\ \Sigma' \end{array}$$

<u>$\rightarrow r$:</u>

$$\begin{array}{c|c} \text{L} & \text{R} \\ \hline \Delta & \Sigma' \\ & A \rightarrow B_\alpha \\ & \Lambda \end{array} \quad \longrightarrow \quad \begin{array}{c|c} \text{L} & \text{R} \\ \hline \Delta & \Sigma' \\ & A \rightarrow B_\alpha \\ \dots & \dots \\ A_\beta & B_{\alpha+\beta} \end{array}$$

(β new)

<u>Implication Introduction ($\rightarrow I$)</u>

$$\begin{array}{l} \Delta \\ \ulcorner A_\beta \quad \urcorner \\ \vdots \\ \llcorner B_{\alpha+\beta} \lrcorner \\ A \rightarrow B_\alpha \;\; \rightarrow I \\ \Sigma' \end{array}$$

[For P: β new and $\max(\beta) \geq \max(\alpha)$]

<u>$\rightarrow l$:</u>

$$\begin{array}{r|l} \text{L} & \text{R} \\ \hline \Delta' & \Sigma \\ A \rightarrow B_\alpha & \\ \Delta'' & \end{array} \quad \longrightarrow \quad \begin{array}{cc|cc} \multicolumn{2}{r|}{\text{L}} & \multicolumn{2}{c}{\text{R}} \\ \hline \multicolumn{2}{r|}{\Delta'} & \multicolumn{2}{c}{\Sigma} \\ \multicolumn{2}{r|}{A \rightarrow B_\alpha} & & \\ \hline 1. & \multicolumn{1}{|c|}{2.} & 1. & \multicolumn{1}{|c}{2.} \\ & \multicolumn{1}{|c|}{B_{\alpha+\beta}} & A_\beta & \multicolumn{1}{|c}{} \end{array}$$

(any β)

<u>Implication Elimination ($\rightarrow E$)</u>

$$\begin{array}{l} \Delta' \\ A \rightarrow B_\alpha \\ \Delta'' \\ \vdots \\ A_\beta \\ B_{\alpha+\beta} \;\; \rightarrow E \\ \vdots \\ \Sigma \end{array}$$

[For P: for any β with $\max(\beta) \geq \max(\alpha)$]

The particular form of the result of the transformation is selected to work with the convention for linearizing tableaux that is chosen.

$\rightarrow$ transmission:[42]

$$
\begin{array}{r|l}
\text{L} & \text{R} \\ \hline
\Delta' & \Sigma' \\
A \rightarrow B_\alpha & \\
\Delta'' & \\
\dots & \dots \\
\Delta''' & \Sigma''
\end{array}
\quad \longrightarrow \quad
\begin{array}{r|l}
\text{L} & \text{R} \\ \hline
\Delta' & \Sigma' \\
A \rightarrow B_\alpha & \\
\Delta'' & \\
\dots & \dots \\
\Delta''' & \Sigma'' \\
\dots & \dots \\
A \rightarrow B_\alpha &
\end{array}
$$

Reiteration (Reit)

$$
\begin{array}{l}
\Delta' \\
A \rightarrow B_\alpha \\
\Delta'' \\
\ulcorner \Delta''' \urcorner \\
\quad A \rightarrow B_\alpha \ \text{Reit} \\
\llcorner \Sigma'' \lrcorner \\
\Sigma'
\end{array}
$$

$\&r$:

$$
\begin{array}{c|c}
\text{L} & \text{R} \\ \hline
\Delta & \Sigma' \\
 & A\&B_\alpha \\
 & \Lambda
\end{array}
\quad \longrightarrow \quad
\begin{array}{cc|cc}
\multicolumn{2}{r|}{\text{L}} & \multicolumn{2}{c}{\text{R}} \\ \hline
\multicolumn{2}{r|}{\Delta} & \multicolumn{2}{c}{\Sigma'} \\
\multicolumn{2}{r|}{} & \multicolumn{2}{c}{A\&B_\alpha} \\ \hline
1. & \multicolumn{1}{|c|}{2.} & 1. & \multicolumn{1}{|c}{2.} \\
 & \multicolumn{1}{|c|}{} & A_\alpha & \multicolumn{1}{|c}{B_\alpha}
\end{array}
$$

Conjunction Introduction ($\&I$)

$$
\begin{array}{l}
\Delta \\
\vdots \\
A_\alpha \\
\vdots \\
B_\alpha \\
A\&B_\alpha \quad \&I \\
\Sigma'
\end{array}
$$

$\&l$:

$$
\begin{array}{r|l}
\text{L} & \text{R} \\ \hline
\Delta' & \Sigma \\
A\&B_\alpha & \\
\Delta'' &
\end{array}
\quad \longrightarrow \quad
\begin{array}{r|l}
\text{L} & \text{R} \\ \hline
\Delta' & \Sigma \\
A\&B_\alpha & \\
\Delta'' & \\
A_\alpha & \\
B'_\alpha &
\end{array}
$$

Conjunction Elimination ($\&I$)

$$
\begin{array}{l}
\Delta' \\
A\&B_\alpha \\
\Delta'' \\
A_\alpha \quad \&E \\
B_\alpha \quad \&E \\
\vdots \\
\Sigma
\end{array}
$$

[42]The last line in both the right tableaux and the natural deduction proofs are cut off.

$\underline{\vee r}$:

$$
\begin{array}{c|c}
\text{L} & \text{R} \\ \hline
\Delta & \Sigma' \\
 & A \vee B_\alpha \\
 & \Lambda
\end{array}
\quad \longrightarrow \quad
\begin{array}{cc|cc}
 & \text{L} & \text{R} & \\ \hline
 & \Delta & \Sigma' & \\
 & & A \vee B_\alpha & \\ \hline
1. & \multicolumn{1}{||c|}{2.} & 1. & \multicolumn{1}{||c}{2.} \\
 & \multicolumn{1}{||c|}{} & A_\alpha & \multicolumn{1}{||c}{B_\alpha}
\end{array}
$$

<u>Disjunction Introduction ($\vee I$)</u>

$$
\begin{array}{lll}
\Delta & & \text{or } \Delta \\
\vdots & & \vdots \\
A_\alpha & & B_\alpha \\
A \vee B_\alpha & & A \vee B_\alpha \\
\Sigma' & & \Sigma'
\end{array}
$$

The double vertical lines indicate that only one of the subtableaux need be closed for the tableaux to be closed. This form is chosen so that deductive tableaux rules are always applied on the right first.

$\underline{\vee l}$:

$$
\begin{array}{r|l}
\text{L} & \text{R} \\ \hline
\Delta' & \Sigma' \\
A \vee B_\alpha & C_{\alpha+\beta} \\
\Delta'' &
\end{array}
\quad \longrightarrow \quad
\begin{array}{cc|cc}
 & \text{L} & \text{R} & \\ \hline
 & \Delta' & \Sigma' & \\
 & A \vee B_\alpha & C_{\alpha+\beta} & \\
 & \Delta'' & & \\ \hline
\multicolumn{1}{c|}{1.} & 2. & \multicolumn{1}{c|}{1.} & 2. \\
\multicolumn{1}{c|}{A_\alpha} & B_\alpha & \multicolumn{1}{c|}{C_{\alpha+\beta}} & C_{\alpha+\beta}
\end{array}
$$

<u>Disjunction Elimination ($\vee E$)</u>

$$
\begin{array}{ll}
\Delta' & \\
A \vee B_\alpha & \\
\Delta'' & \\ \hline
A_\alpha & - \\
\vdots & \\
C_{\alpha+\beta} & - \\ \hline \hline
B_\alpha & - \\
\vdots & \\
C_{\alpha+\beta} & - \\ \hline
C_{\alpha+\beta} & \vee E \\
\Sigma' &
\end{array}
$$

<u>$\sim r$:</u>

$$
\begin{array}{c|c}
\mathrm{L} & \mathrm{R} \\
\hline
\Delta & \Sigma' \\
 & \sim A_\alpha \\
 & \Lambda
\end{array}
\quad \longrightarrow \quad
\begin{array}{rr|l}
 & \mathrm{L} & \mathrm{R} \\
\hline
 & \Delta & \Sigma' \\
 & & \sim A_\alpha \\
 & & \\
\mathrm{P}(\alpha+\gamma) & A_\gamma & \sim A_\alpha
\end{array}
$$

<u>Negation Introduction ($\vee E$)</u>

$$
\begin{array}{ll}
 & \Delta \\
\cline{2-2}
P(\alpha+\beta) & A_\gamma \qquad - \\
 & \vdots \\
 & \sim A_\alpha \quad - \\
\cline{2-2}
 & \sim A_\alpha \\
 & \Sigma'
\end{array}
$$

<u>$\sim\sim r$:</u>

$$
\begin{array}{c|c}
\mathrm{L} & \mathrm{R} \\
\hline
\Delta & \Sigma' \\
 & A_\alpha \\
 & \Lambda
\end{array}
\quad \longrightarrow \quad
\begin{array}{c|c}
\mathrm{L} & \mathrm{R} \\
\hline
\Delta & \Sigma' \\
 & A_\alpha \\
 & \sim\sim A_\alpha
\end{array}
$$

<u>Double Negation Elimination ($\vee E$)</u>

$$
\begin{array}{ll}
\Delta & \\
\vdots & \\
\sim\sim A_\alpha & \\
A_\alpha & \sim\sim E \\
\Sigma' &
\end{array}
$$

<u>$\sim l$:</u>

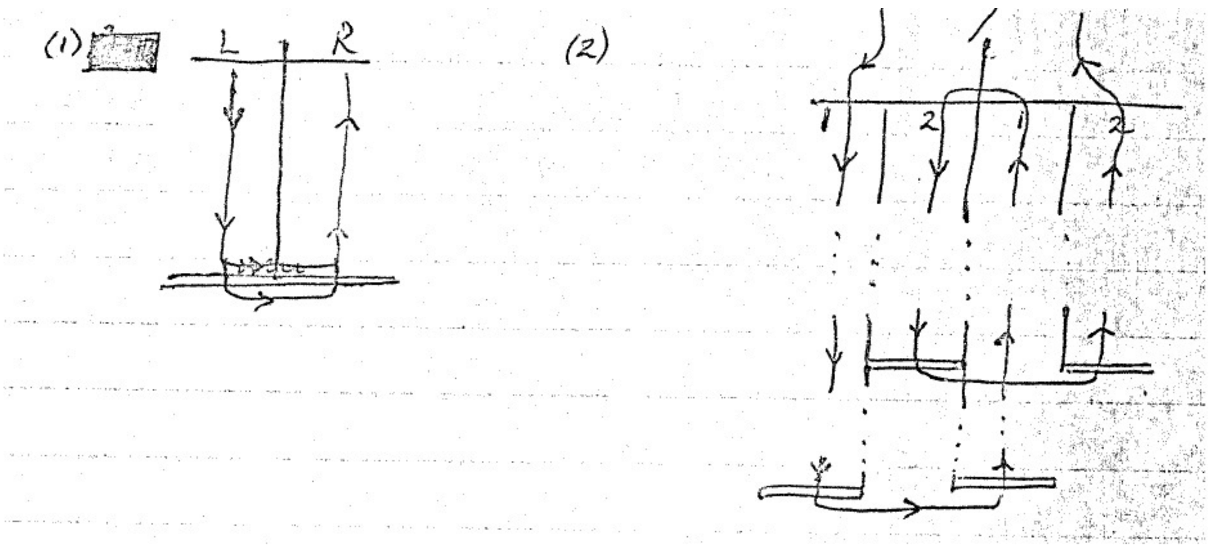
$$
\begin{array}{rr|l}
 & \mathrm{L} & \mathrm{R} \\
\hline
\mathrm{P}(\beta+\gamma) & \Delta' & \Sigma' \\
 & \sim B_\beta & \\
 & \Delta'' &
\end{array}
\quad \longrightarrow \quad
\begin{array}{rr|l}
 & \mathrm{L} & \mathrm{R} \\
\hline
\mathrm{P}(\beta+\gamma) & \Delta' & \Sigma' \\
 & \sim B_\beta & \\
 & \Delta'' & B_\gamma
\end{array}
$$

<u>Negation Elimination ($\sim E$)</u>

$$
\begin{array}{ll}
P(\beta+\gamma) & \Delta \\
 & \sim B_\beta \\
 & \Delta'' \\
 & \vdots \\
 & B_\gamma \\
 & \Sigma'
\end{array}
$$

Once again the negation rules are unsatisfactory.

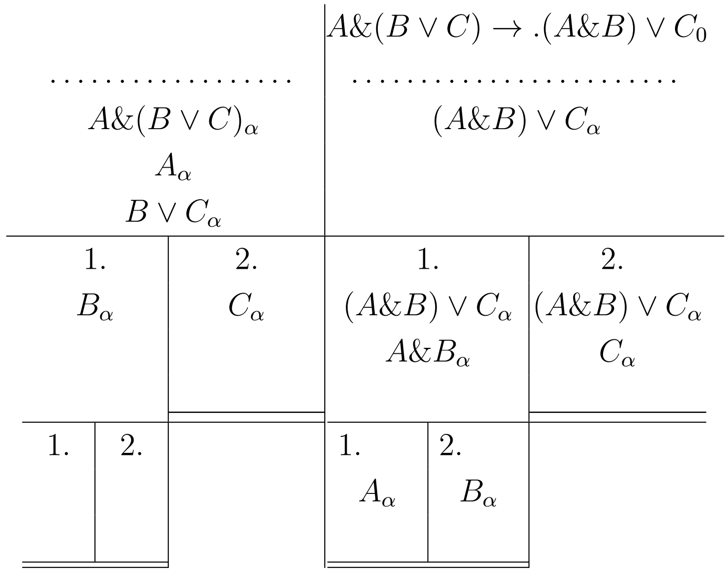
The general method of linearization of closed deductive tableaux is best indicated diagrammatically:



A closed tableaux is transformed into a natural deduction sequence by rewriting the formulae in the tableaux in a vertical arrangement in the order in which they appear along the arrow in this linearization diagram, and by inserting in the resulting vertical arrangement hypotheses introduction and removal signs and marginal notes as to natural deduction rules applied. Both of the later features are listed systematically along with the rules given above for each connective.

To illustrate two important examples of closed deductive tableaux and their transformations in natural deduction form are given.

(i) Distribution (E11)



$\ulcorner A\&(B \vee C)_\alpha \quad \urcorner$	Hyp
A_α	$\&E$
$(B \vee C)_\alpha$	$\&E$
A_α	Rep
$\overline{B_\alpha \quad -}$	Hyp
A_α	Reit
B_α	Rep
$(A\&B)_\alpha$	$\&I$
$\underline{(A\&B) \vee C_\alpha \quad -}$	$\vee I$
$\overline{C_\alpha \quad -}$	Hyp
$\underline{(A\&B) \vee C_\alpha \quad -}$	$\vee I$
$\llcorner (A\&B) \vee C_\alpha \quad \lrcorner$	$\vee E$
$A\&(B \vee C) \rightarrow .(A\&B) \vee C$	$\rightarrow I$

(ii) Contraposition (E13)

		$A \to\sim B \to .B \to\sim A_0$	
$A \to\sim B_\alpha$		$B \to\sim A_\alpha$	
B_β $P(\alpha+\beta+\gamma) \quad A_\gamma$ $A \to\sim B_\alpha$		$\sim A_{\alpha+\beta}$ $\sim A_{\alpha+\beta}$	
1.	2. $\sim B_{\alpha+\gamma}$	1. A_γ	2. B_β

	$\ulcorner A \to\sim B_\alpha \urcorner$	Hyp
	$\ulcorner B_\beta \urcorner$	Hyp
$P(\alpha+\beta+\gamma)$	$\overline{A_\gamma}$	Hyp
	$A \to\sim B_\alpha$	Reit
	A_γ	Rep
	$\sim B_{\alpha+\gamma}$	$\to E$
	B_β	Rep
	$\underline{\sim A_{\alpha+\beta}}$	
	$\llcorner \sim A_{\alpha+\beta} \lrcorner$	$\sim I$
	$\llcorner B \to\sim A_\alpha \lrcorner$	$\to I$
	$A \to\sim B \to .B \to\sim A_0$	$\to I$

For R-deductive tableaux, the $\rightarrow$-transmission rules is deleted and the implication rule $\rightarrow r$ is amended to the following:

$\underline{\rightarrow r}$:

$$\begin{array}{c|c} \text{L} & \text{R} \\ \hline \Delta & \Sigma' \\ & A \rightarrow B_\alpha \\ & \Lambda \end{array} \quad \longrightarrow \quad \begin{array}{c|c} \text{L} & \text{R} \\ \hline \Delta & \Sigma' \\ & A \rightarrow B_\alpha \\ A_\beta & B_{\alpha+\beta} \end{array}$$

(β new)

Implication Introduction ($\rightarrow I$)

$$\begin{array}{ll} \Delta & \\ \ulcorner A_\beta \quad \text{Hyp}\urcorner & \\ \vdots & \\ \llcorner B_{\alpha+\beta} \qquad \lrcorner & \\ A \rightarrow B_\alpha & \rightarrow I \\ \Sigma' & \end{array}$$

The remaining rules are just as for E-deductive tableaux.

Theorem. *If the L-s-construction for A is closed then the L-deductive tableaux construction for A is closed.*

The Fitch-style natural deduction systems, E*, P* and R*, introduced differ from those of Anderson and Belnap, in particular in that two sets of hypotheses are admitted. (However the systems of Anderson and Belnap are taken for granted as background knowledge; see especially [?]). In stating the rules there for E* one takes as central; and differences and qualifications needed for P* and R* are noted where needed. Such standard features of natural deduction systems as vertical arrangement subproof arrangement are taken for granted.

(i) Structural Rules:

New World Hypotheses (N. Hyp). A step B_α may be introduced as the new world hypothesis of a new subproof, where each new hypothesis B receives a new subscript α from N. The introduction of such subscripted hypothesis is marked by the sign $\ulcorner \cdots \urcorner$ written above it, and 'N. Hyp' written to its right, and the hypothesis is eliminated with the paired sign $\llcorner \ldots \lrcorner$.

Ordinary Hypothesis (O. Hyp). A step B_α may be introduced as the ordinary hypothesis in the application of an extensional logical rule of a new subproof. The introduction of such a hypothesis is marked by the signs '$\overline{\quad\quad -}$' and its removal by the coupled sign '$\underline{\quad\quad -}$'.[43]

Repetition. B_α by be repeated within a proof or subproof.

Reiteration. B_α may be reiterated, retaining its subscript, in

(i) ordinary hypothetical subproofs, with no restriction;

(ii) new world hypothetical subproofs, provided B has the form $C \rightarrow D$.

In the case of R* B_α may be reiterated into new world hypothetical subproofs whatever its

[43]The signs I have chosen to use differ slightly from those Routley used.

form. Hence the distinction between N. Hypotheses and O. Hypotheses largely vanishes in R and by reshaping the implication rules N. hypotheses can be eliminated altogether from R^*.

(ii) Logical Rules:

These rules have already been exhibited schematically. They are, to summarize, as follows:

(iia) Implicational rules.

$\to I$. From a proof of $B_{\alpha+\beta}$ on hypothesis A_β to infer $A \to B_\alpha$. For P^* it is required that $\max(\beta) \geq \max(\alpha)$. In the case of R^*, where N. Hypotheses are eliminated, the rule is modified to:

$\to I(R)$. From a proof of $B_{\alpha+\beta}$ on hypothesis A_β to infer $A \to B_\alpha$, provided β is a new label from N except in the case of $\vee E$ and $\sim I$ below.

$\to E$. From A_β and $A \to B_\beta$ to infer $B_{\alpha+\beta}$. For P^* it is required that $\max(\beta) \geq \max(\alpha)$.

(iib) Extensional Rules.

$\&I$. From A_α and B_α to infer $A\&B_\alpha$.

$\&E$. From $(A\&B)_\alpha$ to infer both A_α and B_α.

$\vee I$. From A_α to infer $(A \vee B)_\alpha$. From B_α to infer $(A \vee B)_\alpha$.

$\vee E$. From $(A \vee B)_\alpha$ and a proof of $C_{\alpha+\beta}$ on O. hypothesis A_α and a proof of $C_{\alpha+\beta}$ on O. hypothesis A_α to infer $C_{\alpha+\beta}$.

$\sim\sim E$. From $\sim\sim A_\alpha$ to infer A_α.

$\sim I$. From a proof of $\sim A_\alpha$ on O. hypothesis A_γ where $P(\alpha + \gamma)$ to infer $\sim A_\alpha$.

$\sim E$.From $\sim A_\beta$ where $P(\beta + \gamma)$ to infer A_γ.

Theorem. *If the deductive L-tableaux for A is closed then A is a theorem of L^*.*

Proof. Transform the deductive L-tableaux into vertical form. Then no gaps remain since the L-tableaux is closed, so a proof in L^* results. □

Theorem (Anderson Completeness). *If A is a theorem of L^*, then A is a theorem of L. This may be proved either by the methods of Anderson [**?**], or using deduction theorems. Note that the case ____[44] of proof could be applied directly to deductive tableaux.*

[44]Word cannot be deciphered.

8 Reversed Tableaux and Completeness Through Gentzen Methods

9 Independent Gentzen Formulations of the Positive Systems; A Gentzen Form of $\mathbf{R}^+$: $\mathbf{R}_*^+$

[PDF p. 83]

Axiom Scheme $\quad C_\alpha \Vdash C_\alpha$

In the following formulation Γ, Ⓗ etc are sets of subscripted wff. A singular formulation is given.

Structural rules. in antecedent

Weakening (Thinning):

$$\frac{\Gamma \Vdash D_\delta}{C_\alpha, \Gamma \Vdash D_\delta}$$

Contraction:

$$\frac{C_\alpha, C_\alpha, \Gamma \Vdash D_\delta}{C_\alpha, \Gamma \Vdash D_\delta}$$

Interchange:

$$\frac{\Lambda, C_\alpha, D_\beta Ⓗ \Vdash E_\eta}{\Lambda, D_\beta, C_\alpha, Ⓗ \Vdash E_\eta}$$

Logical rules.

in succedent $\qquad$ in antecedent

$$\underline{\to} \quad \frac{A_\alpha, \Gamma \Vdash B_\beta}{\Gamma \Vdash A \to B_{\beta-\alpha}} \qquad \frac{\Delta \Vdash A_\alpha \qquad B_{\alpha+\beta}, \Gamma \Vdash D_\delta}{A \to B_\beta, \Delta, \Gamma \Vdash D_\delta}$$

Provided $\alpha \neq 0$, $\alpha \subseteq \beta$, $\alpha \notin \delta$ for $C_\delta \in \Gamma$.

$$\underline{\&} \quad \frac{\Gamma \Vdash A_\alpha \qquad \Gamma \Vdash B_\alpha}{\Gamma \Vdash (A\&B)_\alpha} \qquad \frac{A_\alpha, \Gamma \Vdash D_\delta}{(A\&B)_\alpha, \Gamma \Vdash D_\delta}$$

$$\frac{B_\alpha, \Gamma \Vdash D_\delta}{(A\&B)_\alpha, \Gamma \Vdash D_\delta}$$

$$\underline{\vee}\quad \frac{\Gamma \Vdash A_\alpha}{\Gamma \Vdash (A \vee B)_\alpha} \qquad\qquad \frac{A_\alpha, \Gamma \Vdash D_\delta \qquad B_\alpha, \Gamma \Vdash D_\delta}{(A \vee B)_\alpha, \Gamma \Vdash D_\delta}$$

$$\frac{\Gamma \Vdash B_\alpha}{\Gamma \Vdash (A \vee B)_\alpha}$$

$$\underline{\text{Cut}}\quad \frac{\Delta \Vdash C_\delta \qquad C_\delta, \Gamma \Vdash D_\gamma}{\Delta, \Gamma \Vdash D_\gamma}$$

A Cut-Free Reformulation. $\mathbf{R}^+_*$

The following changes are made to the first formulation.
Weakening:

$$\frac{\Gamma \Vdash D_\delta}{C_\alpha, \Gamma \Vdash D_\delta} \quad \text{provided } \alpha \subseteq \delta$$

$\underline{\Vdash\to}$

$$\frac{A_\alpha, \Gamma \Vdash B_\beta}{\Gamma \Vdash A \to B_{\beta-\alpha}}$$

provided $\alpha \neq 0$, $\alpha \subseteq \beta$, $\alpha\ disj\ \beta$, for $C_\delta \in \Gamma$

$\underline{\to\Vdash}$

$$\frac{\Gamma \Vdash A_\alpha \qquad B_{\alpha+\beta}, \Gamma \Vdash D_\delta}{A \to B_\beta, \Gamma \Vdash D_\delta}$$

Cut is omitted.

Lemma 34. *If $\Gamma \Vdash D_\delta$ and $C_\alpha \in \Gamma$ then $\alpha \subseteq \delta$.*

Proof. Proof is by induction over the rules. The one case that is not immediate is $\to\Vdash$. Suppose $C_\gamma \in \Gamma$. Then $\gamma \subseteq \beta$. But also $\alpha \subseteq \beta$ and $\alpha\ disj\ \gamma$; hence $\gamma \subseteq \beta - \alpha$. □

Theorem (Elimination theorem for $\mathbf{R}^+_*$). *If $\Delta \Vdash C_\delta$ and $C_\delta, \Gamma \Vdash D_\gamma$ then $\Delta, \Gamma \Vdash D_\gamma$.*

Proof. Cut may be replaced by the following rule Mix:

$$\frac{\Delta \Vdash M_\delta \qquad \Sigma \Vdash D_\gamma}{\Delta, \Sigma_{M_\delta} \Vdash D_\gamma} \ \text{(Mix)},$$

where $M_\delta \in \Sigma$ and sequence Σ_{M_δ} is obtained from Σ by suppressing all occurrences of M_δ.

Cut follows from Mix, and thus:

$$\dfrac{\dfrac{\Delta \Vdash C_\delta \qquad C_\delta, \Gamma \Vdash D_\gamma}{\Delta, \Gamma_{C_\delta} \Vdash D_\gamma}\ \text{Mix}}{\Delta, \Gamma \Vdash D_\gamma}$$

by weakening, since, by a premise , $\delta \subseteq \gamma$.

$$\dfrac{\Delta \Vdash M_\delta \qquad \dfrac{\Sigma \Vdash D_\gamma}{M_\delta, \Sigma_{M_\delta} \Vdash D_\gamma}}{\Delta, \Sigma_{M_\delta} \Vdash D_\gamma}$$

Proof that all cases of Mix may be eliminated follows Kleene [**?**, p. 54 ft]. Kleene's definition of <u>rank</u> is applied: M is replaced by M_δ in the definitions since M_δ is now the mix formula.[45]

The case structure of the double induction is same as in Kleene, but some cases treated by Klenne no longer occur in a simpler formulation. In some remaining cases subscript induction has to be established. The mix to be established is written:

$$\dfrac{\Pi \Vdash M_\delta \qquad \Sigma \Vdash D_\gamma}{\Pi, \Sigma_{M_\delta} \Vdash D_\gamma}$$

or briefly

$$\dfrac{S_1 \qquad S_2}{S_3}$$

where $M_\delta \in \Sigma$.

A. Preliminary cases

<u>Case 1a.</u> $M_\delta \in \Pi$. If $C_\alpha \in \Pi$ then, in view of S_1, $\alpha \subseteq \delta$, since $M_\delta \in \Sigma$, $\delta \subseteq \gamma$; so $\alpha \subseteq \delta$. Hence S_3 arises from S_2 by weakening.

<u>Case2a.</u> does not occur, but case 2b does:—

$$\dfrac{\Pi \Vdash D_\delta \qquad \dfrac{Ⓗ \Vdash D_\gamma}{M_\delta, Ⓗ \Vdash D_\gamma}\ T \quad \text{with } M_\delta \in Ⓗ}{\Pi, Ⓗ_{M_\delta} \Vdash D_\gamma}\ \text{Mix}$$

Since $\Pi \Vdash M_\delta$, if $C_\alpha \in \Pi$, $\alpha \subseteq \gamma$. Thus the proof figure may be converted to eliminate Mix, thus:

$$\dfrac{Ⓗ \Vdash D_\gamma}{\Pi, Ⓗ_{M_\delta} \Vdash D_\gamma}\ T \quad (Ⓗ_{M_\delta} = Ⓗ)$$

B. Further cases. These cases differ from those in Kleene in only the matter of showing that relevant conditions are satisfied. Main examples:—

B1, where rank is 2.

<u>Case 3</u>

$$\dfrac{\dfrac{A_\alpha, \Pi \Vdash B_{\alpha+\beta}}{\Pi \Vdash A \to B_\beta}\ \text{(with conditions)} \qquad \dfrac{\Gamma \Vdash A_\alpha \qquad B_{\alpha+\beta}, \Gamma \Vdash D_\delta}{A \to B_\beta, \Gamma \Vdash D_\delta}}{\Pi, \Gamma \Vdash D_\delta}\ \text{Mix}$$

[45] There may be an addition cut off at this point.

Since $\alpha \subseteq \beta + \alpha \subseteq \delta$ the figure may be amended to (after change of subscripts perhaps)

$$\dfrac{\dfrac{\dfrac{\Gamma \Vdash A_\alpha \qquad A_\alpha, \Pi \Vdash B_{\alpha+\beta}}{\Gamma, \Pi_{A_\alpha} \Vdash B_{\alpha+\beta}}\text{ Mix} \qquad B_{\alpha+\beta}, \Gamma \Vdash D_\delta}{\Gamma, \Pi_{A_\alpha}, \Pi_{B_{\alpha+\beta}} \Vdash D_\delta}\text{ Mix}}{\Pi, \Gamma \Vdash D_\delta}$$

B2, where rank exceeds 2.

B2.1, the left rank ≥ 2. So M_δ occurs in the antecedent of at least one of the premises for the inference of S_1

<u>Case 4a.</u> S_1 is by an antecedent structural rule δ.

<u>Case 4.ⓐ</u>

ⓑ similar

$$\dfrac{\dfrac{\Gamma \Vdash A_\alpha \qquad \Gamma \Vdash B_\alpha}{\Gamma \Vdash (A\&B)_\alpha} \qquad \dfrac{A_\alpha, \Gamma \Vdash D_\delta}{(A\&B)_\alpha, \Gamma \Vdash D_\delta}\text{ Hence } \alpha \subseteq \delta}{\Pi, \Gamma \Vdash D_\delta}\text{ Mix}$$

Alter to

$$\dfrac{\dfrac{\Pi \Vdash A_\alpha \qquad A_\alpha, \Gamma \Vdash D_\delta}{\Pi, \Gamma_{A_\alpha} \Vdash D_\delta}\text{ Mix}}{\Pi, \Gamma \Vdash D_\delta}\text{ using } \alpha \subseteq \delta$$

<u>Case 5 ⓐ</u>

$$\dfrac{\dfrac{\Pi \Vdash A_\alpha}{\Pi \Vdash (A \vee B)_\alpha} \qquad \dfrac{\Gamma, A_\alpha \Vdash D_\delta \qquad \Gamma, B_\alpha \Vdash D_\delta}{\Gamma, A \vee B_\alpha \Vdash D_\delta}+\text{ conditions}}{\Pi, \Gamma \Vdash D_\delta}\text{ Mix}$$

Alter to

$$\dfrac{\dfrac{\Pi \Vdash A_\alpha \qquad \Gamma, A_\alpha \Vdash D_\delta}{\Pi, \Gamma_{A_\alpha} \Vdash D_\delta}\text{ Mix}}{\Pi, \Gamma \Vdash D_\delta}\text{ Since } \alpha \subseteq \delta$$

Thus all conditions are met automatically.

<u>Case 3:</u> General Form

$$\dfrac{\dfrac{A_\gamma, \Pi, \Vdash D_{\gamma+\beta}}{\Pi \Vdash A \to B_\beta} \qquad \dfrac{\Gamma \Vdash A_\alpha \qquad B_{\alpha+\beta}, \Gamma \Vdash D_\delta}{A \to B_\beta, \Gamma \Vdash D_\delta}}{\Pi, \Gamma \Vdash D_\delta}\text{ Mix}$$

By hypothesis there is a proof without mix of $A_\alpha, \Pi \Vdash B + \gamma + \beta$, where $\xi \ disj \ \gamma$ for every $C_\xi \in \Pi$. If $C_\alpha \in \Pi$ for some C choose a new distinct subscript η and change α to η throughout the proof of $A_\alpha, \Pi \Vdash B_{\gamma+\beta}$. As for lemma 35 in Kleene the new figure is a proof. The procedure eliminates all occurrences of α from Π's subscripts. Finally γ to α throughout the proof figure. Then the same figure as before, only relettered, provides a proof of $A_\alpha, \Pi \Vdash B_{\alpha+\beta}$, satisfying the conditions for $\Vdash\!\to$.

Then the figure on the left is replaced by the figure on the right.

$$\dfrac{\dfrac{\Pi_1 \Vdash M_\delta}{\Pi \Vdash M_\delta}\,\delta \qquad \Sigma \Vdash D_\gamma}{\Pi, \Sigma_{M_\delta} \Vdash D_\gamma}\,\text{Mix} \qquad\qquad \dfrac{\dfrac{\Pi_1 \Vdash M_\delta \qquad \Sigma \Vdash D_\gamma}{\Pi_1, \Sigma_{M_\delta} \Vdash D_\gamma}\,\text{Mix}}{\Pi, \Sigma_{M_\delta} \Vdash D_\gamma}$$

In case the rule δ is weakening by C_α, note that $\alpha \subseteq \delta \subseteq \gamma$, since $M_\delta \in \Sigma$. The new figure reduced the rank of the mix by one.

Case11a: S_1 is by a one premise logical rule L, either $\Vdash\to$, $\& \Vdash$, of $\Vdash \vee$. Since the formulation is singular and the left rank ≥ 2 only the following case can occur with $\Lambda_\alpha = A_\alpha$ of B_α. It is altered as shown on the right.

$$\dfrac{\dfrac{\Lambda_\alpha, \Gamma \Vdash M_\delta}{(A\&B)_\alpha, \Gamma \Vdash M_\delta}\,\&\Vdash \qquad \Sigma \Vdash D_\gamma}{(A\&B)_\alpha, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\,\text{Mix} \qquad\qquad \dfrac{\dfrac{\Lambda_\alpha, \Gamma \Vdash M_\delta \qquad \Sigma \Vdash D_\gamma}{\Lambda_\alpha, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\,\text{Mix}}{(A\&B)_\alpha, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\,\&\Vdash$$

Case 12: S_1 is by a two-premise logical rule L, either $\to\Vdash$ of $\vee \Vdash$ since $\Vdash \&$ is impossible.

$\vee \Vdash$:

$$\dfrac{\dfrac{A_\alpha, \Gamma \Vdash M_\delta \qquad B_\alpha, \Gamma \Vdash M_\delta}{(A \vee B)_\alpha, \Gamma \Vdash M_\delta}\,L \qquad \Sigma \Vdash D_\gamma}{(A \vee B)_\alpha, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\,\text{Mix}$$

Alter to the following figure which reduced the rank of the mix.

$$\dfrac{\dfrac{A_\alpha, \Gamma \Vdash M_\delta \qquad \Sigma \Vdash D_\gamma}{A_\alpha, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\,\text{Mix} \qquad \dfrac{B_\alpha, \Gamma \Vdash M_\delta \qquad \Sigma \Vdash D_\gamma}{B_\alpha, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\,\text{Mix}}{(A \vee B)_\alpha, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\,L$$

$\to\Vdash$:

$$\dfrac{\dfrac{\Gamma \Vdash A_\alpha \qquad B_{\alpha+\beta}, \Gamma \Vdash M_\delta}{A \to B_\beta, \Gamma \Vdash M_\delta}\,L \qquad \Sigma \Vdash D_\gamma}{A \to B_\beta, \Gamma, \Sigma_{M\delta} \Vdash D_\gamma}\,\text{Mix}$$

Alter to:

$$\dfrac{\dfrac{\Gamma \Vdash A_\alpha \qquad \dfrac{B_{\alpha+\beta}, \Gamma \Vdash M_\delta \qquad \Sigma \Vdash D_\gamma}{B_{\alpha+\beta}, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\,\text{Mix}}{A \to B_\beta, \Gamma, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\,L}{A \to B_\beta, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\,\text{C}$$

B2.2: The right rank ≥ 2, so M_δ occurs in the antecedent of it before one of the premises for the inference.

Cases 4b and 10b: S_2 is by an antecedent structural rule δ. The figure is amended as on the right.

$$\dfrac{\Pi \Vdash M_\delta \qquad \dfrac{\Sigma_1 \Vdash D_\gamma}{\Sigma \Vdash D_\gamma}\ \delta}{\Pi, \Sigma_{M_\delta} \Vdash D_\gamma}\ \text{Mix} \qquad\qquad \dfrac{\dfrac{\Pi \Vdash M_\delta \qquad \Sigma_1 \Vdash D_\gamma}{\Pi, \Sigma_{1M_\delta} \Vdash D_\gamma}\ \text{Mix}}{\Pi, \Sigma_{M_\delta} \Vdash D_\gamma}\ \delta$$

In case M_δ is an interchanged or contracted formula in application of δ, Σ_{1M_δ} is Σ_{M_δ} and the last δ step is unnecessary. In case some formula C_α, not M_δ, is introduced by application of δ then $\alpha \subseteq \gamma$, so δ can be applied after mix.

Case 11b: S_2 is by a one premise logical rule L. The rule has the form

$$\dfrac{\Lambda_1, \Gamma \Vdash \Omega, \Lambda_2}{\Xi_1, \Gamma \Vdash \Omega, \Xi_2}\ L$$

where each of Λ_1, Λ_2 is either a side formula of empty and one of Ξ_1, Ξ_1 is the principal formula while the other is empty, and at best one of Ω and Λ_2 and of Ω and Ξ_2 is empty.

Subcase 1: Ξ_1 is not M_δ, so $M_\delta \in \Gamma$.

$$\dfrac{\Pi \Vdash M_\delta \qquad \dfrac{\Lambda_1, \Gamma \Vdash \Omega, \Xi_2}{\Xi_1, \Gamma \Vdash \Omega, \Xi_2}\ L}{\Pi, \Xi_1, \Gamma_{M_\delta} \Vdash \Omega, \Xi_2}\ \text{Mix}$$

The altered proof figure is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi \Vdash M_\delta \qquad \Lambda_1, \Gamma \Vdash \Omega, \Lambda_2}{\Pi, \Lambda_1, \Gamma_{M_\delta} \Vdash \Omega, \Lambda_2}\ \text{Mix}}{\Lambda_1, \Pi, \Gamma_{M_\delta} \Vdash \Omega, \Lambda_2}\ \text{I}}{\Xi_1, \Pi, \Gamma_{M_\delta} \Vdash \Omega, \Xi_2}\ \text{L}}{\Pi, \Xi_1, \Gamma_{M_\delta} \Vdash \Omega, \Xi_2}\ \text{I}$$

The new mix is of rank one less than the original. It remains to show in case L in $\rightarrow\Vdash$ that for each $C_\xi \in \Pi$ $\xi\ disj\ \alpha$, where $\Lambda_1 = A_\alpha$. By S_1, $\xi \subseteq \alpha$ and by the original premise for application of L, $\delta\ disj\ \alpha$ since $M_\delta \in \Gamma$. Hence $\xi\ disj\ \alpha$.

Subcase 2: Ξ_1 is M_δ. Then Ξ_2 is empty, Ω is D_γ and Λ_2 is empty. Also Λ_1 is not M_δ, so $M_\delta \in \Gamma$. (Thus L can only be $\& \Vdash$, but the more general case is given to reduce later new cases in extensions of R^+.)

$$\dfrac{\Pi \Vdash M_\delta \qquad \dfrac{\Lambda_1, \Gamma \Vdash D_\gamma}{M_\delta, \Gamma \Vdash D_\gamma}\ L}{\Pi, \Gamma_{M_\delta} \Vdash D_\gamma}\ \text{Mix}$$

Alter to:

$$\dfrac{\dfrac{\Pi \Vdash M_\delta \qquad \dfrac{\dfrac{\dfrac{\Pi \Vdash M_\delta \qquad \Lambda_1, \Gamma \Vdash D_\gamma}{\Pi, \Lambda_1, \Gamma_{M_\delta} \Vdash D_\gamma}\ \text{Mix}}{\Lambda_1, \Pi, \Gamma_{M_\delta} \Vdash D_\gamma}\ \text{I}}{M_\delta, \Pi, \Gamma_{M_\delta} \Vdash D_\gamma}\ L}{\Pi, \Pi, \Gamma_{M_\delta} \Vdash D_\gamma}\ \text{Mix}}{\Pi, \Gamma_{M_\delta} \Vdash D_\gamma}\ \text{C}$$

<u>Case12b:</u> S_2 is by a two premise logical rule L. The rule has the form

$$\dfrac{\Lambda_{11}, \Gamma \Vdash \Omega\Lambda_{12} \qquad \Lambda_{21}, \Gamma \Vdash \Omega\Lambda_{22}}{\Xi_1, \Gamma \Vdash \Omega, \Xi_2}\ L$$

<u>Subcase 1:</u> Ξ_1 is not M_δ, so $M_\delta \in \Gamma$.

$$\dfrac{\Pi \Vdash M_\delta \qquad \dfrac{\Lambda_{11}, \Gamma \Vdash \Omega\Lambda_{12} \qquad \Lambda_{21}, \Gamma \Vdash \Omega\Lambda_{22}}{\Xi_1, \Gamma \Vdash \Omega, \Xi_2}\ L}{\Pi, \Xi_1, \Gamma_{M_\delta} \Vdash \Omega, \Xi_2}\ \text{Mix}$$

The amended proof figure is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi \Vdash M_\delta \qquad \Lambda_{11}, \Gamma \Vdash \Omega, \Lambda_{12}}{\Pi, \Lambda_{11}, \Gamma_{M_\delta} \Vdash \Lambda_{12}}\ \text{Mix}}{\Lambda_{11}, \Pi, \Gamma_{M_\delta} \Vdash \Lambda_{12}}\ \text{I} \qquad \dfrac{\dfrac{\Pi \Vdash M_\delta \qquad \Lambda_{21}, \Gamma \Vdash \Omega, \Lambda_{22}}{\Pi, \Lambda_{21}, \Gamma_{M_\delta} \Vdash \Lambda_{22}}\ \text{Mix}}{\Lambda_{21}, \Pi, \Gamma_{M_\delta} \Vdash \Lambda_{22}}\ \text{I}}{\Xi_1, \Pi, \Gamma_{M_\delta} \Vdash \Omega, \Xi_2}\ L}{\Pi, \Xi_1, \Gamma_{M_\delta} \Vdash \Omega, \Xi_2}\ \text{I}$$

<u>Subcase 2:</u> Ξ_1 is M_δ. The case reduces to

$$\dfrac{\Pi \Vdash M_\delta \qquad \dfrac{\Lambda_{11}, \Gamma \Vdash D_\gamma \qquad \Lambda_{21}, \Gamma \Vdash D_\gamma}{M_\delta, \Gamma \Vdash D_\gamma}\ L}{\Pi, \Gamma_{M_\delta} \Vdash D_\gamma}\ \text{Mix}$$

$M_\delta \in \Gamma$ (The rule can only be $\Vdash \vee$.)
The amended proof figure is:

$$\dfrac{\dfrac{\Pi \Vdash M_\delta \qquad \dfrac{\dfrac{\dfrac{\Pi \Vdash M_\delta \qquad \Lambda_{11}, \Gamma \Vdash D_\gamma}{\Pi, \Lambda_{11}, \Gamma_{M_\delta} \Vdash D_\gamma}\ \text{Mix}}{\Lambda_{11}, \Pi, \Gamma_{M_\delta} \Vdash D_\gamma}\ \text{I} \qquad \dfrac{\dfrac{\Pi \Vdash M_\delta \qquad \Lambda_{21}, \Gamma \Vdash D_\gamma}{\Pi, \Lambda_{21}, \Gamma_{M_\delta} \Vdash D_\gamma}\ \text{Mix}}{\Lambda_{21}, \Pi, \Gamma_{M_\delta} \Vdash D_\gamma}\ \text{I}}{M_\delta, \Pi, \Gamma_{M_\delta} \Vdash D_\gamma}\ L}{\Pi, \Pi, \Gamma_{M_\delta} \Vdash D_\gamma}\ \text{Mix}}{\Pi, \Gamma_{M_\delta} \Vdash D_\gamma}\ \text{C}$$

$\square$

Corollary 10.

(i) R^+_* *proofs without cut have the subformula property.*

(ii) The separation theorem holds.

(iii) The decidability theorem holds.

Weakening is available in a form which does not affect Kleene's cognation class argument. (In fact in ${}_{\prime}\mathrm{R}^{+}_{*}$ weakening is available with qualification.)

Theorem (The Elimination Theorem for ${}_{|}\mathbf{R}^{+}_{*}$ (without cut)).

Proof. Proof is as for R^+_*, except that in B2.2 it is assumed both that the right rank exceeds 1 and that the left rank = 1. In this way the restrictions on $\Vdash\rightarrow$ needed in case 11b are guaranteed by the form of S_1. □

Theorem (Equivalence Theorem for $\mathbf{R}^+$ Systems).

$$\begin{aligned} \vdash_R A \textit{ iff } &\Vdash A_0 \textit{ in } R^+_* \\ \textit{iff } &\Vdash A_0 \textit{ in } {}_{|}R^+_* \end{aligned}$$

Proof. One half amounts to direct demonstration of the axioms of R^+; for modus ponens follows using Cut and adjunction follows from $\Vdash$ &. For the converse the sequent $\Gamma \Vdash A_\delta$ of the Gentzen system is interpreted as $\Gamma \Vdash_R A_\delta$, i.e. as an R^+-proof of A_δ from hypotheses Γ. Then the axiom scheme holds, and in the case of each rule, if the premises hold the the conclusion holds, using the deduction theorems already established, & their corollaries. Hence, if $\Vdash A_0$ in R^+_* then there is an R^+-proof of A from null hypotheses, so $\vdash_R A$. □

A Cut-Free Formulation $\Box$ $\mathbf{R}^+_*$

The following rules are added to R^+_*

$$\frac{A_\alpha, \Gamma \Vdash D_\delta}{\Box A_\alpha, \Gamma \Vdash D_\delta}\ \Box \Vdash$$

$$\frac{\Box\Gamma \Vdash D_\delta}{\Box\Gamma \Vdash \Box D_\delta}\ \Vdash \Box$$

$\Box\Gamma$ is the sequence of subscripted wff forms by prefixing $\Box$ to each wff in sequence Γ.

Lemma 35. *If* $\Gamma \Vdash D_\delta$ *and* $C_\alpha \in \Gamma$*, then* $\alpha \subseteq \delta$*.*

Theorem (Elimination theorem for $\Box\mathbf{R}^+_*$).

There are the following new cases.

B1, where rank is 2.

Case 6:

$$\frac{\dfrac{\Box\Pi \Vdash C_\gamma}{\Box\Pi \Vdash \Box C_\gamma} \qquad \dfrac{C_\gamma, \Gamma \Vdash D_\delta}{\Box C_\gamma, \Gamma \Vdash D_\delta}}{\Box\Pi, \Gamma \Vdash D_\delta}\ \mathrm{Mix}$$

Amend the figure to:

$$\frac{\Box\Pi \Vdash C_\gamma \qquad C_\gamma, \Gamma \Vdash D_\delta}{\Box\Pi, \Gamma \Vdash D_\delta}\ \mathrm{Mix}$$

B2, where rank exceeds 2.

Case 11a: Only the following new case can occur.

$$\frac{\dfrac{A_\alpha, \Gamma \Vdash M_\delta}{\Box A_\alpha, \Gamma \Vdash M_\delta}\ \Box\Vdash \qquad \Sigma \Vdash D_\gamma}{\Box A_\alpha, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\ \mathrm{Mix}$$

Amend to

$$\frac{\dfrac{A_\alpha, \Gamma \Vdash M_\delta \qquad \Sigma \Vdash D_\gamma}{A_\alpha, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\ \mathrm{Mix}}{\Box A_\alpha, \Gamma, \Sigma_{M_\delta} \Vdash D_\gamma}\ \Box\Vdash$$

Case 11b: Already treated generally: except one case.[46]

Case11b subcase 1: $M_\delta \in \Box\Gamma$; so M_δ is $\Box N_\delta$. The proof figure to be amended is:

[46]The corollary following these subcases was originally put after this line. I have moved the corollary to after the subcases for convenience. Routley had "see p10", the next page, at the location of this footnote. In moving the corollary I have obviated the need for this reference.

$$\dfrac{\Pi \Vdash \Box N_\delta \qquad \dfrac{\Box\Gamma \Vdash D_\gamma}{\Box\Gamma \Vdash \Box D_\gamma}\ \Vdash\Box}{\Pi, \Box\Gamma_{\Box N_\delta} \Vdash \Box D_\gamma}\ \text{Mix}$$

Case a: The left rank is 1. Then $\Box N_\delta$ must have been introduced by $\Vdash \Box$. Thus Π is of the form $\Box Ⓗ$. The proof figure is amended as follows:

$$\dfrac{\dfrac{\Box Ⓗ \Vdash \Box N_\delta \qquad \Box\Gamma \Vdash D_\gamma}{\Box Ⓗ, \Box\Gamma_{\Box N_\delta} \Vdash D_\gamma}\ \text{Mix}}{\Box Ⓗ, \Box\Gamma_{\Box N_\delta} \Vdash \Box D_\gamma}\ \Vdash\Box$$

Case b: The left rank exceeds 1, so ≥ 2, is already treated under B2.1

Corollary 11. *Separation & Decidability theorems for $\Box R^+_*$, and hence for $\Box R^+$.*

Gentzen Forms of $\mathbf{E}^+$

Add to the forms for R^+ the further proviso on $\Vdash\rightarrow$. Provided every member of Γ is a subscripted entailment.

The elimination theorem holds (for case B2.2, assume also that left rank = 1).

Cut Free Formulations of Parts of P

(I) P_I*

$\boxed{\text{Formulation 1}}$. Axiom $C_\alpha \Vdash C_\alpha$

Structural rules: Contraction and interchange (as for R^+)

$$\frac{\Gamma \Vdash D_\delta}{C_\alpha \Gamma \Vdash D_\delta}\text{Weakening}$$

provided $\alpha \subseteq \delta$

Logical Rules:

$$\frac{\Gamma \Vdash A_\alpha \qquad B_{\alpha+\beta}, \Gamma \Vdash D_\delta}{A \to B_\beta, \Gamma \Vdash D_\delta}\to\Vdash$$

provided $\max(\alpha) \geq \max(\beta)$

$$\frac{A_\alpha, \Gamma \Vdash B_\beta}{\Gamma \Vdash A \to B_{\beta-\alpha}}\Vdash\to$$

provided $\alpha \neq 0$, $\alpha \leq \beta$, and for $C_\delta \in \Gamma$
$\alpha\ disj\ \delta$ and $\max(\alpha) > \max(\delta)$.

Lemma 36. *If $\Gamma \Vdash D_\delta$ and $C_\alpha \in \Gamma$ then*

(i) $\alpha \subseteq \delta$

(ii) $max(\alpha) \leq max(\delta)$

New details in the elimination theorem.
(i) For $\to$. In cases 12a and 12b the restrictions in the original figures carry over to the amended figures. In case 12b subcase 1, the further restriction on $\to\Vdash$ is derived thus. For $C_\xi \in \Pi$, $\max(\xi) \leq \max(\delta)$ by the premise S_1. But $M_\delta \in \Gamma$ so $\max(\alpha) > \max(\delta$; hence (α) $> \max(\xi)$.

$\boxed{\text{Formulation 2}}$ As per above but

(a) proviso on weakening removed

(b) $\Vdash\rightarrow$ replace by [the] following rule

$$\frac{A_\alpha, \Gamma \Vdash B_\beta}{\Gamma \Vdash A \rightarrow B_{\beta-\alpha}} \Vdash\rightarrow$$

provided $\alpha \neq 0$, $\max(\alpha) \in \beta$,
and $\max(\alpha) > \max(\delta)$ for $C_\delta \in \Gamma$.

Since rule elimination theorem holds, P_I is... the methods of Kleene.[47]

(II) $\mathrm{P}^*_{I\&}$

Add to the formulations the subscripted rules for $\underline{\&}$. Everything holds

(III) P^*_+

Add to $\mathrm{P}_{I\&}{}^*$ the subscripted rules for $\underline{\vee}$.

Then everything holds but the equivalence theorem. It breaks down because I have not been able to prove primeness, i.e. (here).

(?!) if $\Gamma, A_\beta \Vdash_P C_\delta$ and $\Gamma, B_\beta \Vdash_P C_\delta$ then $\Gamma(A \vee B)_\beta \Vdash_P C_\delta$

There is a Gentzen formulation using primeness in the form I have managed to establish, but then the proof of the elimination theorem breaks down.

[47] Cut off of page

Covering Note **Confidential**

There are several gaps in the argument, and no doubt many invalid moves — I should be grateful if you would point out *all* those you think I have failed to see. The chief gaps & deficiencies are these:

1) A proof of the following primeness theorem is still outstanding in the area of E and P: If $\gamma, A_\alpha \Vdash C_\beta$ and $\gamma, B_\alpha \Vdash C_\beta$ then $\gamma, (A \vee B)_\alpha \Vdash C_\beta$.

2) A separation theorem for the positive logics of R, □R, E and P is still lacking. I've put very little work into looking for one: The lack of one is symptomatic of the next.

3) The lack of a satisfactory treatment of negation. Two reasons:

i) an inadequate logic of negation

ii) The implication evaluation function is not quite right. There are clearly <u>lots</u> of variations on the implication rule with the right sort of features.

I started out e.g. with the following functions:
$h(A \to B, H_1) = T$ iff for every H_2, H_3 if $H_1R(H_2, H_3)$ and $h(A, H_2) = T$ then $h(B, H_3) = T$.
$h(\sim A, H) = T$ iff $h(A, H^*) = F$.
Then the condition for contraposition is the not unpleasing : if $HR(H_1, H_2)$ then $HR(H_2^*, H_1^*)$. But the conditions for implication theorems get quite complicated. It should be possible to guess a suitable simplification. An improved implication rule might enable a solution of 1) too. For all the trouble. . . [48]

4) The later parts — §6 on — are sketchy and even transparently deficient, but I would hope the deficiencies & gaps can be repaired when other problems (1)& 3)) have been solved.

5) The simple rule for negation on the right:
If $\sim A_\alpha$ is in the right of H put A_α on the left <u>appears</u> to work fine, but I haven't been able to show its adequacy. But its mate for negation on the right would sanction Disjunctive Syllogism, if in any case it is not adequate for that thorn Contraposition.

6) I'm still unhappy about disjunctions behavior in $\{\to \& f\}$ formulations of R. There's more to this than has met my eye.

7) All of my "proofs" that simplified models will work for R have broken down. I now <u>think</u> a proof will result using the methods of the sketchy §7: at least it seems to follow that models for R^+ may be simplified as in §2.3.

8) The basic idea of §8, which I have only in rough form, is that deductive tableaux rewritten, from bottom to top, provide a Gentzen cut-free proof method. Thus completeness follows using an interpretation theorem for the Gentzen system. You'll see how the positive Gentzen systems look in §9.

[48] Cut off page.

Transcriber's Note

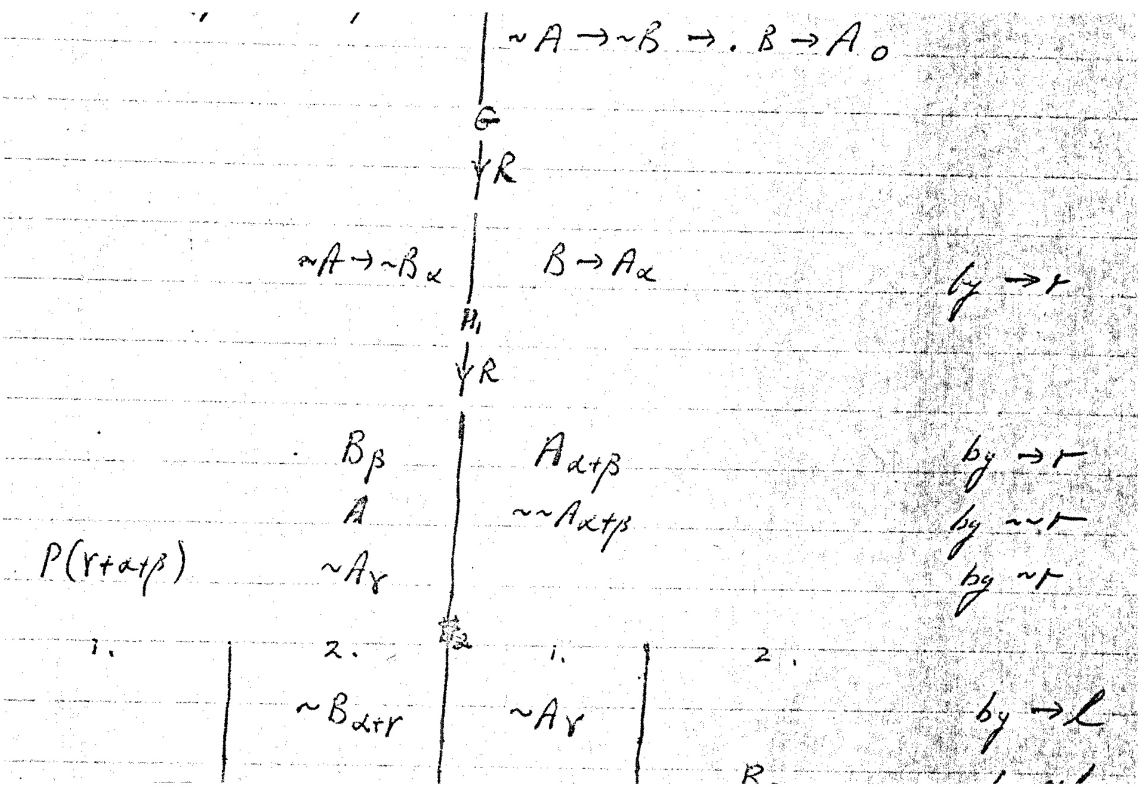
In transcribing this work of Routley's, I have taken the liberty to correct a couple obvious typographical errors. In each case I have added a footnote indicating the correction. Furthermore, the only corrections I have made are small, but significant, errors; e.g. writing $\alpha \in \nabla$ when he means $\alpha \notin \nabla$.

All of the footnotes in this document are my own, and they take note of various things. Some examples include the material of Routley's (single) footnote and notational alterations I have made to ease the transcription to LaTeX.

I have used the latex citation in place of Routley's [ ] notation. The result, however, is very similar.

There are a number of words that I have been unable to decipher. Some of these words were guessed at, and other I have marked with an underlined blank space. In each case there is an accompanying footnote.

My recreation of the example tableaux in section 6 leaves a lot to be desired. Below is a copy of the tableaux from Routley.



In formatting the document I have tried to balance the formatting, and formatting notes, of the original with considerations for readability.

–Nicholas Ferenz