Two temporal logics of contingency

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Abstract

This work concerns the use of operators for past and future contingency in Priorean temporal logic. We will develop a system named $\mathbf{C}_t$, whose language includes a propositional constant and prove that (i) $\mathbf{C}_t$ is complete with respect to a certain class of general frames and (ii) the usual operators for past and future necessity are definable in such system. Furthermore, we will introduce the extension $\mathbf{C}_t lin$ that can be interpreted on linear and transitive general frames. The theoretical result of the current work is that contingency can be treated as a primitive notion in reasoning about temporal modalities.

1 Introduction

Systems of modal logic are usually based on a language in which there is a primitive notion of necessity or possibility, while the remaining modalities are introduced by means of auxiliary definitions. The reason is that an exhaustive picture of modalities can be easily obtained when an operator for necessity or possibility and truth-functional connectives are available. For instance, if the basic language of a system contains an operator $\Box$, such that $\Box\alpha$ means "α is necessary", possibility can be defined as $\neg\Box\neg\alpha$, impossibility as $\Box\neg\alpha$, contingency (in the sense of two-sided possibility: to be contingent is to be neither necessary nor impossible) as $\neg\Box\alpha \land \neg\Box\neg\alpha$, etc. On the other hand, the task of defining necessity and possibility from different modal notions presents some technical difficulties and is interesting to explore in order to answer the philosophical question whether all modal notions are on the same conceptual level. Here we are especially interested in languages containing a primitive notion of contingency or absoluteness, i.e. non-contingency: we will

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summarize the main contributions to this topic available in the literature and extend some results of definability obtained in a monomodal context to the bimodal context of tense logic.

2 Contingency logic

Montgomery and Routley considered in [7] a monomodal language where the primitive operator was chosen between $\triangledown$, representing contingency, and $\triangle$, representing non-contingency (or absoluteness), and were able to axiomatize systems deductively equivalent to **KT**, **S4** and **S5**.[1] For instance, they proved that adding to **PC** (Propositional Calculus) the axioms $\triangle\ \alpha \equiv \triangle\ \neg\alpha$ and $\alpha \to (\triangle\ (\alpha \to \beta) \to (\triangle\ \alpha \to \triangle\ \beta))$, the rule $\vdash \alpha \implies\ \vdash \triangle\ \alpha$[2] and the definition of $\Box\alpha$ as $\alpha \wedge \triangle\ \alpha$ (to be necessary is to be true and non-contingent), one gets a system of non-contingency logic deductively equivalent to **KT**. Furthermore, they showed that the result of adding to this basis $\triangle\ \alpha \to \triangle\triangle\ \alpha$ is a system equivalent to **S4**, while adding $\triangle\triangle\ \alpha$ one gets a system equivalent to **S5**. However, the problem of axiomatizing contingency and non-contingency versions of normal systems weaker than **KT**, which turns out to be, from a philosophical perspective, the problem of providing a general treatment of contingency as a primitive modality, was not addressed by Montgomery and Routley.

Some decades after, Humberstone explored in [3] this crucial issue, presenting a complex axiomatization for the minimal logic of non-contingency, called **NC**, and his solution was refined by Kuhn in [5]. For our purposes we choose to focus on Kuhn's minimal system **K∆**, which has the following axiomatic basis:[3]

A0	all the theorems of **PC**
A1	$\triangle\ \alpha \to \triangle\ \neg\alpha$
A2	$(\triangle\ \alpha \wedge \triangledown(\alpha \wedge \beta)) \to \triangledown\beta$
A3	$(\triangle\ \alpha \wedge \triangledown(\alpha \vee \beta)) \to \triangle\ (\neg\alpha \vee \gamma)$
R1	if $\vdash \alpha$, then $\vdash \triangle\ \alpha$

[1] We say that two systems S and S′ are deductively equivalent iff all the theorems of S are derivable in S′ and all the theorems of S′ are derivable in S.

[2] This rule says that all theorems are non-contingent: if α is derivable, then $\triangle\ \alpha$ is derivable too. It is a weakening of the necessitation rule used in normal modal systems with a primitive operator of necessity.

[3] A remark on A3: any absolute proposition α is either necessary or impossible. In the first case it is necessarily implied by any other proposition, in the second case it necessarily implies any other proposition. Furthermore, since what is necessary is also absolute, we get: $\triangle\ \alpha \to (\triangle\ (\beta \to \alpha) \vee \triangle\ (\alpha \to \gamma))$, which is equivalent to A3.

R2 if $\vdash \alpha \equiv \beta$, then $\vdash \triangle\, \alpha \equiv \triangle\, \beta$
MP if $\vdash \alpha$ and $\vdash \alpha \to \beta$, then $\vdash \beta$

The operators for contingency and non-contingency are interdefinable by means of the clause $\nabla\alpha := \neg \triangle \alpha$, so the system is based on a monomodal language. An interesting problem is that, in spite of the elegance of Kuhn's axiomatization, it is not possible to provide a definition of necessity in $\mathbf{K}\triangle$. As a matter of fact, Cresswell proved in [1] that (i) the equivalence between $\Box\alpha$ and $\alpha\wedge \triangle \alpha$ used by Montgomery and Routley holds only for systems of non-contingency logic equivalent to **KT** or to its normal extensions and (ii) for many systems weaker than **KT** there is no formula $f(\alpha)$ containing only $\triangle$ (or ∇) and truth-functional operators such that $f(\alpha) \equiv \Box\alpha$. Here we reproduce the first part of Cresswell's result in order to get acquainted with the semantic notions that will be used in the following.

Consider a modal language with $\Box$ as primitive operator and the definitions $\triangle\, \alpha := \Box\alpha \vee \Box\neg\alpha$ and $\nabla\alpha := \neg \triangle \alpha$. Such language is interpreted on relational frames $\mathfrak{F} = \langle W, \mathfrak{R}\rangle$, where W is a non-empty set of worlds and $\mathfrak{R} \subseteq W \times W$ an accessibility relation. If v is a valuation function that maps propositional variables to sets of worlds, we say that $\mathfrak{M} = \langle \mathfrak{F}, v\rangle$ is a relational model built on $\mathfrak{F}$. The notion of truth at a world w of a model $\mathfrak{M}$ for formulas whose main operator is $\triangle$ or ∇ is specified below:

$w \vDash \nabla\alpha$ iff for some w' such that $w\mathfrak{R}w'$, $w' \vDash \alpha$ and for some w'' such that $w\mathfrak{R}w''$, $w'' \nvDash \alpha$;
$w \vDash \triangle\, \alpha$ iff either (i) for all w' such that $w\mathfrak{R}w'$, $w' \vDash \alpha$ or (ii) for all w' such that $w\mathfrak{R}w'$, $w' \nvDash \alpha$.

Take an arbitrary frame $\mathfrak{F}$ where **KT** is valid. Since axiom $\Box\alpha \to \alpha$ is true at all points for any valuation v, we know from correspondence theory that such frame is reflexive. For *reductio*, suppose that for some model $\mathfrak{M}$ built on $\mathfrak{F}$ there is a world w such that $w \nvDash \Box\alpha \equiv (\alpha\wedge \triangle \alpha)$. There are two cases: (i) $w \vDash \Box\alpha$ and (ii) $w \nvDash \Box\alpha$. With reference to the first case, we know that $w' \vDash \alpha$ for all w' such that $w\mathfrak{R}w'$; hence $w \vDash\triangle \alpha$. In addition, since the frame is reflexive, $w \vDash \alpha$ and from this it follows immediately that $w \vDash \alpha\wedge \triangle \alpha$. Regarding the second case, we know that there is a w' such that $w\mathfrak{R}w'$ and $w' \nvDash \alpha$. Even if $w \vDash \alpha$, from reflexivity it follows that $w \nvDash\triangle \alpha$ and, finally, that $w \nvDash \alpha\wedge \triangle \alpha$. In both cases we contradict the assumption $w \nvDash \Box\alpha \equiv (\alpha\wedge \triangle \alpha)$. Consider, instead, a frame $\mathfrak{F}'$ containing only a point u such that $\neg u\mathfrak{R}u$ (u is a dead-end); then there is a model $\mathfrak{M}'$ on $\mathfrak{F}'$ such that for some atomic propostion p we have $u \nvDash p$ and $u \vDash \Box p$, hence $u \nvDash \Box\alpha \equiv (\alpha\wedge \triangle \alpha)$. This completes the first part of Cresswell's result. In the second part, as we said, he proved that a definition of necessity in terms of non-contingency is usually not available for systems that can be interpreted on non-reflexive frames.[4] Hence, the important conclusion to draw is that a straightforward use of contingency as a primitive notion is not possible in all modal contexts.

3 A general definition of necessity

In recent years two different solutions have been proposed to overcome the difficulties of defining necessity from contingency or absoluteness in systems weaker than **KT**. The first solution, due to Zolin and presented in [12], consists in defining in **K**$\triangle$ a new operator of necessity $\boxtimes$ behaving like $\Box$ even if not in general truth-implying. Zolin's $\boxtimes$ is introduced via an infinitary conjunction over a subset of **K**$\triangle$-formulas; the idea is that, given a world w and a formula α, $w \vDash \boxtimes\alpha$ iff $w \vDash\triangle (\beta \to \alpha)$ for every formula β containing only $\triangle$ and truth-functional operators. Here we suggest an alternative definition with finitary means:[5]

Def $\boxtimes$ $\qquad \boxtimes\alpha :=\triangle \alpha \land (\nabla\beta \to\triangle (\beta \to \alpha))$

Such definition is based on a semantic consideration. From the perspective of a world w every formula β can be classified as necessary, impossible or contingent: if $w \vDash \Box\beta$, then we have $w \vDash\triangle (\beta \to \alpha)$ iff $w \vDash\triangle \alpha$ (and we already know this from the first conjunct of the definition); if $w \vDash \Box\neg\beta$ then $w \vDash\triangle (\beta \to \alpha)$ as a trivial consequence and if $w \vDash \nabla\beta\land \triangle \alpha$, then $w \vDash\triangle (\beta \to \alpha)$ iff $w \vDash \Box\alpha$. Hence, the meaning of $\boxtimes\alpha$ is: (i) α is absolute and (ii) if there is a contingent proposition, α is also necessary. This explains the fact that, as Zolin remarks, the truth of $\Box\alpha$ implies the truth of $\boxtimes\alpha$, but not the other way round, so $\boxtimes$ is weaker than $\Box$. Indeed, if one considers a frame with two worlds, w and w', such that the only relation of accessibility is $w\Re w'$, there can be a model where for some proposition p we have $w \vDash \boxtimes p$ (p cannot be contingent at w), but $w' \nvDash p$, whence $w \nvDash \Box p$.

The second solution is provided by Pizzi in [8], where the author adds to the axiomatic basis of **K**$\triangle$ a postulate for the existence of at least one contingent proposition, i.e. $\exists p\nabla p$, enriching the language with propositional

[4] See [1] for some exceptions, which are anomalous normal systems.

[5] *Update*: a similar definition for $\Box$ is independently formulated by Fan, Wang and van Ditmarsch in [2] to prove results of "almost definability" (local definability on a frame) of necessity in terms of contingency. Their article suggested some variations in the proof of completeness for the systems $\mathbf{C}_t$ and $\mathbf{C}_t lin$.

quantifiers. In [9] and [10] this strategy is refined with the use of a propositional constant, which we can represent as c.[6] The remarkable point is that adding to $\mathbf{K}\triangle$ the axiom $\triangledown c$ or the weaker $\triangledown c \vee \triangle\, \alpha$ it can be proved that the following definitions hold:[7]

Def $\Box$ $\quad \Box\alpha := \triangle\, \alpha \wedge \triangle\, (c \to \alpha)$
Def $\Diamond$ $\quad \Diamond\alpha := \triangledown\alpha \vee \triangledown(c \wedge \alpha)$

Hence, one gets an elegant definition of necessity and possibility relative to a contingent stock of information, c, whose interpretation depends on the context.[8] Pizzi proves also that the c-free fragment of $\mathbf{K}\triangle + \triangledown c$ is deductively equivalent to **KD**, whereas the c-free fragment of $\mathbf{K}\triangle + (\triangledown c \vee \triangle\, \alpha)$ is deductively equivalent to **K**. For further analysis of Zolin's and Pizzi's approaches to the problem of definability we invite the reader to see [4].

4 From a monomodal to a bimodal language

Keeping in mind what we said in the previous sections, we will explore the use of contingency and non-contingency operators in the *bimodal language* of Priorean temporal logic, seeking to get also analogous results of definability for a temporal notion of necessity. However, before doing that we have to recall some fundamental aspects of the new modal context. The language of tense logic includes two primitive operators, $\blacksquare$ ("it has always been the case that") for past necessity and $\Box$ ("it will always be the case that") for future necessity. The minimal system of tense logic is attributed to Lemmon (see [11]) and called $\mathbf{K}_t$. The axiomatic basis of $\mathbf{K}_t$ includes an axiom of distribution and a rule of theorem necessitation for both operators: $\blacksquare(\alpha \to \beta) \to (\blacksquare\alpha \to \blacksquare\beta)$, $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$, $\vdash \alpha \implies \vdash \blacksquare\alpha$ and $\vdash \alpha \implies \vdash \Box\alpha$; additionally, there are two *bridge-axioms* expressing an intuitive connection between past and future, i.e. $\alpha \to \blacksquare\neg\Box\neg\alpha$ and $\alpha \to \Box\neg\blacksquare\neg\alpha$.[9] The role of these axioms will be clarified afterwards.

[6] The original notation is τ but we would like to avoid confusion with the temporal setting in the following sections.

[7] The weaker axiom says that either c is contingent or every formula is necessary or impossible (which would be a trivialization of modalities).

[8] In [10] some philosophically interesting interpretations of c are proposed, such as "the moral laws are respected" or "general relativity is accepted by the scientific community". These interpretations allow to clarify the idea of "being necessary in relation to something", which underlies Def $\Box$.

[9] See [11] and [6] for a detailed presentation of the most important systems of temporal logic. Here we use an alternative notation for temporal operators, the standard one is H

$\mathbf{K}_t$ is interpreted on frames $\mathfrak{F} = \langle T, <, > \rangle$ where T is a non-empty set of instants, $<$ is the temporal relation "before" ($t < t'$ means that t is before t') and $>$ is the temporal relation "after" ($t > t'$ means that t is after t'). A model $\mathfrak{M} = \langle \mathfrak{F}, v \rangle$ consists of a frame $\mathfrak{F}$ and a valuation function v mapping propositional variables to subsets of T. The notion of truth at an instant t of a model $\mathfrak{M}$ is defined as below:

$t \vDash p$ iff $t \in v(p)$ for every propositional variable p
$t \vDash \neg\alpha$ iff $t \nvDash \alpha$
$t \vDash \alpha \to \beta$ iff $t \nvDash \alpha$ or $t \vDash \beta$
$t \vDash \blacksquare\alpha$ iff for every t' such that $t' < t$, $t' \vDash \alpha$
$t \vDash \square\alpha$ iff for every t' such that $t' > t$, $t' \vDash \alpha$

Going back to the axioms $\alpha \to \blacksquare\neg\square\neg\alpha$ and $\alpha \to \square\neg\blacksquare\neg\alpha$, they define the class of relevant frames for tense logic as those where the relations $<$ and $>$ are *mutually inverse*: $\forall t, t'(t < t' \equiv t' > t)$. Such property, which seems to be essential for time series, means that if an instant t is before an instant t', then t' is after t and the other way round. Here it is worth reproducing this result of frame definition to focus on the fundamental difference between temporal and modal structures.

Consider a frame $\mathfrak{F} = \langle T, <, > \rangle$ and suppose that for every model $\mathfrak{M}$ build on $\mathfrak{F}$, $\alpha \to \blacksquare\neg\square\neg\alpha$ holds at all instants. Take an arbitrary instant t: if $t \nvDash \alpha$ then the validity of the axiom does not require any particular condition for the frame, since $\blacksquare\neg\square\neg\alpha$ can be either true or false at t. If $t \vDash \alpha$, instead, $\blacksquare\neg\square\neg\alpha$ is required to be true. In such case, if there is no t' such that $t' < t$, then $\blacksquare\neg\square\neg\alpha$ turns out to be trivially true. Otherwise, if there is at least a t' such that $t' < t$, it is required that there exists a t'' such that $t'' > t'$ and $t'' \vDash \alpha$. However, if $>$ is not the inverse relation of $<$, there is nothing ensuring that this requirement is satisfied, because even if we have $t \vDash \alpha$, we don't have $t > t'$. Hence, the assumption that $\alpha \to \blacksquare\neg\square\neg\alpha$ holds at t could be falsified. This means that from the validity of $\alpha \to \blacksquare\neg\square\neg\alpha$ in a frame it follows that $>$ is the inverse relation of $<$. A similar analysis shows that from the validity of $\alpha \to \square\neg\blacksquare\neg\alpha$ in a frame it follows that $<$ is the inverse relation of $>$. Taking the two parts of the proof together, we can conclude that $<$ and $>$ are mutually inverse in the class of $\mathbf{K}_t$-frames.

for past necessity and G for future necessity. Priorean languages for temporal logic are based on the distinction between "past", "present" and "future" instead of the distinction between "earlier" and "later", even though the latter notions play a fundamental role from a semantic perspective.

5 The system $\mathbf{C}_t$

In order to develop a system of temporal logic with two primitive operators of contingency, which we will represent as $\blacktriangledown$ and $\triangledown$, the mutual inversity of $<$ and $>$ must be granted in some way, because this is the distinctive feature of temporal frames. A possible approach, following the suggestions in [10], consists in introducing a propositional constant k and using it to formulate some axioms able to force that property on frames. We will consider a system named $\mathbf{C}_t$, which stands for "temporal contingency", whose axiomatic basis is defined below:

A0 All the theorems of **PC**
A1.1 $\blacktriangledown\alpha \to \blacktriangledown\neg\alpha$
A1.2 $\triangledown\alpha \to \triangledown\neg\alpha$
A2.1 $(\blacktriangle\alpha \wedge \blacktriangle\beta) \to \blacktriangle(\alpha \wedge \beta)$
A2.2 $(\vartriangle\alpha \wedge \vartriangle\beta) \to \vartriangle(\alpha \wedge \beta)$
A3.1 $(\blacktriangle\alpha \wedge \blacktriangledown(\alpha \vee \beta)) \to \blacktriangle(\neg\alpha \vee \gamma)$
A3.2 $(\vartriangle\alpha \wedge \triangledown(\alpha \vee \beta)) \to \vartriangle(\neg\alpha \vee \gamma)$
A4.1 $\alpha \to \blacktriangle(\beta \vee \triangledown\alpha \vee \triangledown(k \wedge \alpha))$
A4.2 $\alpha \to \vartriangle(\beta \vee \blacktriangledown\alpha \vee \blacktriangledown(k \wedge \alpha))$
A5 $\blacktriangledown k \wedge \triangledown k$
R1.1 if $\vdash \alpha$ then $\vdash \blacktriangle\alpha$
R1.2 if $\vdash \alpha$ then $\vdash \vartriangle\alpha$
R2.1 if $\vdash \alpha \equiv \beta$ then $\vdash \blacktriangle\alpha \equiv \blacktriangle\beta$
R2.2 if $\vdash \alpha \equiv \beta$ then $\vdash \vartriangle\alpha \equiv \vartriangle\beta$
MP if $\vdash \alpha$ and $\vdash \alpha \to \beta$ then $\vdash \beta$

As in the monomodal case, we assume a definition of non-contingency operators, $\blacktriangle\alpha := \neg\blacktriangledown\alpha$ and $\vartriangle\alpha := \neg\triangledown\alpha$. Some comments on this list: A1.1-A3.2 and R1.1-R2.2 represent, with few variations, the bimodal version of the axiomatic basis for the system $\mathbf{K}\vartriangle$ in [5]; A4.1-A4.2 are our *bridge-axioms* for past and future modalities, playing the same role as $\alpha \to \Box\neg\blacksquare\neg\alpha$ and $\alpha \to \blacksquare\neg\Box\neg\alpha$ in $\mathbf{K}_t$, as we will see below; finally, A5 says that k is contingent before and after an arbitrary point, i.e., it is the bimodal version of one of the axioms used in [10] to grant definability. Any stock of information which corresponds to a proposition contingent in the past and in the future of every instant can be a suitable interpretation for k.[10]

[10]For instance, k may represent the well-known Aristotelian proposition that a sea-battle is taking place: in such case $\triangledown\alpha$ would mean "it is contingent whether a sea-battle takes place in the future" and $\blacktriangledown\alpha$ "it is contingent whether a sea-battle took place in the past". However, it must be noted that in order to provide an accurate formalization of Aristotle's

Given the presence of a propositional constant, we want the system $\mathbf{C}_t$ to be interpreted on structures where only some valuations for k are admitted. Hence we make use of *general frames* $\mathfrak{G} = \langle T, <, >, \Pi \rangle$ where T, $<$ and $>$ are defined as usual and Π is a set of admissible valuations for propositional terms (variables and constants). The set Π is closed under the boolean operations and the additional operations $h_\blacktriangledown$ and $h_\triangledown$:

- if $X \in \Pi$, then $T - X \in \Pi$
- if $X, Y \in \Pi$, then $X \cap Y \in \Pi$
- if $X \in \Pi$, then $h_\blacktriangledown(X) \in \Pi$
- if $X \in \Pi$, then $h_\triangledown(X) \in \Pi$

where $h_\blacktriangledown(X) = \{t \in T | \exists t', t''(t' < t \wedge t' \in X \wedge t'' < t \wedge t'' \notin X)\}$ and $h_\triangledown(X) = \{t \in T | \exists t', t''(t' > t \wedge t' \in X \wedge t'' > t \wedge t'' \notin X)\}$. The following *admissibility conditions* define valuations for our class of general frames:[11]

- $v(p) \in 2^T$, for every propositional variable p;
- $\forall t \exists t', t''((t' < t \wedge t' \in v(k)) \wedge (t'' < t \wedge t'' \notin v(k)))$ and $\forall t \exists t', t''((t' > t \wedge t' \in v(k)) \wedge (t'' > t \wedge t'' \notin v(k)))$.

A model $\mathfrak{M} = \langle \mathfrak{G}, v \rangle$ is obtained with a valuation v which maps propositional terms to elements of Π in agreement with the admissibility conditions. Appealing to the terminology used in [10], any admissible valuation will be called k-forked. Truth-conditions for $\mathbf{C}_t$-formulas whose main operator is $\blacktriangledown$ or $\triangledown$ are specified below:

$t \vDash \blacktriangledown\alpha$ iff for some t' such that $t' < t$, $t' \vDash \alpha$ and for some t'' such that $t'' < t$, $t'' \nvDash \alpha$

argument about future contingencies one needs a language with *metric operators* [11]: indeed it is required not only to say that a given proposition is contingent in the future, but also that there are different possibilities related to the same future instant. In particular, the non-metric operator of future contingency turns out to be either too strong or too weak. Take $\triangledown_1 s$ for "it is contingent whether a sea battle takes place *tomorrow*": if we define $\triangledown s$ as $\forall i, 1 \leq i, \triangledown_i s$, then $\triangledown s \to \triangledown_1 s$, but not vice versa; if we define $\triangledown s$ as $\exists i, 1 \leq i, \triangledown_i s$, then $\triangledown_1 s \to \triangledown s$, but not vice versa.

[11] The admissibility condition on $v(p)$, for any propositional variable p, is trivial, whereas the admissibility condition on $v(k)$ ensures that all interpretations of k satisfy A5. Note that this axiom is specific for k (not closed under uniform substitution) and requires models to have at least two instants. Indeed, minimal models for A5 are built on a general frame where $T = \{t_1, t_2\}$, the relations $<$ and $>$ are universal and, for any valuation v, either $v(k) = \{t_1\}$ or $v(k) = \{t_2\}$.

$t \vDash \triangledown\alpha$ iff for some t' such that $t' > t$, $t' \vDash \alpha$ and for some t'' such that $t'' > t$, $t'' \nvDash \alpha$

We will prove that $\mathbf{C}_t$ is complete with respect to the class of general frames where the relations $<$ and $>$ are mutually inverse (as required by the temporal context), every instant is preceded and followed by at least two instants and only k-forked valuations are admitted. For the sake of simplicity, such class of general frames will be called Θ_0.

Theorem 5.1 $\mathbf{C}_t$ is sound with reference to the class Θ_0.

Proof Take an arbitrary frame $\mathfrak{G}$ of the class defined. Axioms A1.1-A1.2 are valid as an immediate consequence of the truth conditions assigned to $\blacktriangledown$ and $\triangledown$ and of bivalence. For axiom A2.1 consider a model on $\mathfrak{G}$ where for some instant t we have $t \vDash \blacktriangle\alpha \wedge \blacktriangle\beta$ and suppose $t \nvDash \blacktriangle(\alpha \wedge \beta)$. Then for every t' such that $t' < t$ α has the same value and β has the same value. But this means that either $\alpha \wedge \beta$ is always true or always false in the past of t, hence $\blacktriangle(\alpha \wedge \beta)$ is true at t. Axiom A2.2 can be tested in a similar way. Regarding axiom A3.1 suppose we have a model where $t \vDash \blacktriangle\alpha \wedge \blacktriangledown(\alpha \vee \beta)$ and $t \nvDash \blacktriangle(\neg\alpha \vee \gamma)$. Then either α is always true or always false in the past of t but, if we want $\blacktriangledown(\alpha \vee \beta)$ to be true at t, α must be always false in its past. Hence, $\neg\alpha$ is always true in the past of t, as well as $\neg\alpha \vee \gamma$, and $\blacktriangle(\neg\alpha \vee \gamma)$ is true at t. Axiom A3.2 can be tested in a similar way. For A4.1 suppose $t \vDash \alpha$ and $t \nvDash \blacktriangle(\beta \vee \triangledown\alpha \vee \triangledown(k \wedge \alpha))$. At every instant t' such that $t' < t$ we have that α is possible in the future, but this means either $t' \vDash \triangledown\alpha$ or $t' \vDash \triangledown(k \wedge \alpha)$. Since t' is an arbitrary instant preceding t, then $\beta \vee \triangledown\alpha \vee \triangledown(k \wedge \alpha)$ is necessary in the past of t and $t \vDash \blacktriangle(\beta \vee \triangledown\alpha \vee \triangledown(k \wedge \alpha))$. The same holds, *mutatis mutandis*, for A4.2. The validity of A5, Modus Ponens, R1.1 and R1.2 is straightforward. For rules R2.1 and R2.2 consider that $\alpha \equiv \beta$ is valid in a model built on $\mathfrak{G}$ in three cases: (i) α and β hold at every instant, (ii) α and β hold at no instant, (iii) there are some instants where both α and β hold and at all other instants neither α nor β hold. In cases (i) and (ii) we have that, at an arbitrary instant t, $\blacktriangle\alpha$ and $\blacktriangle\beta$, as well as $\triangle\, \alpha$ and $\triangle\, \beta$, hold. In case (iii) if for some instant t we have $t \nvDash \blacktriangle\alpha$ we also have $t \nvDash \blacktriangle\beta$ and if for some instant t' we have $t' \nvDash \triangle\, \alpha$ we also have $t' \nvDash \triangle\, \beta$. Therefore, in all cases $\blacktriangle\alpha \equiv \blacktriangle\beta$ and $\triangle\, \alpha \equiv \triangle\, \beta$ hold.
Q.E.D.

The definition of canonical models for logics of contingency is due to [3] and [5]. Here we appeal to the specific technique used in [8] for a logic containing a propositional constant. Let $\mathfrak{G}_c = \langle T_c, <_c, >_c, \Pi\rangle$ be the general frame of the canonical model, where only k-forked valuations are allowed and any instant $t \in T_c$ is a maximal set of formulas consistent with $\mathbf{C}_t$. The relations "earlier" and "later" can be defined thanks to the functions $\mathbf{p}$ and $\mathbf{f}$: if t is a maximal consistent set of formulas then $\mathbf{p}(t) = \{\alpha | \blacktriangle\alpha \wedge \blacktriangle(\alpha \vee k) \in t\}$ and $\mathbf{f}(t) = \{\alpha | \vartriangle \alpha \wedge \vartriangle (\alpha \vee k) \in t\}$. We say that $t' < t$ iff $\mathbf{p}(t) \subseteq t'$ and $t'' > t$ iff $\mathbf{f}(t) \subseteq t''$. The canonical model of $\mathbf{C}_t$, $\mathfrak{M}_c = \langle \mathfrak{G}_c, v_c, \rangle$, is such that:

- v_c is a k-forked valuation (it satisfies the admissibility conditions defined above);
- $t \vDash p$ iff $p \in t$, for every propositional variable p and instant t.

Lemma 5.2 For every formula α and every instant t in $\mathfrak{M}_c$, $t \vDash \alpha$ iff $\alpha \in t$.

Proof This fact holds by definition of $\mathfrak{M}_c$ for propositional variables and is preserved by boolean operators. The interesting cases are $\alpha \equiv \vartriangle \beta$ and $\alpha \equiv \blacktriangle\beta$. We do the first case. Suppose $\vartriangle \beta \in t$. Then either $\beta \in \mathbf{f}(t)$ or $\neg\beta \in \mathbf{f}(t)$, otherwise we would have, according to the definition of $\mathbf{f}(t)$, $\triangledown(\beta \vee k) \in t$ and $\triangledown(\neg\beta \vee k) \in t$, which would make t inconsistent. Indeed, $\beta \vee k$ and $\neg\beta \vee k$ are together contingent in the future of t only if β is neither always true nor always false after t, which means $\triangledown\beta \in t$, whence $\vartriangle \beta \notin t$, contrarily to the initial hypothesis. If $\beta \in \mathbf{f}(t)$ then, for induction on the complexity of β, for any instant t' such that $t' > t$, $t' \vDash \beta$ and this entails $t \vDash \vartriangle \beta$; if $\neg\beta \in \mathbf{f}(t)$, instead, for any instant t' such that $t' > t$, $t' \nvDash \beta$, and $t \vDash \vartriangle \beta$ as well.

Conversely, suppose $\vartriangle \beta \notin t$. Then the sets $t' = \mathbf{f}(t) \cup \{\beta\}$ and $t'' = \mathbf{f}(t) \cup \{\neg\beta\}$ are consistent. Indeed if t' were not consistent there would be $\{\gamma_1, ...\gamma_n\} \subseteq \mathbf{f}(t)$ such that $\gamma_1 \wedge ... \wedge \gamma_n \to \neg\beta$ and we could derive $\neg\beta \in \mathbf{f}(t)$, whence $\vartriangle \beta \in t$. The same argument shows the consistency of t''. Furthermore, for induction on the complexity of β, $t' \vDash \beta$ and $t'' \nvDash \beta$, which means $t \vDash \triangledown\beta$, that is $t \nvDash \vartriangle \beta$.
Q.E.D.

Theorem 5.3 $\mathbf{C}_t$ is complete with reference to the class Θ_0.

Proof The fact that every instant is preceded and followed by two instants is an immediate consequence of the validity of axiom A5. Thus, it rests for us to prove that $<$ and $>$ are mutually inverse. Suppose $t' < t$ for some t and t' belonging to $\mathfrak{M}_c$: this means that $\mathbf{p}(t) \subseteq t'$. Additionally, suppose that for some formula γ, $\vartriangle \gamma \wedge \vartriangle (\gamma \vee k) \in t'$: we must prove $\gamma \in t$ to obtain $t > t'$. For *reductio*, assume $\gamma \notin t$, then $\neg\gamma \in t$. In such case $\neg\gamma \to \blacktriangle(\beta \vee \triangledown\gamma \vee \triangledown(k \wedge \neg\gamma)) \in t$, as a consequence of the fact that axiom

A4.2 belongs to every maximal set of formulas consistent with $\mathbf{C}_t$. We need to consider some possible substitutions of β, which is an arbitrary formula in A4.2. For $\beta = k$, since $\blacktriangledown k \in t$, then $k \vee \triangledown\gamma \vee \triangledown(k \wedge \neg\gamma)$ must be true at every instant preceding t and, in particular, it must be true at t'. We can safely assume $\neg k \in t'$ and concentrate on the other two disjuncts; otherwise, if $k \in t'$, we may use the very same argument for the substitution $\beta = \neg k$. If $\triangledown\gamma \in t'$ then $\vartriangle \gamma \notin t'$, but we assumed $\vartriangle \gamma \in t'$, so there is a contradiction. If $\triangledown(k \wedge \neg\gamma) \in t'$, then γ is somewhere false in the future of t'; but we also know that $\vartriangle \gamma \in t'$, so we must say that γ is always false in the future of t'. However, in this situation we get $\triangledown(k \vee \gamma) \in t'$, because $\triangledown k \in t'$. Hence, $\vartriangle (k \vee \gamma) \notin t'$, which contradicts our assumption. We must conclude that $\gamma \in t$ and this means $t > t'$. A similar argument can be used to prove $t' < t$ from $t > t'$: the relations "before" and "after" are mutually inverse in $\mathfrak{M}_c$ and its frame belongs to the class Θ_0.
Q.E.D.

Corollary 5.4 The operators $\blacksquare$ and $\square$ are definable in $\mathbf{C}_t$.

Proof A straightforward adaptation to the temporal context of a result already obtained in [8]; the definability of past/future necessity in terms of past/future non-contingency in $\mathbf{C}_t$ is granted once it is proved that $\blacksquare\alpha \equiv (\blacktriangle\alpha \wedge \blacktriangle(k \vee \alpha))$ and $\square\alpha \equiv (\vartriangle \alpha \wedge \vartriangle (k \vee \alpha))$ are valid in every model for $\mathbf{C}_t$. Suppose there is an instant t of a model $\mathfrak{M}$ at which $\blacksquare\alpha$ and $\blacktriangle\alpha \wedge \blacktriangle(k \vee \alpha)$ have different values. There are two cases to be analysed: what happens when $\blacksquare\alpha$ is true and what happens when it is false. In the first case, α is always true in the past of t, hence $\blacktriangle\alpha$ is true at t, as well as $\blacktriangle(k \vee \alpha)$ and we easily get a contradiction with the initial hypothesis that the two formulas involved in the definition have different truth values. In the second case there is an instant t' preceding t and such that α is false at t'. We can infer that either α is not possible or it is contingent in the past of t; if it is contingent, then $\blacktriangle\alpha \wedge \blacktriangle(k \vee \alpha)$ cannot be true, whereas if it is not possible, we have $\blacktriangle\alpha$ true at t. But what about $\blacktriangle(k \vee \alpha)$? Given the validity of axiom A5, k is sometimes true and sometimes false in the past of t, hence we have $\blacktriangledown(k \vee \alpha)$ true at t and this implies that $\blacktriangle\alpha \wedge \blacktriangle(k \vee \alpha)$ is false at t, contrarily to our initial assumption. A similar analysis shows that $\square\alpha$ and $\vartriangle \alpha \wedge \vartriangle (k \vee \alpha)$ cannot have different values at any instant of a model for $\mathbf{C}_t$.
Q.E.D.

6 The system $\mathbf{C}_t lin$

In this section we will be concerned with an extension of $\mathbf{C}_t$ interpreted on transitive and linear structures. Within a deterministic view of time the property of temporal contingency can be ascribed to propositions which change their truth value in the past or in the future of a given instant, even if there is no actual branching of possibilities. Let the axiomatic basis of $\mathbf{C}_t$ be enriched with the following formulas:[12]

A6.1 $\blacktriangle\alpha \rightarrow \blacktriangle(\blacktriangle\alpha \vee \beta)$
A6.2 $\triangle\ \alpha \rightarrow\triangle\ (\triangle\ \alpha \vee \beta)$
A7.1 $(\blacktriangle\alpha \wedge \blacktriangle(\alpha \vee k)\wedge \triangle\ \alpha\wedge \triangle\ (\alpha \vee k) \wedge \alpha) \rightarrow \blacktriangle(\beta\vee \triangle\ (\alpha \vee \gamma))$
A7.2 $(\blacktriangle\alpha \wedge \blacktriangle(\alpha \vee k)\wedge \triangle\ \alpha\wedge \triangle\ (\alpha \vee k) \wedge \alpha) \rightarrow\triangle\ (\beta \vee \blacktriangle(\alpha \vee \gamma))$

Hereafter the resulting system will be called $\mathbf{C}_t lin$. Such a system is interpreted in the subclass of Θ_0 containing only transitive and linear general frames. For the sake of simplicity, this class will be named Θ_1.

Theorem 6.1 $\mathbf{C}_t lin$ is sound with reference to the class Θ_1.

Proof Take a model $\mathfrak{M}$ built on a frame $\mathfrak{G} \in \Theta_1$ and suppose for some instant t we have $t \vDash \blacktriangle\alpha$ but $t \nvDash \blacktriangle(\blacktriangle\alpha \vee \beta)$. We know that in the past of t α has the same value and transitivity entails that for every instant t' such that $t' < t$, the past of t' is also the past of t, hence we have $t' \vDash \blacktriangle\alpha$ and $t' \vDash \blacktriangle\alpha\vee\beta$. Then it results $t \vDash \blacktriangle(\blacktriangle\alpha \vee \beta)$, contrarily to our assumption. An analogous proof can be given for A6.2. In the case of A7.1 suppose for an instant t we have $t \vDash (\blacktriangle\alpha \wedge \blacktriangle(\alpha \vee k)\wedge \triangle\ \alpha\wedge \triangle\ (\alpha \vee k) \wedge \alpha)$ but $t \nvDash \blacktriangle(\beta\vee \triangle\ (\alpha \vee \gamma))$; this means that $t \vDash \blacktriangledown(\beta\vee \triangle\ (\alpha \vee \gamma))$ and, in particular, there is an instant $t' < t$ such that $t' \vDash \neg(\beta\vee \triangle\ (\alpha \vee \gamma))$, so $t' \vDash \neg \triangle\ (\alpha \vee \gamma)$, i.e. $t' \vDash \triangledown(\alpha \vee \gamma)$; this, in turn, entails that there is an instant $t'' > t'$ such that $t'' \vDash \neg(\alpha \vee \gamma)$, so $t'' \vDash \neg\alpha$. Linearity tells us that either $t'' < t$ or $t'' > t$ or $t'' = t$; however, we know that α is true at t and Corollary 5.4 tells us that α is also true at any instant in the past or in the future of t, so we reached a contradiction. The same holds, *mutatis mutandis*, for A7.2.
Q.E.D.

Theorem 6.2 $\mathbf{C}_t lin$ is complete with reference to the class Θ_1.

[12] A6.1 and A6.2 represent a duplication for past and future modalities of an axiom used by Kuhn in [5] for a logic interpreted on transitive frames.

Proof The canonical model $\mathfrak{M}_c$ for $\mathbf{C}_t lin$ is built in the same way as for $\mathbf{C}_t$. We have to demonstrate that the canonical model is linear and transitive. The proof of transitivity is a slight adaptation to the temporal context of a result in [5]. Here we analyse only the case of transitivity towards the past, since the temporal relations are mutually inverse. Let t, t' and t'' be three instants in $\mathfrak{M}_c$ such that $t < t'$ and $t' < t''$. Suppose $\blacktriangle\alpha \wedge \blacktriangle(\alpha \vee k) \in t''$ but $\alpha \notin t$, so $\neg\alpha \in t$. Given that t'' satisfies A6.1, it follows that $\blacktriangle(\blacktriangle\alpha \vee \beta) \in t''$ as well as $\blacktriangle(\blacktriangle(\alpha \vee k) \vee \beta) \in t''$. Consider the substitution $\beta = k$ and suppose $k \notin t'$ (otherwise the same argument for $\beta = \neg k$). Since $\blacktriangledown k \in t''$, then $\blacktriangle(\alpha \vee k) \in t'$ and $\blacktriangle\alpha \in t'$. From the definition of temporal precedence in the canonical model we get $\alpha \in t$, contrarily to our assumption, and conclude $t < t''$.

As far as linearity is concerned we must prove that for every three instants t, t' and t'' such that $t < t'$ and $t < t''$ either $t' < t''$ or $t'' < t'$ or $t' = t''$. Assume that none of these three conditions hold: then we must have, for some formulas ϕ, ψ and χ, that the set $\Gamma = \{\vartriangle \phi \wedge \vartriangle (\phi \vee k), \blacktriangle\psi \wedge \blacktriangle(\psi \vee k), \chi\}$ is a subset of t' but ϕ, ψ and χ do not belong to t''. This means $\neg(\phi \vee \psi \vee \chi) \in t''$. Let α be a shorthand for $\phi \vee \psi \vee \chi$, then it can be easily proved that also the set $\Gamma' = \{\vartriangle \alpha \wedge \vartriangle (\alpha \vee k), \blacktriangle\alpha \wedge \blacktriangle(\alpha \vee k), \alpha\}$ is a subset of t'. From A7.1 we can infer $\blacktriangle(\beta \vee \vartriangle (\alpha \vee \gamma)) \in t'$. Consider the substitution $\beta = k$ and assume $k \notin t$ (otherwise the same argument for $\beta = \neg k$): since $\blacktriangledown k \in t'$, then $\vartriangle (\alpha \vee \gamma) \in t$; now, consider the substitution $\gamma = k$ and assume $k \notin t''$ (otherwise the same argument for $\gamma = \neg k$): since $\triangledown k \in t$, then $\alpha \in t''$, which means $\phi \vee \psi \vee \chi \in t''$, contrarily to our assumption. As a conclusion, either $t' = t''$ or one precedes the other. The proof of linearity towards the past is analogous, so the frame of $\mathfrak{M}_c$ is in Θ_1.
Q.E.D.

7 Final remarks

We developed two systems of Priorean temporal logic based on a language with two primitive operators of contingency, one for past reference and the other for future reference, ensuring the fundamental property of the relations “before” and “after”, i.e. mutual inversity. The stronger system is interpreted on linear and transitive structures. The proofs of completeness were based on general frames in order to point out the restriction on admissible valuations for the propositional constant k; thanks to this constant and its characteristic axiom we were able to introduce the usual operators of past necessity and future necessity. Such result shows that the notion of contingency can be used to define (in an indirect way) the other modalities even in a temporal context. Further research can be done in order to develop systems of temporal contingency of different strength, which may be suitable for reasoning on other classes of temporal structures and to establish their relation with traditional systems of tense logic.

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