Fixed Point Models for Theories of Properties and Classes

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Abstract

There is a vibrant (but minority) community among philosophical logicians seeking to resolve the paradoxes of classes, properties and truth by way of adopting some non-classical logic in which trivialising paradoxical arguments are not valid. There is also a long tradition in theoretical computer science—going back to Dana Scott's fixed point model construction for the untyped λ-calculus [35, 36]—of models allowing for fixed points. In this paper, I will bring these traditions closer together, to show how these model constructions can shed light on what we could hope for in a non-trivial model of a theory for classes, properties or truth featuring fixed points.

1 The Target

One well-worn motivation for *non-classical* mathematical theories is found in the *class struggle*—our attempt to give a coherent theory of classes that allows for a naïve class abstraction principle

$$a \in \{x : \phi(x)\} \text{ iff } \phi(a)$$

according to which any predicate $\phi(x)$ (with the variable x free) determines a class of all and only those things satisfying that predicate. It is well known (since Russell's response in 1902 to Frege's use of the class abstraction principle, or Frege's "Basic Law V" [23]) that class abstraction leads to paradox in any theory of classes in which membership is a predicate available for use in class abstraction. Substituting the Russell class, $\{x : x \notin x\}$, into the abstraction principle gives us

$$\{x : x \notin x\} \in \{x : x \notin x\} \text{ iff } \{x : x \notin x\} \notin \{x : x \notin x\}$$

a biconditional between a formula ($\{x : x \notin x\} \in \{x : x \notin x\}$) and its negation, which is unsatisfiable in classical or in intuitionistic logic. In those logics, the theory of class

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abstraction has no models. The defender of class abstraction in its full form must look elsewhere to find models for the theory.

The same holds for theories of properties, too. Just as it seems plausible to collect those objects satisfying the predicate $\phi(x)$ in a class, it seems plausible that for any predicate $\phi(x)$, there is a *property* borne by all and only those things satisfying that predicate. (What property? Why not the property of *satisfying* $\phi(x)$.) We have the following *property* abstraction scheme.

$$a \mathrel{\varepsilon} \lambda x.\phi(x) \text{ iff } \phi(a)$$

asserting that the property $\lambda x.\phi(x)$ of satisfying $\phi(x)$ is borne by all and only the objects that satisfy $\phi(x)$. Here there is analogue of Russell's paradox for classes—the heterological paradox concerning the property of non-self-application, $\lambda x.(x \mathrel{\not\varepsilon} x)$.

$$\lambda x.(x \mathrel{\not\varepsilon} x) \mathrel{\varepsilon} \lambda x.(x \mathrel{\not\varepsilon} x) \text{ iff } \lambda x.(x \mathrel{\not\varepsilon} x) \mathrel{\not\varepsilon} \lambda x.(x \mathrel{\not\varepsilon} x)$$

Paradoxes like these have led many to the conclusion that what is needed in a non-classical logic for classes and properties is a different, non-classical treatment of *negation.* This is true enough, but is not the whole story. Haskell Curry noted [18] that *the conditional* gives rise to similar paradoxes. For any statement p, the abstraction scheme gives us

$$\{x : x \in x \to p\} \in \{x : x \in x \to p\} \text{ iff } \{x : x \in x \to p\} \in \{x : x \in x \to p\} \to p$$

$$\lambda x.(x \mathrel{\varepsilon} x \to p) \mathrel{\varepsilon} \lambda x.(x \mathrel{\varepsilon} x \to p) \text{ iff } \lambda x.(x \mathrel{\varepsilon} x \to p) \mathrel{\varepsilon} \lambda x.(x \mathrel{\varepsilon} x \to p) \to p$$

which are just as paradoxical as Russell's paradox in classical or intuitionist logic—it furnishes a proof of p, whatever p happens to be. Curry's paradox is another example of what is, in fact a properly general scheme. For *any* sentence context $F(-)$ we have

$$\{x : F(x \in x)\} \in \{x : F(x \in x)\} \text{ iff } F(\{x : F(x \in x)\} \in \{x : F(x \in x)\})$$

$$\lambda x.F(x \mathrel{\varepsilon} x) \mathrel{\varepsilon} \lambda x.F(x \mathrel{\varepsilon} x) \text{ iff } F(\lambda x.F(x \mathrel{\varepsilon} x) \mathrel{\varepsilon} \lambda x.F(x \mathrel{\varepsilon} x))$$

We can form a statement $\{x : F(x \in x)\} \in \{x : F(x \in x)\}$ (call this f) that is a fixed point for the context $F(-)$, in the sense that f holds iff $F(f)$. Russell's Paradox is formed with $F(q)$ set to $\neg q$, Curry's when $F(q)$ is $q \to p$, and there are many other contexts like these that don't have fixed points in classical logic or intuitionist logic. The search for models for theories of classes and properties involves finding formal systems that allow fo the right kind of *fixed points.* In the rest of this paper, I will attempt to motivate some strategies and techniques for finding such models.

However, before we continue, it is worth considering the difference between *class* theories and *property* theories. Classes are determined by their members. Two classes are identical if they have the same elements. They differ only when they can be distinguished by way of their members. *Properties*, on the other hand, need not satisfy a condition of extensionality: if it turned out that all renates are cordates, and all cordates are renates, then while the class of cordates would be identical to the class of renates (since they have exactly the same members), there is no need to conclude that the property of being a renate and the property of being a cordate are identical. There are challenges to the formalisation of extensionality in non-classical theories of classes, and they are not the focus of this paper. One formulation which I have defended elsewhere [33] is this sequent rule:

$$\frac{\Gamma, x \in a \succ x \in b, \Delta \quad \Gamma, x \in b \succ x \in a, \Delta}{\Gamma \succ a = b, \Delta}$$

but the rest of this paper will be concerned with model constructions rather than proof systems. In these model constructions, the semantic value of a class term will be determined by its extension (in an appropriate sense), so at least the *spirit* of extensionality will be respected. I will leave for another time the investigation of what form an extensionality condition is satisfied. One reason for the difficulty in expressing an axiom of extensinonality in non-classical systems has been the intensionality of the conditional in the underlying logic and the appropriate formulation of the conditional connection in the coextensiveness condition, and then betewen coextensiveness and identity. (See Section A.7 "Extensionality and Identity Determinables" of Routley's *Jungle Book* [34] for a helpful discussion). *That*, however, is not the problem for a straightforward examination of extensionality in the models I will explore here. Rather, it is the more fundamental issue to the effect that it is not clear how best to model *identity* in these models.

2 Model Construction

The study of non-classical theories of classes and properties has featured various kinds of model construction [10–13, 20, 21, 25, 28, 29, 32], chiefly with an aim to show that the theory is non-trivial. If a theory has models (and if those models aren't trivial—if they distinguish at least some truths from untruths) then the theory is at least, coherent. In the face of Russell's and Curry's Paradoxes, and the ever present threat of trivialisation through fixed points, such a result is welcome. However, averting the threat of triviality is not all that we might hope to gain from a model construction. Models can be used for more than this, and when evaluating the models we seek to use for a theory of properties and classes, we would do well to keep in mind the variety of ways that models play a role in the study of theories.

- A central use of models, prior to the role of proving non-triviality is in the definition of *validity*. According to the model theoretic analysis of logical consequence, an argument is valid if and only if every model of the premises is also a model of the conclusion. Consequence, relativised to a particular theory can be analysed in a similar way. Given a theory T, an argument is T-valid if and only if every T-model satisfying the premises aso satisfies the conclusion. Accordingly, if there is *some* T-model in which some claims are satisfied and others are not, then at least some arguments are not T-valid, and the theory is non-trivial. The proof of non-triviality relies on the underlying analysis of consequence at least in the weak sense that if we possess a model in which the premises P are true and the conclusion C is untrue, then the argument from P to C is invalid. A rich class of models of a theory provides the materials for an analysis of T-consequence, and the more we understand the structure of such models, the more hope there is that we can understand T-consequence and the structure of the theory T itself.

- Models for theories also provide a way to relate distinct theories. If we can build models of a theory T out of models of another theory T' (for example, encoding models of hereditarily finite sets out of models of arithmetic), then this can provide way—given sufficient expressive resources—to relate the theories T and T'. We find ways of *relatively interpreting* T in T'. Another example of this work is in the geometric models of non-Euclidean geometries, which provide relative interpretability and consistency results for exotic theories [26]. This is another way that model constructions can be of use in the analysis of a theory.

- In a related fashion, constructions of different models for T can give us a greater understanding of what the theory T can be taken to be *about*. Providing a large class of models for a theory gives us a range of possible ways to interpret the primitive vocabulary of the theory, and hence, a range of different structures in which the tools of the theory may be applied.

- The fact that a particular kind of model can be understood as a model of a theory might also motivate the theory itself. Instead of taking the theory as given and the model as generated by that theory, we may start with a structure and design a language to appropriately describe that structure and axiomatise it. The relation between *model* and *theory* may be used in both directions, to motivate or to ground either the model or the theory.

Consider how these distinctions play out in different models of theories of sets and classes. Consider the conception of models of classical set theories (ZFC and its cousins) motivated by the iterative conception of set [9,31], according to which sets are constructed in 'stages' up the ordinal hierarchy to form ranks R_α at each level α (see Figure 1). A model construction of this kind not only provides an intuitive motivation for the *kind* of set concept under consideration, it also provides the raw materials for relating these models to other kinds of models, for example, Church–Oswald set theories with a Universal set [16,22], in which sets can be defined not just by what they *include*, but also by what the *exclude*. This understanding of the structure of models of ZFC has also played an important role in the study of ZFC itself, in particular, the proof of the independence of the continuum hypothesis, and the strength of large cardinal axioms and forcing conditions.

Models for set theories can take other forms, too. The study of models of set theories without the axiom of foundation in terms of directed graphs (see Figure) play a number of roles in the study of set theories without foundation [2]. They show the relationship

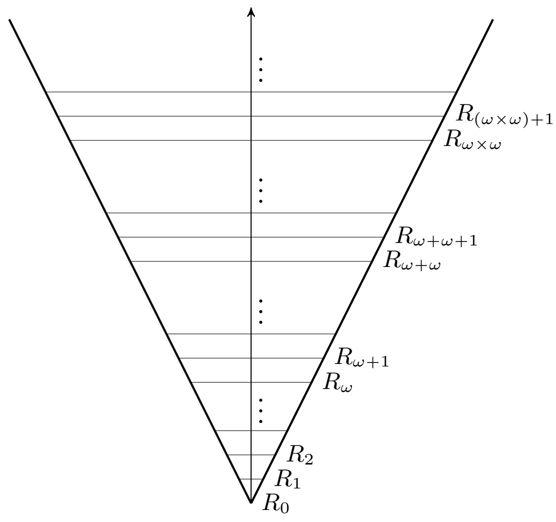
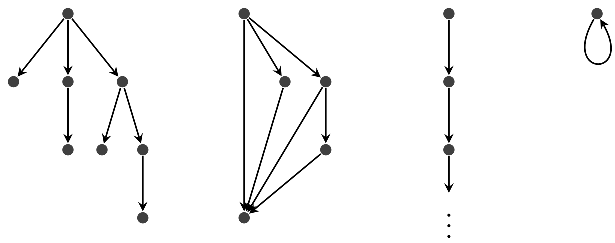


Figure 1: The Iterative Hierarchy



$\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$ an a where $a = \{a\}$

Figure 2: Sets as graphs in models of non-wellfounded sets

between wellfounded and non-wellfounded set theories (by showing how sets in one universe can be modelled inside the other). They also motivate a choice of a replacement to the anti-foundation axiom. (In Figure 2, the two graphs at the left are in some sense 'observationally equivalent'—they are bisimilar—as are the two graphs on the right. One form of anti-foundation axiom takes any two sets that are observationally equivalent to be identical as sets, so in this case there is only one set a which is identical to its own singleton $\{a\}$.) Finally, these models also show where one might expect models of non-wellfounded set theories to be applicable—where the membership structure allows for loops like these [8]. Models for traditional, classical set theory—in both its wellfounded and nonwellfounded forms—provides a rich vein of tools and techniques for the study and application of those theories.

The same is true for the untyped λ-calculus. This calculus, since its introduction in the work of Schönfinkel, Church, Curry and others [14, 15, 17–19][1] stood in need of a model theory, not only because the consistency or coherence of the theory was in question, but because it was altogether unclear how to understand the objects of the theory. In the untyped calculus, the domain of objects is understood both as a class of functions and the domain of application of those very functions. In the basic calculus, terms consist of *variables* $(x, y, \ldots)$, *application* terms (where M and N are terms, MN is M *applied* to N) and *abstraction* terms (where M is a term, $(\lambda x.M)$ is a term, in which the variable x is now *bound*), with the constraint that $(\lambda x.M)N = M[x := N]$, the result of applying an abstraction term to another term is formed by substitution in the variable bound by the λ abstractor. In such a calculus, for any term M, the self-application MM is well-formed. Each object of the calculus can be viewed as a function, and as the domain of those functions. Each object is in the domain D, and also serves as a function $D \to D$. This makes finding models for the calculus a subtle matter, because the collection of *all* functions $D \to D$ has a strictly higher cardinality than D itself—and so, no such set of *all* such functions can serve as a model. If we are to find a model in which λ-terms are indeed modelled by *functions*, we must be more selective about which functions to choose.

Dana Scott's construction of models for the untyped λ-calculus was an ingenious solution to the puzzle of finding a suitable domain of functions to interpret the calculus. Given a starting set D_0, ordered by $\sqsubseteq$, we can consider D_1, the set $[D_0 \to D_0]$ of *order preserving* functions on D_0, which are themselves ordered pointwise, using the ordering on D_0. (We set $f \sqsubseteq g$ iff $(\forall x)(f(x) \sqsubseteq g(x))$.) We can lift D_1 into a larger set D_2, and so on, by setting D_{i+1} to be $[D_i \to D_i]$ and the embedding sends D_i into D_{i+1}. This construction has a limit D_∞, and it is not difficult to show that D_∞ is isomorphic (as a partially ordered set) to $[D_\infty \to D_\infty]$, the set of continuous functions from D to D under the partial order topology, and the domain is closed under the operations of the untyped λ-calculus, and this model construction, and others like it, have given insight into how the λ-calculus can be interpreted and applied.

Another example of a model construction in a field closely related to theories of classes and properties with fixed points is the fixed point construction for theories of truth, due to Kripke [27] and Martin and Woodruff [30, 37]. The construction is now standard. Take a language $\mathcal{L}$, interpreted in a model $\mathfrak{M}$ on domain D which includes a denotation for each quotation name $\langle A \rangle$ for every formula A in $\mathcal{L}[T]$, the language $\mathcal{L}$ extended with a truth predicate T. We extend $\mathfrak{M}$ to interpret T in a three-valued evaluation. We start in $\mathfrak{M}_0$ by assigning $T\langle A \rangle$ the value true (1) for all sentences A true in $\mathfrak{M}$, the value false (0) for all sentences false in $\mathfrak{M}$, and n otherwise—in particular, sentences A which *use* the truth predicate are uninterpreted in $\mathfrak{M}$, so $T\langle A \rangle$ is not yet assigned a classical truth value. We can repeat the process (as sketched in Figure 3), sending a model $\mathfrak{M}_\alpha$ to its successor $\mathfrak{M}_{\alpha+1}$ in which more truth sentences are assigned classical values 1 and 0,

Given that the underlying logic satisfies an important preservation property [expand] this feature extends to the whole language. More sentences are assigned 1 and 0 at each

[1] See Alama's overview [3] for a brisk introduction, or Barendregt's classic text for details [7].

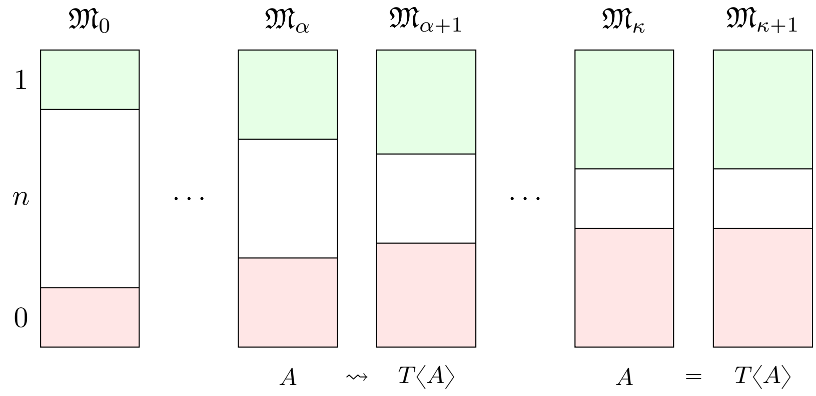


Figure 3: Fixed Point construction for truth.

stage. We can extend the hierarchy to limit ordinals, by assigning 1 (or 0) to $T\langle A\rangle$ at $\mathfrak{M}_\beta$ iff $T\langle A\rangle$ was assigned 1 (or 0) at an earlier stage $\mathfrak{M}_\alpha$ for $\alpha < \beta$. Given that the language $\mathcal{L}[T]$ has *some* cardinality, it will be outrun by some ordinal, and given that the evaluations of classically assigned values are monotonically increasing, they will come to a fixed point. At this stage ($\mathfrak{M}_\kappa$ in Figure 3) we have a truth evaluation for the language $\mathcal{L}[T]$ for which $T\langle A\rangle$ has the same semantic value as A, for every sentence A, including sentences featuring the truth predicate. This construction has proved important in understanding the behaviour of truth predicates which allow for fixed points, in which the semantic value of a truth predication $T\langle A\rangle$ is identical to that of the sentence A. This construction is agnostic between different understandings of the semantic value n assigned in the evaluation. While Kripke and Martin and Woodruff each conceived of the intermediate value 'n' as a truth value 'gap'—or rather, a failure to assign a genuine truth value—the construction itself does not necessitate this understanding. It is equally coherent to conceive of the intermediate value n as a truth value 'glut' as a truth value 'gap' or to have some other conception of the three values in the construction. In fact, as we will see in Section 4, there is no need to restrict the interpretation to the three values $0, n, 1$ for the construction to work. Provided that the underlying space of semantic values are partially ordered and the connectives and quantifiers are appropriately order preserving so as to allow for preservation of semantic value from stage to stage, this construction will work. It is a robust technique with many applications.

As is now well known, the same kind of construction works for theories of classes and properties [10, 11, 13, 25].[2] We proceed in the same way, except instead of assigning the extension of the truth predicate, we assign the extension of the membership predicate '$\in$', at each stage by assigning '$a \in \{x : \phi(x)\}$' the value in $\mathfrak{M}_{\alpha+1}$ the value of the sentence '$\phi(a)$' in $\mathfrak{M}_\alpha$. Notice that in this construction, it is the extension of the membership

[2]Gilmore's 1974 paper was presented at a 1967 UCLA meeting on Set Theory, and Brady's work on consistency and non-triviality proofs for set theories in intensional logics was independent.

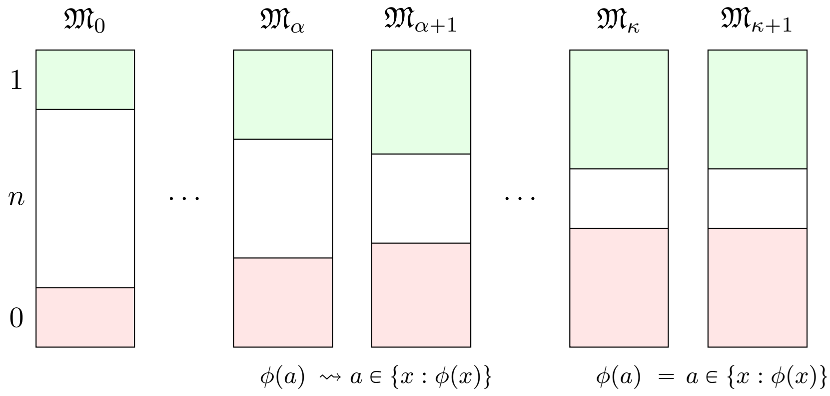


Figure 4: Fixed Point construction for membership.

predicate that varies from stage to stage, not the underlying domain D of the models $\mathfrak{M}_\alpha$. The domain of evaluation, as far as this construction is concerned, is constant. It is a *term* model, in which the domain consists of the terms $\{x : \phi(x)\}$ for each formula $\phi(x)$. This construction shows what the theory is *about* in only a very weak sense. It gives no insight into the structure of the domain of objects under consideration—but it does give us a way to understand the interpretation of the membership relation, or how the membership facts could be grounded.

When we reflect on the usefulness of model constructions, it seems clear that this kind of construction does not tell us much about what the theories of classes or properties can be *about*, or how to relatively interpret one theory in another. While it gives some insight into the membership predicate, and it reassures us that the theories are coherent, the construction gives us little insight into the nature or structure of classes themselves. There is an opportunity, therefore, to explore further the world of model constructions in order to gain insight into the possibilities for class and property theories with fixed points, and even doing so without prejudging the precise strength or expressive power of the logic underlying that construction. After all, model constructions give us constraints on the kinds of logics interpretable on them.[3] So, in the rest of this paper I turn to model constructions that attempt to do for class theories what Scott's construction did for the untyped lambda calculus.

[3] Think of Kripke frames for intuitionist logic, in which propositions are interpreted as upwardly closed sets of points—in these models, boolean negation is not definable, because it violates the upward closure condition, while a weak paraconsistent negation ($-A$ holds at point x if and only if A fails to hold at some point $y \leqslant x$) *is* definable. Here the model constrains the logic interpretable on that model, but does not sharply define it.

3 Classifying Class Theories

To present a richer conception of a model construction for theories of classes and properties with fixed points, we should be a little more specific about how the uuderlying logic is to be conceived. As we saw in the first section, negation is an issue in the paradoxes, but it is not the only issue. The most pressing issue is the availability of *fixed points.* For any formula context $F(-)$ we need to allow for the presence of a formula A which is, in some sense, *equivalent to* $F(A)$, for the comprehension scheme tells us that

$$\{x : F(x \in x)\} \in \{x : F(x \in x)\} \text{ iff } F(\{x : F(x \in x)\} \in \{x : F(x \in x)\})$$

The first crucial distinction to be drawn, then, is not between truth value gaps or gluts, whether the language is paraconsistent or paracomplete, or what connectives and quantifiers are present—but the status of the *equivalence* in the comprehension scheme itself. What is the status of the 'if and only if' in comprehension? One answer is that given in the constructions for truth or classes in the previous section (or equivalently for Scott's construction for the untyped λ-calculus), and that is semantic *identity.* In our models, we will take the membership claim $a \in \{x : \phi(x)\}$ to have the *same* semantic value as the predication $\phi(a)$. A different answer is to settle for a *weaker* semantic connection between membership and predication, perhaps according to which the deduction between $a \in \{x : \phi(x)\}$ and $\phi(a)$ is *valid* though not preserving semantic value in every sense,[4] or perhaps the connection is even weaker still, in that the material biconditional $a \in \{x : \phi(x)\} \equiv \phi(a)$ is at least true, despite the fact that we may have one side designated and the other undesignated.[5] There are reasons to consider all sorts of conceptions of the status of the comprehension scheme, none the least that it is a difficult area and one in which it seems important to consider all options to evaluate their costs and benefits, but in this paper I will consider only the strong conception of the comprehension scheme, according to which $a \in \{x : \phi(x)\}$ has the same semantic value as $\phi(a)$, whatever the field of semantic values for formulas turns out to be. For the time being, let us think of the domain of semantic values of formulas as a collection Ω. These could be thought of as truth values, or some other structured collection of propositional values. The details of Ω are not yet impotant.

So, to keep the analogy with Scott's construction to the fore, we are looking for a construction in which we model classes by what they *do*. For the λ calculus, a term was interpreted as a function in $[D \to D]$, on a domain D isomorphic to the class of functions $[D \to D]$ itself. In our case, a class can also be conceived of as a function, but the function is not from a domain D to itself, but from a domain to Ω. Given an underlying domain D, we can consider a class of D objects as a function assigning to each D object d a value in Ω, where that value is the semantic value of the statement 'the object d is in this class.' If the values in Ω were simply the Boolean values $\{0, 1\}$ such a

[4] Perhaps the sentences are either both designated or both undesignated, so the deduction from one to the other preserves designation status, even though they might differ in semantic value

[5] In the paraconsistent logic LP, for example, we could have one side both true and false and the other false only [32].

function would be a characteristic function of the class. In a non-classical universe, the function is richer.

So, if we are classifying objects in some given D and we are working with an untyped universe of classification where we can also classify those classifications themselves, we are looking for a domain C of classes such that

$$C \cong [(C \cup D) \to \Omega]$$

That is, the classes C can be considered in two ways: as objects themselves, or as functions assigning for each object (either a class in C or an underlying domain object D) a semantic value in Ω. As with the untyped λ-calculus, Cantor's theorem forces us to not consider *every* possible function to Ω,[6] but to look to a restricted class of such functions. If we can find such a class, and if each formula $\phi(x)$ in the language of the theory determines a well-behaved function in that class, then we have a universe in which the strong comprehension scheme is satisfied. In the next section I will unpack this claim in some more detail, and examine some of its features in the case of a particular choice of the field Ω of semantic values.

However, before we get there, let us consider a little of what remains of the distinction between properties and classes, and the status of an *extensionality* condition in this setting. Recall, an extensionality condition is the claim that coextensive classes are identical. In this setting, we have said nothing about the internal conception of identity of classes. (Such a conception could be modelled by a function in $[C \times C \to \Omega]$, sending a pair of classes to a semantic value, but we have made no claims about such a predicate yet, and nor will we.) The choices for how to interpret such a notion are difficult and contested. The *external* conception of extensionality seems clearer. At the very least, classes in this model are totally determined (identified with) the classification functions in $[(C \cup D) \to \Omega]$. If it turned out that the 'cordate' classification function classified objects in just the same way that the 'renate' classification function, then these are *identical* objects in any model of this form. At least in this external sense, extensionality is satisfied in the stronger sense than in term models for classes. In term models, the term $\{x : Fx \wedge Gx\}$ is *distinct* as a term to the term $\{x : Gx \wedge Fx\}$, for example, even though they may have the same members—and do in any model in which $Fx \wedge Gx$ always receives the same semantic value as $Gx \wedge Fx$. In this sense, extensionality is externally respected to some degree in these models for class theories, but the way that this could to be modelled *internally* is an active research question.

4 Order and Continuity

In order to examine the properties of classes C such that

$$C \cong [(C \cup D) \to \Omega]$$

[6] Well, it forces us in the case where Ω has at least two members. A monistic or nihilistic universe in which there is at most one semantic value for statements seems beyond the scope of this paper.

it is impotant to say a little more about the properties of field Ω of semantic values. It is important to say *something*, but it helps to not be too specific. As we have seen, one of the features of the term model construction is an underlying syntactic sensitivity. The domain of evaluation, such as it is, is determined by the syntax of class terms. One small change to the underlying syntax (adding a new connective or even replacing one choice of primitive connectives by another) and the domain of evaluation changes completely. That is a cost of the approach using term models. Hopefully, we can do better, in finding a model construction in which the generation of the domain depends on different *features* of the class Ω of semantic values, but does not depend so tightly on an arbitrary choice of syntax. The core idea behind the fixed point constructions of truth and abstraction is the *preservation* properties of the vocabulary of the underlying logic. The core feature is this: the semantic values in Ω are ordered by some partial order $\sqsubseteq$ such that all connectives and quantifiers in the vocabulary are appropriately $\sqsubseteq$-preserving. We call $\sqsubseteq$ the *refinement* ordering on Ω, but we hold little store on any particular interpretation of refinement. In the case of a binary connective $\sharp$, if $x \sqsubseteq x'$ and $y \sqsubseteq y'$ then $x \sharp y \sqsubseteq x' \sharp y'$, and this generalises to other operators and arities. In the case of the three valued logic with semantic values $0, *, 1$ (where we use the ambiguous label '$*$' for the intermediate value, and not n, to clarify that we *really* do not care if the value is to be thought of as '*neither* true nor false' or '*both* true and false'). The underlying ordering on this choice of Ω is

$$\begin{tikzcd} 0 & & 1 \\ & * \arrow[ul] \arrow[ur] & \end{tikzcd}$$

where this ordering is not to be confused with the entailment ordering on Ω. As regards entailment, we have $0 < * < 1$. For *refinement* we have $* \sqsubset 0$ and $* \sqsubset 1$. The usual truth evaluations for negation, conjunction and disjunction on a three valued logic,

$\wedge$	0	$*$	1
0	0	0	0
$*$	0	$*$	$*$
1	0	$*$	1

$\vee$	0	$*$	1
0	0	$*$	1
$*$	$*$	$*$	1
1	1	1	1

$\neg$	
0	1
$*$	$*$
1	0

are refinement preserving, as are many other operators. However, the conditionals for the three valued logic Ł$_3$ and RM3 are *not* refinement preserving, and neither are the strong or weak negation operators $-$ and $\sim$.[7]

$\to_{Ł_3}$	0	$*$	1
0	1	1	1
$*$	$*$	1	1
1	0	$*$	1

$\to_{RM3}$	0	$*$	1
0	1	1	1
$*$	0	$*$	1
1	0	0	1

$-$	
0	1
$*$	0
1	0

$\sim$	
0	1
$*$	1
1	0

[7] In $* \to_{Ł_3} *$ is 1, but $1 \to_{Ł_3} 0$ is 0. This is a counterexample to the preservation of refinement, since $* \sqsubseteq 1$ and $* \sqsubseteq 0$, but $(* \to_{Ł_3} *) \not\sqsubseteq (1 \to_{Ł_3} 0)$. Similarly, $1 \to_{RM3} *$ is 0, but $1 \to_{RM3} 1$ is 1. With the negations, $-*$ is 0 while -0 is 1. Similarly, $\sim *$ is 1, while ~ 1 is 0.

For the construction, we do not care *which* connectives and quantifiers are present in our language, only that those that are preserve refinement.

This structure is not the only candidate for Ω that can do the job. Other choices are possible: for example, we could order the four valued set $\{0, 1, \star, *\}$ familiar from first degree entailment, as follows:

$$\begin{tikzcd} & \star & \\ 0 \arrow[ur] & & 1 \arrow[ul] \\ & * \arrow[ul] \arrow[ur] & \end{tikzcd}$$

or indeed, we could restrict our attention to the Boolean values $\{0, 1\}$, and order them like this: $0 \sqsubset 1$. Then $\wedge, \vee$, 0, 1 are order preserving, but $\neg$ and $\supset$ are *not* order preserving. Richer structures are possible, too. However, for the rest of this note, we will focus on $3 = \{0, *, 1\}$ as our choice for Ω where being specific makes any difference.

5 Order Models

In this section we will look at the behaviour of *order* models for class and property theories, for these have distinct properties, not shared by all models of class and property theories. Here is the precise definition, first of order models themselves, and then we will give an account of how to interpret a formal language in an order model.

DEFINITION 1: Given an order algebra $\langle \Omega, \sqsubseteq \rangle$ of semantic values, and a domain D of urelements, $\langle C, \sqsubseteq, \Uparrow, \Downarrow \rangle$ is a $\langle D, \Omega, \sqsubseteq \rangle$-ORDER MODEL iff the following three conditinos are satisfied:

- $\sqsubseteq$ is a partial order on C.
- $\Uparrow : C \to [C \cup D \to \Omega]$ is order preservering and invertible, where $[C \cup D \to \Omega]$ is the set of order preserving functions from $C \cup D$ to Ω.[8]
- $\Downarrow : [C \cup D \to \Omega] \to C$, where $\Downarrow = \Uparrow^{-1}$, is also order preserving.

For shorthand, we write '$\Uparrow(c)$' as '$c_{\Uparrow}$' and '$\Downarrow(f)$' as '$f_{\Downarrow}$.' So, for each $c \in C$, $c_{\Uparrow\Downarrow} = c$ and for each $f \in [C \cup D \to \Omega]$, $f_{\Downarrow\Uparrow} = f$.

To understand the significance of $\Uparrow$ and $\Downarrow$, if $b \in C \cup D$ and $c \in C$, then $c_{\Uparrow}(b)$ is the semantic value of the statement to the effect that the object b (either a class in C or an urelement in D) is in the class c. In other words, we could think of the membership fact $c_{\Uparrow}(b)$ as a binary membership relation over $C \times (C \cup D)$. As the folloing result shows, this relation is quite well behaved.

[8]This allows for the possibility that D, too, is ordered under refinement, but no assumptions are made here about that ordering. In particular, D may be completely *unordered* by refinement.

FACT 1: *In any $\langle D, \Omega, \sqsubseteq\rangle$-order model $\langle C, \sqsubseteq, \Uparrow, \Downarrow\rangle$, membership is order preserving in both coodinates, in the sense that for any $x, x' \in C$ and any $y, y' \in C \cup D$, if $x \sqsubseteq x'$ and $y \sqsubseteq y'$ then $x_\Uparrow(y) \sqsubseteq x'_\Uparrow(y')$.*

Proof: We have $x_\Uparrow(y) \sqsubseteq x_\Uparrow(y')$, since $y \sqsubseteq y'$ and $x_\Uparrow$ is order preserving as a function of x. But we have $x_\Uparrow \sqsubseteq x'_\Uparrow$, since $x \sqsubseteq x'$ and $\Uparrow$ is order preserving. It follows that $x_\Uparrow(y') \sqsubseteq x'_\Uparrow(y')$, by the definition of $\sqsubseteq$ for functions. ■

It turns out, then, that any $\langle D, 3, \sqsubseteq\rangle$-order model $\langle C, \sqsubseteq, \Uparrow, \Downarrow\rangle$ may be used to interpret a language in which the membership predicate $\in$ is explicit. An evaluation for such a language may be given as follows:

DEFINITION 2: For a language $\mathcal{L}$ involving the predicate $\in$, an INTERPRETATION of $\mathcal{L}$ an order model $\mathfrak{M} = \langle C, \sqsubseteq, \Uparrow, \Downarrow\rangle$ relates sentences to models by way of an assignment of the values to the variables, where each assignment α, takes variables to values in $C \cup D$. Relative to such an assignment α of values and variables, we can define the extensions $[\![t]\!]_{\mathfrak{M},\alpha}$ and $[\![A]\!]_{\mathfrak{M},\alpha}$ of a terms and formulas as follows:

- $[\![x]\!]_{\mathfrak{M},\alpha} = \alpha(x)$ is the interpretation of the variable x. (We abbreviate this as '$[\![x]\!]$' when the choice of $\mathfrak{M}$ and α is clear.)
- $[\![s \in t]\!]_{\mathfrak{M},\alpha}$ is $[\![t]\!]_\Uparrow([\![s]\!])$ when $[\![t]\!] \in C$, and is 0 when $[\![t]\!] \in D$.[9]
- For any binary connective $\sharp$ interpreted as a refinement order preserving function $[\![\sharp]\!]$ on 3, $[\![A \sharp B]\!]_{\mathfrak{M},\alpha} = [\![A]\!]_{\mathfrak{M},\alpha}\ [\![\sharp]\!]\ [\![B]\!]_{\mathfrak{M},\alpha}$. Connectives of other arities and quantifiers are interpreted in the same way, as usual.

We can expand the language to explicitly include abstraction terms:

$$\{x : \phi(x)\}$$

for each formula $\phi(x)$. Since $[\![\phi(x)]\!]_{\mathfrak{M},\alpha[x:=v]}$ is order preserving in v we can use that function, in $[C \cup D \to 3]$, to choose as the extension $[\![\{x : \phi(x)\}]\!]$ of the abstraction term. That is,

$$[\![\{x : \phi(x)\}]\!]_{\mathfrak{M},\alpha} = (\lambda v.[\![\phi(x)]\!]_{\mathfrak{M},\alpha[x:=v]})_\Downarrow$$

This choice of extension of the abstraction term is suitable because of the following result.

FACT 2: *For any order model, the strong comprehension axiom is satisfied.*

[9]This is the point at which the selection of Ω as 3 is important, at least in the choice of 0 as a distingushed 'false' value.

Proof: Here is a proof of the comprehension equivalence in the strong form. We show that in every order model, $[\![t \in \{x : \phi(x)\}]\!]$ is the same value as $\phi(t)$.

$$\begin{array}{rcl} [\![t \in \{x : \phi(x)\}]\!]_{\mathfrak{M},\alpha} & = & [\![\{x : \phi(x)\}]\!]_{\alpha\Uparrow}([\![t]\!]_\alpha) \\ & = & (\lambda v.[\![\phi(x)]\!]_{\alpha[x:=v]})_{\Downarrow\Uparrow}([\![t]\!]_\alpha) \\ & = & (\lambda v.[\![\phi(x)]\!]_{\alpha[x:=v]})([\![t]\!]_\alpha) \\ & = & [\![\phi(x)]\!]_{\alpha[x:=[\![t]\!]_\alpha]} \\ & = & [\![\phi(t)]\!]_{\mathfrak{M},\alpha} \end{array}$$ ∎

In order models, since the domain $[(C \cup D) \to \Omega]$ contains *all* order preserving functions, we know that the domain contains particular special functions, in particular the constant functions on Ω. So, in the case where Ω is 3, we have the three logical constant functions, whose outputs are 0, $*$ and 1 respectively, and the corresponding classes

$$\Lambda \qquad \mathbb{X} \qquad \mathrm{V}$$

since those functions into Ω are order preserving. The point wise ordering on these functions generates the corresponding ordering on the classes, as follows:

$$\begin{array}{ccc} 0 & & 1 \\ & \nwarrow \quad \nearrow & \\ & * & \end{array} \qquad\qquad \begin{array}{ccc} \Lambda & & \mathrm{V} \\ & \nwarrow \quad \nearrow & \\ & \mathbb{X} & \end{array}$$

In fact, we not only have $\mathbb{X} \sqsubseteq \Lambda$ and $\mathbb{X} \sqsubseteq \mathrm{V}$, in our models. We also have $\mathbb{X} \sqsubseteq c$ for every every class $c \in C$, since $\mathbb{X}$ is the constant function returning the $\sqsubseteq$-minimal value $*$, and revinement on classes is defined pointwise. From now, we'll use 'Λ', 'V' and '$\mathbb{X}$' as both the *class terms* in the language, and as their denotations, names for objects in C.

It is beyond the scope of this article to go into the detail of how order models for classes might be constructed (for discussions of the technique, look here [4–6, 28, 29]), but it is appropriate to sketch a little of the behaviour of finite stages of the construction process, to draw our attention beyond the very simple classes Λ, $\mathbb{X}$ and V.

In an iterative process of constructing order preserving functions from $[D_i \to 3]$, the next stage after $D_1 = \{\Lambda, \mathbb{X}, \mathrm{V}\}$ (which are isomorphic to the constant functions from a singleton class to Ω) is D_2, the set of order preserving functions $[D_1 \to 3]$. There are eleven members of D_2. They are ordered by $\sqsubseteq$ as indicated in Figure 5, where the notation for each function has the form s/t where the left of the slash indicates the members of D_1 sent to 0 while the right of the slash indicates the members of D_1 sent to 1. So, $\Lambda\mathrm{V}/$ is the function sending both Λ and V to 0, while sending $\mathbb{X}$ to $*$. This function is refined only by the constant 0 function $\Lambda\mathbb{X}\mathrm{V}/$. The function $\Lambda/$ on the other hand, can be refined by three functions: Λ/V, or $\Lambda\mathrm{V}/$ and further by $\Lambda\mathbb{X}\mathrm{V}/$. It is clear that D_1 is represented within D_2 by way of the constant functions $/$, $\Lambda\mathbb{X}\mathrm{V}/$ and $/\Lambda\mathbb{X}\mathrm{V}$. The functions in $D_3 = [D_2 \to \Omega]$ provide a much richer structure. It turns out that there are 397 members of D_3. They are presented in Figure 6.

The diagrams for D_2 and D_3 bring to light a general phenomenon in order models.

$$\begin{tikzcd}
 & & \Lambda\mathbb{X}\mathrm{V}/ & & /\Lambda\mathbb{X}\mathrm{V} & & \\
\Lambda/\mathrm{V} & & \Lambda\mathrm{V}/ \arrow[u] & & /\Lambda\mathrm{V} \arrow[u] & & \mathrm{V}/\Lambda \\
\Lambda/ \arrow[u] \arrow[urr] & & /\mathrm{V} \arrow[ull] \arrow[urr] & & \mathrm{V}/ \arrow[ull] \arrow[urr] & & /\Lambda \arrow[ull] \arrow[u] \\
 & & & / \arrow[ulll] \arrow[ul] \arrow[ur] \arrow[urrr] & & &
\end{tikzcd}$$

Figure 5: The eleven members of $D_2 = [D_1 \to 3]$

DEFINITION 3 [SHARP CLASSES]: In a model $\mathfrak{M}$, a class c is SHARP if and only if for each object b in $C \cup D$, $c_{\Uparrow}(b)$ takes the value 0 or 1.

In our order models, Λ and V are sharp. But $\mathbb{X}$ is *not* sharp. In fact, almost *no* classes are sharp.

FACT 3: *If* $c_{\Uparrow}(b) = 1$ *and* $c_{\Uparrow}(b') = 0$*, then* $c_{\Uparrow}(\mathbb{X}) = *$.

Proof: Since $\mathbb{X} \sqsubseteq b$, we have $c_{\Uparrow}(\mathbb{X}) \sqsubseteq c_{\Uparrow}(b) = 1$. Since $\mathbb{X} \sqsubseteq b'$, we have $c_{\Uparrow}(\mathbb{X}) \sqsubseteq c_{\Uparrow}(b') = 0$. It follows that $c_{\Uparrow}(\mathbb{X}) = *$. The class c fails to be sharp. ∎

In other words, once a class *includes* something and *excludes* something, it is *indecisive* about $\mathbb{X}$. It follows from this that there are severe constraints on the kind of behaviours we can expect from classes in order models. They cannot play the role of *classical recapture* through the behaviour of sharp classes. One way to provide a rapproachment between classical set theories and non-classical class theories is to hope to find the classical sets inside the larger non-classical universe. A straightforward hope would be that we could find a universe of well-behaved classical sets whose membership predicate behaves classically, and any non-classical behaviour is relegated to the purely paradoxical objects.[10] Any such hope is vain in order models. There are no properly classical sets at all other than Λ and V. The set $\mathbb{X}$ is properly paradoxical in that is weaker in the refinement ordering than *every* set, and hence (by the order preservation of membership), once you decide on the membership status of $\mathbb{X}$, you have decided the membership status of *everything*.

Nonetheless, even though there are no properly *classical* objects, you can see in Figures 5 and 6 that there are objects that behave classically *enough*. These are what

[10] See Section 3 of my "Note on Naïve Set Theory in LP" for a discussion of this [32].

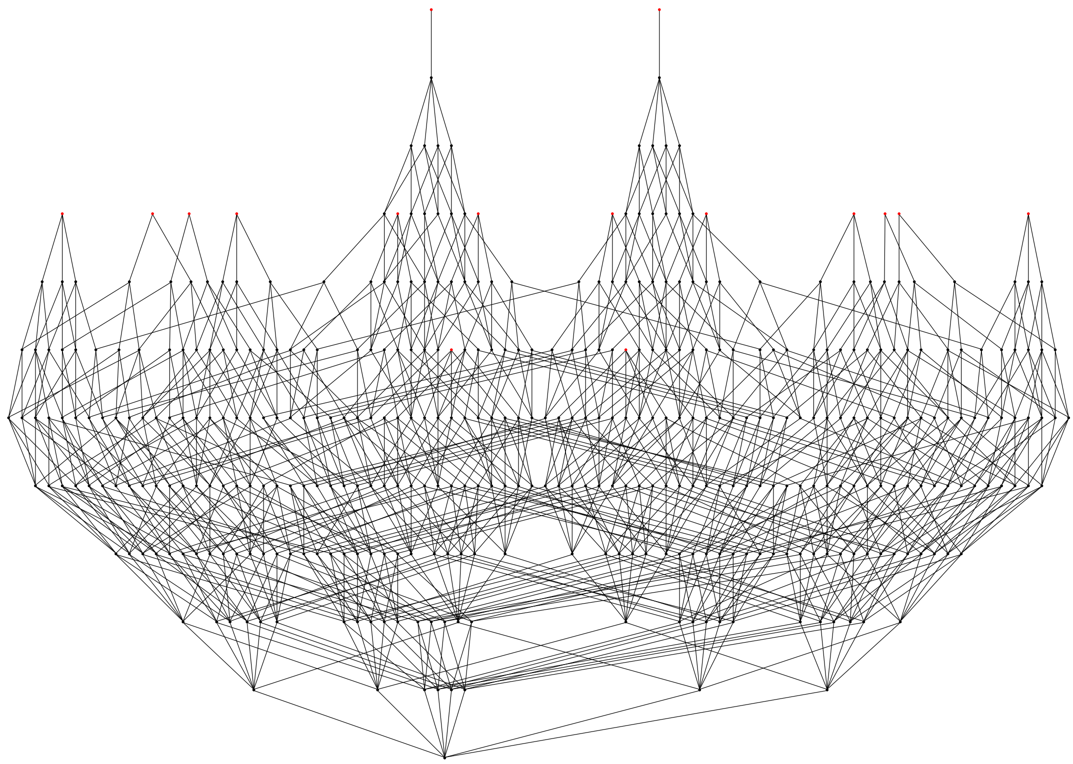


Figure 6: The 397 members of $D_3 = [D_2 \to 3]$

we might call the *crown* of the diagram, in Figure 5 for D_2, they are the four elements Λ/V, ΛXV/, /ΛXV, V/Λ which determine their membership as much as possible—these functions are at the top of the refinement ordering. In Figure 6 for D_3, there are sixteen such elements.[11]

In an order model, special sets are are what we will call the *singletons* and the *antisingletons*.

DEFINITION 4 [SINGLETONS AND ANTISINGLETONS]: In any $\langle D, 3, \sqsubseteq\rangle$-order model, the singleton $\{t\}$ and anti-signetons $\}t\{$ of an object t are as follows:

Singletons: $[\![\, \{t\}\,]\!]_\alpha$ is the class representative of the function that (1) assigns 1 to x iff $[\![t]\!]_\alpha \sqsubseteq x$; (2) assigns 0 to x iff there is no z where $x \sqsubseteq z$ and $[\![t]\!]_\alpha \sqsubseteq z$, (3) and assigns $*$ otherwise.

Antisingletons: $[\![\, \}t\{\,]\!]_\alpha$ is the class representative of the function that (1) assigns 0 to x iff $[\![t]\!]_\alpha \sqsubseteq x$, (2) assigns 1 to x if there is no z where $x \sqsubseteq z$ and $[\![t]\!]_\alpha \sqsubseteq z$, (3) and assigns $*$ otherwise.

[11]You may need to look *very* closely to find them.

These classes are not crisp (except in the very special case of $\{\mathbb{X}\}$ which is identical to V and $\}\mathbb{X}\{$ which is identical to $\varnothing$—at least in the pure theory of classes, in which there are no urelements), but, for object t, the singleton $\{t\}$ contains t (and any object it refines into) and it excludes as much as possible, consistent with that fact. Similarly, the antisingleton *excludes* t (and any object it refines into) and it includes as much as possible, consistent with that fact. As you can see, in the special case of $\{\mathbb{X}\}$, which contains every class, and $\}\mathbb{X}\{$, which excludes every class, there are some very strange 'singletons' and there are many other classes like them, waiting to be discovered in order models.

6 Limitations and Future Work

There is much left to do to gain a comprehensive understanding of order models for class and property theories. They seem to provide a rich mathematical structure which can be studied fruitfully, using a range of mathematical and logical techniques. Here is a list of questions that would reward further study.

- Study *pure* order models (where D is empty),

  ... and *impure* order models for different sets D of urelements.

- Find perspicuous ways to *construct* order models.

- Relate these constructions to other known model constructions, and characeterise their strength and expressive power.

- *Axiomatise* the logic of order models for different choices of the language.

- Characterise identity in order models. What forms of identity can be specified in order models?

- Examine different *motivations* of order models. How are we to understand refinement? In domain models for computation, refinement has a straightforward reading in terms of computation [1]. (In such models, $s \sqsubseteq t$ just when s is the result of a partial computation which can be further elaborated into t.)

These order models provide a rich and significant class of structures, which look to reward further exploration. However, we must be aware of their limitations. For one, order models provide us a significant and rich *external* account of identity of classes (a class is determined by its extension, its map from the universe to Ω), but it seems very difficult to reflect this external notion of identity *inside* an order model [29, §9.10]. And if identity is difficult to state, then an axiom of extensionality eludes us, and we do not have an expressive class theory. Monotonic *surrogates* for the external notion of identity (say, taking x and y to be witnessed to be distinct when we have some member determined to be inside x and outside y, or *vice versa*) seems appropriately monotonic, but weak. What is it for the judgement that $x = y$ to be *true*?

Furthermore, the condition that the logical connectives be monotonic is a strong constraint on the expressive power of the language. If we choose $3 = \{0, *, 1\}$ for Ω, then the connectives $\wedge$, $\vee$ and $\neg$ are refinement preserving, but as we have seen, the conditionals $Ł_3$ or of RM_3 are not. The situation does not improve in richer structures, either. We know of no connective anything like a *conditional* — satisfying identity and *modus ponens* — that is order preserving in a natural structure like Ω. This does not mean, however, that order models have no place in specifying class theories, even if we have a commitment to interpreting conditionals satisfying modus ponens and identity. Generalisations of order preservation and other closure conditions may generalise our results, or we may be able to find other ways to interpret conditionals in the class of propositions defined on these structures. Regardless, they provide a rich and wide terrain for future researchers to explore.

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