A REMARK ON MAKSIMOVA'S VARIABLE SEPARATION PROPERTY IN SUPER-BI-INTUITIONISTIC LOGICS

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ABSTRACT. We provide a sufficient frame-theoretic condition for a super-bi-intuitionistic logic to have Maksimova's variable separation property. We conclude that bi-intuitionistic logic enjoys the property. Furthermore, we offer an algebraic characterization of the super-bi-intuitionistic logics with Maksimova's property.

1. INTRODUCTION

Propositional bi-intuitionistic or Heyting-Brouwer logic is a natural expressive extension of propositional intuitionistic logic which has been explored in places like [9, 8] and much more recently [5]. It comes from adding a binary connective called subtraction (symbolized as $-$) to the language of intuitionistic logic while leaving the class of models of intuitionistic logic untouched (hence producing no new validities in the old language of intuitionistic logic). We call a *super-bi-intuitionistic logic* to any logical calculus which extends axiomatically bi-intuitionistic logic (in the sense that it adds a $-$possibly empty$-$ set of new validities to the theorems of bi-intuitionistic logic). As a referee points out, it is worth mentioning that our super-bi-intuitionistic logics have received other names in the literature, for instance, "super-HB logics" ([11], p. 356).

In [6], the following two open problems were posed:

(i) Does bi-intuitionistic logic have Maksimova's variable separation property?

(ii) Provide an algebraic characterization of the super-bi-intuitionistic logics with the property.

Maksimova's property is a generalization of Halldén completeness (i.e., the property that if two formulas with no propositional variables in common have countermodels then we must be able to find a countermodel to their disjunction) which was originally introduced and studied (for relevant logics) in [7] (see §3). Borrowing from the simple model-theoretic techniques of [7], we give a positive answer to (i) in §4. In passing, we establish an analogue of the

main result from [1] on Halldén completeness but for Maksimova's property. Finally, in §5 we also give an answer to (ii).

2. Bi-intuitionistic logic

A propositional *bi-intuitionistic* or *Heyting-Brouwer* language L contains a list PROP of propositional variables $p, q, r \ldots$, as well as the logical symbols $\perp$, $\vee$, $\wedge$, $\Rightarrow$ and new binary connective $-$ dual to $\Rightarrow$.

An algebraic semantics for L is provided by Heyting-Brouwer algebras (also called double Heyting algebras), that is, structures of the form $\langle A, \vee, \wedge, \Rightarrow, -, 0, 1 \rangle$ where both $\langle A, \vee, \wedge, \Rightarrow, 0, 1 \rangle$ and $\langle A, \vee^\delta, \wedge^\delta, -, 0^\delta, 1^\delta \rangle$ are Heyting algebras, where the dual operation δ is as usual: $\vee^\delta = \wedge, \wedge^\delta = \vee, 0^\delta = 1, 1^\delta = 0$. In $\langle A, \vee^\delta, \wedge^\delta, 0^\delta, 1^\delta \rangle$ the partial order $\leqslant$ (which is defined as always in terms of $\vee$ or $\wedge$) is the converse ordering of $\langle A, \vee, \wedge, 0, 1 \rangle$. The following will hold (for the partial ordering of $\langle A, \vee, \wedge, 0, 1 \rangle$):

$$x \wedge y \leqslant z \text{ iff } y \leqslant x \Rightarrow z,$$
$$x \vee y \geqslant z \text{ iff } y \geqslant z - x.$$

The paper [10] is a place where these algebras have been studied. An equational basis for this class of algebras is given by the equational basis for Heyting algebras plus their dual equations for the dual algebra.

A Kripke frame for L is a structure $\mathscr{F} = \langle W, R \rangle$, where W is a non-empty set and $R \subseteq W \times W$ is a partial order. A model $M = \langle \mathscr{F}, V \rangle$, is a frame together with a valuation function $V : \text{PROP} \longrightarrow \wp(W)$ satisfying the following closure property:

if $w \in V(p)$ and Rwu then $u \in V(p)$.

We define satisfaction at w in M recursively as follows:

$M, w \Vdash \perp$	never,	
$M, w \Vdash p$	iff	$w \in V(p)$,
$M, w \Vdash (\phi \wedge \psi)$	iff	$M, w \Vdash \phi$ and $M, w \Vdash \psi$,
$M, w \Vdash (\phi \vee \psi)$	iff	$M, w \Vdash \phi$ or $M, w \Vdash \psi$,
$M, w \Vdash \phi \Rightarrow \psi$	iff	for every a such that Rwa, if $M, a \Vdash \phi$ then $M, a \Vdash \psi$,
$M, w \Vdash \phi - \psi$	iff	there is a such that Raw and $M, a \Vdash \phi$ while $M, a \nVdash \psi$.

By *bi-intuitionistic logic* (which we will abbreviate as **BiInt**) we will be mean the set of formulas true at every world in all the above defined Kripke models, i.e., the formulas valid in all the above defined frames.

3. Maksimova's property

If $\phi_1, \ldots, \phi_n$ is some list of formulas, by $\text{Var}(\phi_1, \ldots, \phi_n)$ we will mean the collection of propositional variables appearing in any of the ϕ_i $(1 \leqslant i \leqslant n)$.

DEFINITION 1. *A super-bi-intuitionistic logic* **L** *is said to have* Maksimova's *variable separation property if whenever* $\text{Var}(\phi_1, \phi_2) \cap \text{Var}(\psi_1, \psi_2) = \varnothing$, *then*

$(\ast)$ $\phi_1 \wedge \psi_1 \Rightarrow \phi_2 \vee \psi_2 \in \mathbf{L}$ *only if either* $\phi_1 \Rightarrow \phi_2 \in \mathbf{L}$ *or* $\psi_1 \Rightarrow \psi_2 \in \mathbf{L}$.

Maksimova's property is closely related to the Halldén completeness of a logic. Recall that a logic is said to be *Halldén complete* if whenever $\text{Var}(\phi) \cap \text{Var}(\psi) = \varnothing$ and $\phi \vee \psi$ is a theorem, then either ϕ is a theorem or ψ is a theorem.

For any super-bi-intuitionistic logic **L**, Maksimova's property implies Halldén completeness. To see this, define $\top =_{df} \bot \Rightarrow \bot$, noting that $\top \in \mathbf{L}$. Suppose $\phi \vee \psi \in \mathbf{L}$, but then $\top \wedge \top \Rightarrow \phi \vee \psi \in \mathbf{L}$. By Maksimova's property for **L**, either $\top \Rightarrow \phi \in \mathbf{L}$ or $\top \Rightarrow \psi \in \mathbf{L}$. However, $\top \in \mathbf{L}$, so either $\phi \in \mathbf{L}$ or $\psi \in \mathbf{L}$.[1]

4. BI-INTUITIONISTIC LOGIC HAS MAKSIMOVA'S PROPERTY

We start this section by introducing a notion of p-morphism for bi-intuitionistic frames. This idea is well-known from intuitionistic (and modal) logic (cf. [2], p. 30).

DEFINITION 2. *Let* $\mathscr{F}_1 = \langle W_1, R_1 \rangle, \mathscr{F}_2 = \langle W_2, R_2 \rangle$ *be a pair of frames. A* p-morphism *from* $\mathscr{F}_1$ *onto* $\mathscr{F}_2$ *is a surjective map* $f : W_1 \longrightarrow W_2$ *such that*

   (i) $R_1 wv$ *only if* $R_2 f(w) f(v)$
   (ii) $R_2 f(w) f(v)$ *only if there is* $u \in W_1$ *such that* $R_1 wu$ *and* $f(u) = f(v)$
   (iii) $R_2 f(v) f(w)$ *only if there is* $u \in W_1$ *such that* $R_1 uw$ *and* $f(u) = f(v)$

THEOREM 1. (p-morphism theorem) *Let f be a p-morphism from a frame* $\mathscr{F}_1$ *to a frame* $\mathscr{F}_2$. *Then for any formula ϕ of a language L,* $\langle \mathscr{F}_1, V' \rangle, w \Vdash \phi$ *iff* $\langle \mathscr{F}_2, V \rangle, f(w) \Vdash \phi$, *where for each propositional variable p of L,* $f^{-1}(V(p)) = V'(p)$.

Proof. This is as for Theorem 2.15 from [2]. It proceeds by an easy induction on formula complexity and we leave it to the reader. For the case $\phi \to \psi$ we appeal to the property (ii) in Definition 2 whereas for $\phi - \psi$ we use the new condition (iii). $\square$

DEFINITION 3. *A class K of frames is said to be* closed under p-morphic super-fusion *if for any* $\mathscr{F}_1, \mathscr{F}_2 \in K$, *worlds* $w_1, w_1' \in W_1, w_2, w_2' \in W_2$ *such that* $R_1 w_1 w_1'$ *and* $R_2 w_2 w_2'$, *there is* $\mathscr{F}_3 \in K$, *worlds* $w_3, w_3' \in W_3$ *and* p-*morphisms* $f_1 : W_3 \longrightarrow W_1, f_2 : W_3 \longrightarrow W_2$ *such that* $R_3 w_3, w_3'$ *while* $f_1(w_3) = w_1, f_1(w_3') = w_1', f_2(w_3) = w_2$ *and* $f_2(w_3') = w_2'$. *In a picture,*

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[1]The Halldén completeness of **BiInt** is proven via a clever syntactic argument in [6].

$$\mathscr{F}_3$$
$$f_1 \qquad f_2$$
$$\mathscr{F}_2 \qquad\qquad \mathscr{F}_1$$
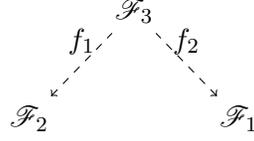
The following result is the analogue of Theorem 1 from [1].

THEOREM 2. *Any super-bi-intuitionistic logic* **L** *determined by a class of frames closed under p-morphic super-fusion enjoys Maksimova's variable separation property.*

Proof. Let $\text{Var}(\phi_1, \phi_2) \cap \text{Var}(\psi_1, \psi_2) = \varnothing$. Let L_{ϕ_1,ϕ_2} and L_{ψ_1,ψ_2} be the languages with only the propositional variables in $\text{Var}(\phi_1, \phi_2)$ and $\text{Var}(\psi_1, \psi_2)$ respectively. We argue for the contrapositive. Thus suppose $\phi_1 \Rightarrow \phi_2 \notin$ **L** and $\psi_1 \Rightarrow \psi_2 \notin$ **L**. Hence, there are models $\langle\mathscr{F}_1, V_1\rangle$, $\langle\mathscr{F}_2, V_2\rangle$ (for L_{ϕ_1,ϕ_2} and L_{ψ_1,ψ_2} respectively) and worlds $w_1 \in W_1, w_2 \in W_2$ such that such that $\langle\mathscr{F}_1, V_1\rangle, w_1 \nVdash \phi_1 \Rightarrow \phi_2$ and $\langle\mathscr{F}_2, V_2\rangle, w_2 \nVdash \psi_1 \Rightarrow \psi_2$. Consequently, there is w_1' such that $R_1 w_1 w_1'$, $\langle\mathscr{F}_1, V_1\rangle, w_1' \Vdash \phi_1$ while $\langle\mathscr{F}_1, V_1\rangle, w_1' \nVdash \phi_2$, and w_2' such that $R_2 w_2 w_2'$, $\langle\mathscr{F}_2, V_2\rangle, w_2' \Vdash \psi_1$ while $\langle\mathscr{F}_2, V_2\rangle, w_2' \nVdash \psi_2$.

By the closure under p-morphic super-fusion assumption, we see that there must be a frame $\mathscr{F}_3$ for **L**, worlds $w_3, w_3' \in W_3$ and p-morphisms $f_1 : W_3 \longrightarrow W_1, f_2 : W_3 \longrightarrow W_2$ such that $R_3 w_3, w_3'$ while $f_1(w_3) = w_1, f_1(w_3') = w_1', f_2(w_3) = w_2$ and $f_2(w_3') = w_2'$. Recalling our assumption that $\text{Var}(\phi_1, \phi_2) \cap \text{Var}(\psi_1, \psi_2) = \varnothing$, we might consider a valuation V_3 on $\mathscr{F}_3$ such that $V_3(p) = f_1^{-1}(V_1(p))$ for $p \in L_{\phi_1,\phi_2}$ and $V_3(p) = f_2^{-1}(V_2(p))$ for $p \in L_{\psi_1,\psi_2}$. By properties of p-morphisms and the closure properties of V_1, V_2, we have that V_3 must have the required closure property to qualify as a valuation.

According to the p-morphism theorem we have that for any formulas θ, σ of L_{ϕ_1,ϕ_2} and L_{ψ_1,ψ_2} respectively,

$$\langle\mathscr{F}_3, V_3\rangle, w \Vdash \theta \text{ iff } \langle\mathscr{F}_1, V_1\rangle, f_1(w) \Vdash \theta,$$
$$\langle\mathscr{F}_3, V_3\rangle, w \Vdash \sigma \text{ iff } \langle\mathscr{F}_2, V_2\rangle, f_2(w) \Vdash \sigma.$$

Hence, $\langle\mathscr{F}_3, V_3\rangle, w_3' \Vdash \phi_1$ and $\langle\mathscr{F}_3, V_3\rangle, w_3' \Vdash \psi_1$, so $\langle\mathscr{F}_3, V_3\rangle, w_3' \Vdash \phi_1 \wedge \psi_1$. On the other hand, $\langle\mathscr{F}_3, V_3\rangle, w_3' \nVdash \phi_2$ and $\langle\mathscr{F}_3, V_3\rangle, w_3' \nVdash \psi_2$, so $\langle\mathscr{F}_3, V_3\rangle, w_3' \nVdash \phi_2 \vee \psi_2$. But then $\langle\mathscr{F}_3, V_3\rangle, w_3 \nVdash \phi_1 \wedge \psi_1 \Rightarrow \phi_2 \vee \psi_2$, so $\phi_1 \wedge \psi_1 \Rightarrow \phi_2 \vee \psi_2 \notin$ **L** as desired. $\square$

Next we present a rather standard model-theoretic construction (see, for instance, [3], p. 413) which becomes very useful in arguments where we would like to glue two models together while preserving some desirable features of both.

DEFINITION 4. *Let $\mathscr{F}_1 = \langle W_1, R_1 \rangle$ and $\mathscr{F}_2 = \langle W_2, R_2 \rangle$ be two frames (which are nothing but structures in the usual sense). We can assume that $W_1 \cap W_2 = \varnothing$. The product of $\mathscr{F}_1$ and $\mathscr{F}_2$ will be the structure $\mathscr{F}_1 \times \mathscr{F}_2 = \langle W_1 \times W_2, R_{1 \times 2} \rangle$ where:*

$\qquad R_{1 \times 2} \langle a, b \rangle \langle c, d \rangle$ *iff $R_1 ac$ and $R_2 bd$.*

Note that $R_{1 \times 2}$ is a partial order. This follows since the fact that $R_{1 \times 2}$ is a partial order is expressible by a Horn theory. Recall that a Horn fomula is a first order formula of the form $Q_0 x_0, \ldots, Q_n x_n \bigwedge_{j<k} \phi_j$, where $Q_i (i \leqslant n)$ is either $\exists$ or $\forall$ and ϕ_j is a basic horn formula, that is, a formula of the form $\bigwedge \Theta \supset \chi$ where Θ is a finite set of atomic formulas (including the empty set, in which case the formula is just χ) and χ is either atomic or a contradiction. But then the theory with the sentences $\forall x, y, z (Rxy \wedge Ryz \supset Rxz), \forall x (Rxx)$ and $\forall x, y (Rxy \wedge Ryx \supset x = y)$ is Horn. So, by Corollary 9.1.6 from [3], $\mathscr{F}_1 \times \mathscr{F}_2$ must satisfy such theory.

PROPOSITION 3. *Any super-bi-intuitionistic logic **L** determined by a class of frames axiomatizable by a set of Horn sentences (in the signature containing only a binary predicate R for the accessibility relation of the frames) is closed under p-morphic super-fusion.*

Proof. Using Corollary 9.1.6 from [3] again, classes of frames axiomatized by a set of Horn sentences are closed under the formation of products. It suffices to check that the projection map is a p-morphism. $\qquad \square$

We are finally in a position to establish the following result.

COROLLARY 4. **BiInt** *has Maksimova's variable separation property.*

Proof. Immediate from Theorem 2 and Proposition 3 together with the fact that **BiInt** is determined by a class of frames axiomatizable by a Horn theory. $\qquad \square$

To end this section, let us look at another example of a super-bi-intuitionistic logic with Maksimova's property. Consider the axiom $(p \rightarrow \bot) \vee ((p \rightarrow \bot) \rightarrow \bot)$, called the *weak law of the excluded middle* (**wem**). Proposition 2.37 from [2] establishes that **wem** corresponds to the frame condition $\forall x, y, z (Rxy \wedge Rxz \supset \exists u (Ryu \wedge Rzu))$. Now, by standard first order manipulations, we can push the existential quantifier out in the previous formula to get the equivalent form: $\forall x, y, z \exists u (Rxy \wedge Rxz \supset Ryu \wedge Rzu)$. In turn, the latter formula is equivalent to $\forall x, y, z \exists u ((Rxy \wedge Rxz \supset Ryu) \wedge (Rxy \wedge Rxz \supset Rzu))$, which is a Horn formula. So, from Theorem 2 and Proposition 3 again, we see that **BiInt** plus the axiom **wem** is a super-bi-intuitionistic logic with Maksimova's property.

5. Algebraic Characterization

We characterize the super-bi-intuitionitic logics having Maksimova's property by means of a result entirely analogous to Theorem 5.32 from [4] (indeed the proof is essentially the same but we include it for completeness). Notation is very much as in [4], except that given an algebra $\mathbf{A}$, we use $\mathrm{dom}(\mathbf{A})$ to denote the domain of $\mathbf{A}$. Given a super-bi-intuitionistic logic $\mathbf{L}$, the symbol $V(\mathbf{L})$ denotes the variety corresponding to $\mathbf{L}$. Given a set of propositional variables X, by $\mathbf{Fm}(X)$ we denote the set of bi-intuitionistic formulas which can be built from X. Given a set of equations X, by $\mathrm{Mod}(X)$ we will mean all the structures satisfying X. Finally given a collection of formulas $\mathbf{Fm}(Y)$ based on a list of propositional variables Y and a logic $\mathbf{L}$, by $\mathbf{Fm}(Y)/\equiv_{\mathbf{L}}$ we denote the standard Lindenbaum algebra of $\mathbf{L}$.

DEFINITION 5. *Let $\mathbf{C_1}, \mathbf{C_2}, \mathbf{C}$ be algebras such that $\mathbf{C_1}, \mathbf{C_2} \subseteq \mathbf{C}$. We say that $\mathbf{C_1}, \mathbf{C_2}$ form a strongly well-connected pair of $\mathbf{C}$ if for arbitrary $a_1, a_2 \in \mathrm{dom}(\mathbf{C_1})$ and $b_1, b_2 \in \mathrm{dom}(\mathbf{C_2})$, if $a_1 \wedge_{\mathbf{C}} b_1 \leqslant a_2 \vee_{\mathbf{C}} b_2$ in $\mathbf{C}$ then either $a_1 \leqslant a_2$ or $b_1 \leqslant b_2$ in $\mathbf{C}$.*

THEOREM 5. *For any super-bi-intuitionistic logic $\mathbf{L}$, the following are equivalent:*

(i) *$\mathbf{L}$ has Maksimova's variable separation property.*
(ii) *For any two non-degenerate H-B algebras $\mathbf{A}, \mathbf{B} \in V(\mathbf{L})$ there are H-B algebras $\mathbf{C_1}, \mathbf{C_2}, \mathbf{C} \in V(\mathbf{L})$, $\mathbf{C_1}, \mathbf{C_2} \subseteq \mathbf{C}$ such that $\mathbf{C_1}$ and $\mathbf{C_2}$ form a strongly well-connected pair of $\mathbf{C}$ and surjective homomorphisms $h : \mathrm{dom}(\mathbf{C_1}) \longrightarrow \mathrm{dom}(\mathbf{A}), j : \mathrm{dom}(\mathbf{C_2}) \longrightarrow \mathrm{dom}(\mathbf{B})$. In a picture,*
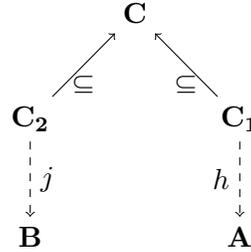
Proof. $(i) \longrightarrow (ii)$: Take any two non-degenerate H-B algebras $\mathbf{A}, \mathbf{B} \in V(\mathbf{L})$. Consider two disjoint sets of variables X, Y with cardinality greater than or equal to $|\mathrm{dom}(\mathbf{A})|$ and $|\mathrm{dom}(\mathbf{B})|$ respectively. Let $Z = X \cup Y$. For a set of variables V, the relation $\equiv_{\mathbf{L}}$ holds between any two $\phi, \psi \in \mathbf{Fm}(V)$ iff $\phi \Rightarrow \psi, \psi \Rightarrow \phi \in \mathbf{L}$. One can see that $\equiv_{\mathbf{L}}$ is a fully invariant congruence relation on $\mathbf{Fm}(V)$. Put $\mathbf{C_1} = \mathbf{Fm}(X)/\equiv_{\mathbf{L}}, \mathbf{C_2} = \mathbf{Fm}(Y)/\equiv_{\mathbf{L}}$, and $\mathbf{C} = \mathbf{Fm}(Z)/\equiv_{\mathbf{L}}$. Clearly, $\mathbf{C_1}, \mathbf{C_2} \subseteq \mathbf{C}$. $\mathbf{C_1}$ is a free algebra generated by X in the variety $\mathrm{Mod}(\{\phi = \psi : \phi \equiv_{\mathbf{L}} \psi\})$ and similarly with $\mathbf{C_2}$, Y, $\mathbf{C}$ and Z. Now, if $\mathbf{D} \in V(\mathbf{L})$, and $\phi \equiv_{\mathbf{L}} \psi$, then $\mathbf{D} \models (\phi \Rightarrow \psi \wedge \psi \Rightarrow \phi) = 1$,

so $\mathbf{D} \models \phi = \psi$, i.e., $\mathbf{D} \in \text{Mod}(\{\phi = \psi : \phi \equiv_\mathbf{L} \psi\})$. On the other hand if $\mathbf{D} \in \text{Mod}(\{\phi = \psi : \phi \equiv_\mathbf{L} \psi\})$, since $\phi \in \mathbf{L}$ iff $\phi \equiv_\mathbf{L} 1$, $\mathbf{D} \in \text{V}(\mathbf{L})$. Hence, $\text{Mod}(\{\phi = \psi : \phi \equiv_\mathbf{L} \psi\}) = \text{V}(\mathbf{L})$. Since $\mathbf{C_1}, \mathbf{C_2}$ are free, any surjective maps $h_0 : X \longrightarrow \text{dom}(\mathbf{A}), j_0 : Y \longrightarrow \text{dom}(\mathbf{B})$ can be extended to surjective homomorphisms $h : \text{dom}(\mathbf{C_1}) \longrightarrow \text{dom}(\mathbf{A})$ and $j : \text{dom}(\mathbf{C_2}) \longrightarrow \text{dom}(\mathbf{B})$ respectively.

All that is left is to show that $\mathbf{C_1}$ and $\mathbf{C_2}$ form a strongly well-connected pair. Take arbitrary $\phi_1/\equiv_\mathbf{L}, \phi_2/\equiv_\mathbf{L} \in \mathbf{C_1}$ and $\psi_1/\equiv_\mathbf{L}, \psi_2/\equiv_\mathbf{L} \in \mathbf{C_2}$. Suppose that $\phi_1/\equiv_\mathbf{L} \wedge_\mathbf{C} \psi_1/\equiv_\mathbf{L} \leqslant \phi_1/\equiv_\mathbf{L} \vee_\mathbf{C} \psi_1/\equiv_\mathbf{L}$ in $\mathbf{C}$, but then $\phi_1 \wedge \psi_1 \Rightarrow \phi_2 \vee \psi_2 \in \mathbf{L}$. Since $\mathbf{L}$ has Maksimova's variable separation property and $\text{Var}(\phi_1, \phi_2) \cap \text{Var}(\psi_1, \psi_2) = \varnothing$, either $\phi_1 \Rightarrow \phi_2 \in \mathbf{L}$ or $\psi_1 \Rightarrow \psi_2 \in \mathbf{L}$. This means that either $\phi_1/\equiv_\mathbf{L} \leqslant \phi_2/\equiv_\mathbf{L}$ or $\psi_1/\equiv_\mathbf{L} \leqslant \psi_2/\equiv_\mathbf{L}$ in $\mathbf{C}$. Thus, $\mathbf{C_1}$ and $\mathbf{C_2}$ form a strongly well-connected pair.

$(ii) \longrightarrow (i)$: Assume that $\text{Var}(\phi_1, \phi_2) \cap \text{Var}(\psi_1, \psi_2) = \varnothing$ and $\phi_1 \Rightarrow \phi_2, \psi_1 \Rightarrow \psi_2 \notin \mathbf{L}$. Hence, there are non-degenerate algebras $\mathbf{A}, \mathbf{B} \in \text{V}(\mathbf{L})$, and valuation maps $f : \mathbf{Fm}(\text{Var}(\phi_1, \phi_2)) \longrightarrow \text{dom}(\mathbf{A})$, $g : \mathbf{Fm}(\text{Var}(\psi_1, \psi_2)) \longrightarrow \text{dom}(\mathbf{B})$ such that $f(\phi_1) \not\leqslant_\mathbf{A} f(\phi_2)$ and $g(\psi_1) \not\leqslant_\mathbf{A} g(\psi_2)$. By assumption (ii), there are are H-B algebras $\mathbf{C_1}, \mathbf{C_2}, \mathbf{C} \in V(\mathbf{L})$, $\mathbf{C_1}, \mathbf{C_2} \subseteq \mathbf{C}$ such that $\mathbf{C_1}$ and $\mathbf{C_2}$ form a strongly well-connected pair and surjective homomorphisms $h : \text{dom}(\mathbf{C_1}) \longrightarrow \text{dom}(\mathbf{A}), j : \text{dom}(\mathbf{C_2}) \longrightarrow \text{dom}(\mathbf{B})$. We take any valuation $k : \mathbf{Fm}(\text{Var}(\phi_1, \phi_2) \cup \text{Var}(\psi_1, \psi_2)) \longrightarrow \mathbf{C}$ such that $k(p) \in h^{-1} \circ f(p)$ for $p \in \text{Var}(\phi_1, \phi_2)$ and $k(p) \in j^{-1} \circ g(p)$ for $p \in \text{Var}(\psi_1, \psi_2)$. Observe that for any formula ϕ with variables in $\text{Var}(\phi_1, \phi_2)$, $h \circ k(\phi) = f(\phi)$ and for any formula ψ with variables in $\text{Var}(\psi_1, \psi_2)$, $j \circ k(\psi) = g(\psi)$. This can be proven by induction on formula complexity, using the facts that h, j, f, g, k are homomorphisms in the inductive hypothesis. Then, $h \circ k(\phi_1) \not\leqslant_\mathbf{A} h \circ k(\phi_2)$ and $j \circ k(\psi_1) \not\leqslant_\mathbf{A} j \circ k(\psi_2)$. Since h, j are surjective homomorphisms and positive first order formulas are preserved under surjective homomorphisms, $k(\phi_1) \not\leqslant_{\mathbf{C_1}} k(\phi_2)$ and $k(\psi_1) \not\leqslant_{\mathbf{C_2}} k(\psi_2)$, so since $\mathbf{C_1}, \mathbf{C_2} \subseteq \mathbf{C}$, $k(\phi_1) \not\leqslant_\mathbf{C} k(\phi_2)$ and $k(\psi_1) \not\leqslant_\mathbf{C} k(\psi_2)$. Given that $\mathbf{C_1}, \mathbf{C_2}$ form a well-connected pair, $k(\phi_1), k(\phi_2) \in \text{dom}(\mathbf{C_1})$ and $k(\psi_1), k(\psi_2) \in \text{dom}(\mathbf{C_2})$, we have that $k(\phi_1) \wedge_\mathbf{C} k(\psi_1) \not\leqslant_\mathbf{C} k(\phi_2) \vee_\mathbf{C} k(\psi_2)$. Since k is a homomorphism, $k(\phi_1 \wedge \psi_1) \not\leqslant_\mathbf{C} k(\phi_2 \vee \psi_2)$. But this means that $\phi_1 \wedge \psi_1 \Rightarrow \phi_2 \vee \psi_2 \notin \mathbf{L}$ as desired. $\square$

6. Conclusion

Any super-bi-intuitionistic logic determined by a class of frames closed under p-morphic super-fusion has Maksimova's property. In particular, bi-intuitionistic logic satisfies this condition. Moreover, a super-bi-intuitionistic logic has Maksimova's property iff any two non-degenerate algebras in the variety of the logic are homomorphic images of two algebras in that variety forming a strongly well-connected pair. The problem of finding other (more tailored to this context) algebraic characterizations remains open, though.

Acknowledgments

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