FDE Circumscription

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Abstract

In his article "Reassurance via Translation" Marcel Crabbé proposed a formalism to obtain reassurance and classical recapture in the setting of minimal FDE. His formalism proved to be general enough to be extended in order to formalize other forms of non-monotonic systems based on preference relations. It is the aim of this article to show how his result can be extended in a natural way by combining two different reasoning systems, namely minimal FDE and circumscription, in order to get a paraconsistent and paracomplete version of circumscription, which we will call paracomplistent circumscription, which has the advantages of FDE and circumscription but is neither explosive nor lacks modus ponens in consistent contexts. Furthermore, we will complete a proof Crabbé left unfinished.

Keywords. circumscription, minimal LP, minimal FDE, paraconsistent and paracomplete reasoning, combining logics, reassurance, classical recapture

1 Introduction

In 1979 Graham Priest published an article, "The Logic of Paradox" [17], where he proposed a simple generalization of classical logic which is paraconsistent, namely LP. This logic introduced an at that time new approach of dealing with paradoxes. Unfortunately, the semantics of LP leads to the invalidity of modus ponens and the disjunctive syllogism. To overcome these shortcomings and to construct a paraconsistent logic that recaptures classical logic in consistent situations Priest developed in [18], "Minimally Inconsistent LP", a modified version of LP, which has the default assumption of consistency and is non-monotonic. He introduced a way of measuring the degrees of inconsistency via a partial ordering of models in a certain way, which works perfectly fine on the propositional level. However, on the First-order level there are some problems regarding properties like reassurance, i.e., the existence of minimal non-trivial models for consistent premise sets, and classical recapture, i.e., obtaining classical entailment for consistent premise sets. That is why Priest improved his original idea in a revised version of "Minimally Inconsistent LP" [19]. Nevertheless, some problems, namely the lack of reassurance, still remain in First-order languages, FOL languages, with function symbols and equality. Marcel Crabbé in "Reassurance for the Logic of Paradox" [6] and "Reassurance via translation" [7] then proposed a slightly altered definition of minimal LP and extended it to first-degree entailment (FDE) in order to guarantee reassurance and classical recapture in FOL languages with function symbols but without equality.

Since Marcel Crabbé's results prove to be very general, it is the aim of this paper to show how these results can be used to combine logics with preference relations on their models with the example of a paraconsistent and paracomplete, i.e., a paracomplistent logic and the non-monotonic reasoning system circumscription.

It should be mentioned that combining non-monotonic and paraconsistent as well as paracomplete reasoning is not new. The whole tradition of inconsistency-adaptive and paracomplete adaptive logics can be seen as a very prominent example, see [3]. But in contrast to the adaptive framework we minimize here the abnormality, incompleteness and inconsistency of objects instead of formulas. Another notable example can be found in the work of Ofer Arieli and Arnon Avron, cf. [1] and follow-up papers. There, they generalize the KLM preference semantics to the paraconsistent case, but without the intention of combining it with circumscription.

But, the only work that we are aware of which combines circumscription with LP has been presented by Zuoquan Lin, cf. [10] and [11]. In the appendix you may find a brief discussion of his work.

Below and in the sections 2 and 3 we will give a brief overview of First-order languages, the non-monotonic logic circumscription, as well as FDE. In section 4 we will present results from Marcel Crabbé's paper "Reassurance via translation" [7] because the results of this paper shall prove to be crucial for our work. In his work, Marcel Crabbé showed how reassurance and classical recapture of a minimal version of FDE can be proved. The final section generalizes Crabbé's results to introduce a slightly changed definition of circumscription and finally it is shown how one can construct paracomplistent circumscription. Furthermore, in the appendix you will find a brief discussion of Zuoguan Lin's paper "Paraconsistent Circumscription" [12]. It turns out that his approach contains, unfortunately, some flaws.

Definition 1. Let $\mathcal{L}$ be a First-order language. It consists of a countable number of n-ary relation symbols P, m-ary function symbols F and constants $c_1, c_2, \ldots$. Furthermore, we have the following connectives $\neg, \wedge, \vee$ and $\rightarrow$ as well as the universal quantifier $\forall$ and the existential quantifier $\exists$. Well-formed formulas (wff) are defined as usual. Identity will be omitted.

A structure $\mathfrak{A}$ is called an FOL interpretation if it contains a non-empty set of objects, the universe $|\mathfrak{A}|$, and every n-ary relation $P(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are terms, is interpreted by its extension $P^+_{\mathfrak{A}}$, with $P^+_{\mathfrak{A}} \subseteq |\mathfrak{A}|^n$ and its anti-extension $P^-_{\mathfrak{A}} = |\mathfrak{A}|^n \backslash P^+_{\mathfrak{A}}$. We refer to $P(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are terms, as an n-ary atomic formula. Constants and function symbols are as usual interpreted by objects $o_{c_1}, o_{c_2} \ldots$ and functions, respectively.

A valuation $v_{\mathfrak{A}}$ is a function from the set of variables and terms to $|\mathfrak{A}|$. Truth and falsehood of a wff H, in symbols $(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^+ H$ and $(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^- H$ respectively, in an interpretation $\mathfrak{A}$ with respect to a valuation $v_{\mathfrak{A}}$ are defined inductively, as follows: Let H and G be wff and $P(t_1, \ldots, t_n)$ an n-ary atomic formula:

$(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^+ P(t_1, \ldots, t_n)$	iff	$(v_{\mathfrak{A}}(t_1), \ldots, v_{\mathfrak{A}}(t_n)) \in P^+_{\mathfrak{A}}$
$(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^- P(t_1, \ldots, t_n)$	iff	$(v_{\mathfrak{A}}(t_1), \ldots, v_{\mathfrak{A}}(t_n)) \in P^-_{\mathfrak{A}}$
$(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^+ \neg H$	iff	$(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^- H$
$(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^- \neg H$	iff	$(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^+ H$
$(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^+ H \wedge G$	iff	$(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^+ H$ and $(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^+ G$
$(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^- H \wedge G$	iff	$(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^- H$ or $(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^- G$
$(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^\pm H \vee G$	iff	$(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^\pm \neg(\neg H \wedge \neg G)$
$(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^\pm H \rightarrow G$	iff	$(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^\pm \neg(H \wedge \neg G)$
$(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^+ \forall x H$	iff	$(\mathfrak{A}, v_{\mathfrak{A}}[x/o]) \Vdash^+ H$ for all $o \in \lvert\mathfrak{A}\rvert$
$(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^- \forall x H$	iff	$(\mathfrak{A}, v_{\mathfrak{A}}[x/o]) \Vdash^- H$ for at least one $o \in \lvert\mathfrak{A}\rvert$
$(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^\pm \exists x H$	iff	$(\mathfrak{A}, v_{\mathfrak{A}}[x/o]) \Vdash^\pm \neg\forall x \neg H$

Where $v_{\mathfrak{A}}[x/o]$ is the modified valuation in which the function value of x is o and where $\Vdash^\pm$ stands for either $\Vdash^+$ or $\Vdash^-$.

Remark 2. In [7, p. 8] one can find a definition of FDE, which Marcel Crabbé calls bivalent logic. We used his definition with the modification that $P^-_{\mathfrak{A}} = |\mathfrak{A}|^n \backslash P^+_{\mathfrak{A}}$ to define an FOL interpretation.

Definition 3. A wff H is

1. satisfiable iff there are $\mathfrak{A}$ and $v_{\mathfrak{A}}$ with $(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^+ H$.
2. valid in $\mathfrak{A}$ iff $(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^+ H$ for all $v_{\mathfrak{A}}$. In this case $\mathfrak{A}$ is called a model of H.
3. generally valid iff $(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^+ H$ for all $\mathfrak{A}$ and $v_{\mathfrak{A}}$. In this case H is called a tautology.

Let Σ be a set of wff. Σ entails H ($\Sigma \vDash H$), iff if $(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^+ G$ for all $G \in \Sigma$ then $(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^+ H$, for all $\mathfrak{A}$ and $v_{\mathfrak{A}}$.

A structure $\mathfrak{A}$ is a model of Σ iff $(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^+ G$ for all $G \in \Sigma$ and all $v_{\mathfrak{A}}$.

Definition 4. Let Σ be a set of wff. Σ is FOL trivial iff $\Sigma \vDash H$ for all H of our language

2 Circumscription

Circumscription, introduced by John McCarthy [14], is a form of non-monotonic reasoning. It attempts to model a certain assumption of common sense reasoning, namely that things behave like we expect they do, unless we're told otherwise. If we know, for example, that usually all PhD candidates have good job opportunities, then we would expect that a certain PhD candidate called Dan has good job opportunities, unless we are told Dan is a logician. So, circumscription is a kind of defeasible reasoning. It was introduced syntactically and semantically, even though it is well-known that semantical and syntactical circumscription are not equivalent (see for example [4],[20]). Roughly speaking, syntactical circumscription can be defined via a translation into a second order language while semantical circumscription makes use of a preference relation on the set of all FOL models of a set of sentences. A sentence can then be evaluated only with respect to certain preferred models. In the following we will only deal with semantical circumscription.

One of the core ideas of circumscription is the selection of certain preferred models via a preference relation. A model is then called minimal if the extensions of so called abnormal predicates in this model are minimal. Usually abnormal predicates appear as relation symbols (e.g. $A(x)$) in our premise set within conditionals, like $\forall x(L(x) \rightarrow A(x))$ or $\forall x((L(x) \wedge \neg A(x)) \rightarrow G(x))$. Those conditionals can be used to formalize exceptions from the general case, while the main idea is that if nothing is stated about the abnormality of a certain object, then this object should not have the property of being abnormal. The above formulas can be read as: if someone is a logician he/she is abnormal or if someone is a logician and not abnormal then he/she has good job opportunities. Circumscription itself has been developed in great detail, at least in the field of computer science, and a lot of work has been done to extend McCarthy's original approach. But in this article, we will deal only with the simplest case of circumscription [14], i.e., we will not discuss pointwise circumscription, prioritized circumscription, autocircumscription etcetera (cf. [5],[8],[9],[15],[16]). The following definition is based on the definition found in [9]. We changed the order relation on models to a preorder, while the definition of minimal models is still based on a strict partial order. The reason for presenting the definition of circumscription in this way is to make it similar to the definition of minimal models Marcel Crabbé gave in [7].

Definition 5. Let Δ be a set of n-ary open atomic formulas $P(x_1, \ldots, x_n)$ and $\mathfrak{A}$ a model of some set Σ of wff. We define the true part of Δ in $\mathfrak{A}$ as follows:

$$\Delta^+_{\mathfrak{A}} = \{\langle P, (o_1, \ldots, o_n)\rangle | (o_1, \ldots, o_n) \in P^+_{\mathfrak{A}}, P \text{ occurs in } \Delta\}$$

Definition 6. Let Δ be a set of open atomic formulas and Σ be a set of wff.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two FOL models of Σ. A preorder $\prec_{=\Delta}$ on the set of all FOL models of Σ is defined as $\mathfrak{A} \prec_{=\Delta} \mathfrak{B}$ iff

1. $|\mathfrak{A}| = |\mathfrak{B}|$
2. $\Delta^+_{\mathfrak{A}} \subseteq \Delta^+_{\mathfrak{B}}$

A model $\mathfrak{A}$ of Σ is called $=\Delta$-minimal iff for all models $\mathfrak{B}$ of Σ if $\mathfrak{B} \prec_{=\Delta} \mathfrak{A}$, then $\Delta^+_{\mathfrak{A}} \subseteq \Delta^+_{\mathfrak{B}}$. Furthermore,

$$\Sigma \models^{=\Delta} H$$
iff
all $=\Delta$-minimal models of Σ are also models of H

Remark 7. Even though $\prec_{=\Delta}$ is a preorder, a $=\Delta$-minimal model is defined as a minimal model relative to the strict partial ordering "$\mathfrak{B} \prec_{=\Delta} \mathfrak{A}$ and not $\Delta^+_{\mathfrak{A}} \subseteq \Delta^+_{\mathfrak{B}}$". With this definition of minimality we follow [7, pp. 3 & 11].

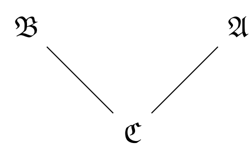
Remark 8. The main idea of circumscription is the minimization of the extension of certain predicates in our language. In other words we prefer models that are minimal with respect to the true part of Δ. Even though circumscription is defined for an arbitrary set of predicates Δ, we want that Δ contains those predicates which represent the abnormalities. But what it means to be abnormal has to be given from the application context.

Example 9. In order to see how circumscription works, we give give a simple example:

Let $\Sigma = \{\forall x((P(x) \wedge \neg A(x)) \rightarrow G(x)), P(\text{Dan})\}$, $\Delta = \{A(x)\}$ and our universe consists only of the object Dan, then we have the following three types of models of Σ,

Model	$P(x)$	$A(x)$	$G(x)$
$\mathfrak{A}$	Dan^+	Dan^+	Dan^+
$\mathfrak{B}$	Dan^+	Dan^+	Dan^-
$\mathfrak{C}$	Dan^+	Dan^-	Dan^+

where Dan^+ and Dan^- mean, Dan is in the extension or the complement of the extension of the respective relation symbol. Obviously we have $\Sigma \not\models G(\text{Dan})$. But after minimizing the abnormalities we get the following order of models:



And it is easy to see that we only have one $=\Delta$-minimal model, namely $\mathfrak{C}$. Therefore we have $\Sigma \models^{=\Delta} G(\text{Dan})$ as wanted.

3 Adding gaps and gluts to FOL (FDE)

Definition 10 (See p. 8 in [7])**.** First-degree entailment can be seen as a natural generalization of FOL. It uses the same language. A structure $\mathfrak{A}$ is called an FDE-interpretation if it contains a non-empty set of objects, the universe $|\mathfrak{A}|$, and every n-ary relation symbol P is interpreted by its extension $P^+_{\mathfrak{A}}$, with $P^+_{\mathfrak{A}} \subseteq |\mathfrak{A}|^n$ and its anti-extension $P^-_{\mathfrak{A}} \subseteq |\mathfrak{A}|^n$. Constants and function symbols are interpreted as usual by objects $o_{c_1}, o_{c_2} \ldots$ and functions.

The truth of a wff in an interpretation $\mathfrak{A}$ with respect to a valuation $v_{\mathfrak{A}}$ is defined as in Definition 1.

Remark 11. The main difference between FDE and FOL is that in FDE $P^+_{\mathfrak{A}}$ and $P^-_{\mathfrak{A}}$ don't need to be complements of each other. In other words, we allow $P^+_{\mathfrak{A}} \cup P^-_{\mathfrak{A}} \neq |\mathfrak{A}|^n$ (gaps) and $P^+_{\mathfrak{A}} \cap P^-_{\mathfrak{A}} \neq \emptyset$ (gluts) for some P.

Definition 12. Let Σ be a set of wff. Σ entails H in FDE ($\Sigma \vDash_{FDE} H$), iff if $(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^+ G$ for all $G \in \Sigma$ then $(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^+ H$, for all $\mathfrak{A}$ and $v_{\mathfrak{A}}$.

Validity, models and triviality for FDE are defined as in Definitions 3 and 4.

Furthermore, an FDE-model $\mathfrak{A}$ is FDE-trivial iff $(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^+ H$ for all $v_{\mathfrak{A}}$ and all H of our language

Remark 13. If we allow objects to be in the extension and anti-extension of a relation symbol or to be neither in one of them, the so defined logic, FDE, is obviously paracomplete and paraconsistent. But this system has one essential disadvantage: modus ponens and disjunctive syllogism are not valid and, moreover, FDE has no tautologies at all. Therefore, this logic can be seen as very weak.
But usually we assume that objects do not display glutty or gappy behavior. For example, it seems to be the case that in many circumstances only a few exceptional sentences are inconsistent. One attempt to formalize this thought in the setting of LP (the logic of paradox [17]) was developed by Graham Priest in [18], [19] and extended to FDE by Marcel Crabbé [7]. The main idea is the default assumption of the consistency of formulas, i.e., a minimal model contains as few contradictions as possible. Necessarily inconsistent formulas could be formulas like $H(x) \wedge \neg H(x)$ but also the wff $H(x)$ as a subformula of formulas of the set $\{H(x) \wedge G(x), \neg H(x)\}$. Like in circumscription we can construct a preference relation on the set of all models of a set of sentences. But unlike in circumscription we here minimize the amount of inconsistency and, since we are in a setting with gluts and gaps, we minimize the amount of incompleteness as well.
The main idea would be like this: we isolate all incomplete and inconsistent sentences, and then for the remaining classical sentences we get all of our FOL inference rules back.

Definition 14. Let $\mathfrak{A}$ be an FDE-model of some set of wff Σ. We define the inconsistent part $\mathfrak{A}!$ of $\mathfrak{A}$ and incomplete part $\mathfrak{A}?$ of $\mathfrak{A}$ in the following way:

$$\mathfrak{A}! = \{\langle P, (o_1, \ldots, o_n)\rangle | (o_1, \ldots, o_n) \in P^+_{\mathfrak{A}} \cap P^-_{\mathfrak{A}}\}$$

$$\mathfrak{A}? = \{\langle P, (o_1, \ldots, o_n)\rangle | (o_1, \ldots, o_n) \notin P^+_{\mathfrak{A}} \cup P^-_{\mathfrak{A}}\}$$

Remark 15. Roughly speaking, $\mathfrak{A}!$ and $\mathfrak{A}?$ can be understood as the set of all couples of some relation symbols P and (tuples of) objects o $(o_1, \ldots o_n)$ that are either in the extension and the anti-extension or neither in the extension nor anti-extension of P.

Definition 16. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two FDE-models of a set of wff Σ. A preorder $\prec_{\supseteq!?}$ on the set of all FDE-models of a set of wff Σ is defined as $\mathfrak{A} \prec_{\supseteq!?} \mathfrak{B}$ iff

1. $|\mathfrak{A}| \supseteq |\mathfrak{B}|$
2. $\mathfrak{A}! \subseteq \mathfrak{B}!$ and $\mathfrak{A}? \subseteq \mathfrak{B}?$

A model $\mathfrak{A}$ of Σ is called $\supseteq!?$-minimal iff for all models $\mathfrak{B}$ of Σ if $\mathfrak{B} \prec_{\supseteq!?} \mathfrak{A}$, then $\mathfrak{A}! \subseteq \mathfrak{B}!$ and $\mathfrak{A}? \subseteq \mathfrak{B}?$. Furthermore,

$$\Sigma \models^{\supseteq!?}_{FDE} H$$
iff
all $\supseteq!?$-minimal models of Σ are also models of H

Remark 17. Crabbé defined minimal FDE slightly differently, cf. Def. 2.1 in [7]. Item 2. of the above definition is as follows $\mathfrak{A}!? \subseteq \mathfrak{B}!?$, where $\mathfrak{A}!? = \mathfrak{A}! \cup \mathfrak{A}?$. The reason we changed Crabbé's original definition is that his proof of proposition 2.2, proposition 48 in this paper, is incomplete. Even though his proof can be completed, the combination of circumscription and minimal FDE is more straightforward with the definition given above. See appendix A for the complete proof of Crabbé's proposition 2.2 based on his definition of minimal FDE as well as an alternative definition of paracomplistent circumscription.

Remark 18. As one can easily see, one obvious difference between circumscription and minimizing in the setting of FDE is the relation between the universes of the models. The use of $\supseteq$ in the above condition(1) seems a bit odd, because usually we want the models to share the same universe in order to be comparable. But as a matter of fact, as Crabbé showed in [6], any other relation or no relation between the models than $\supseteq$ may lead to a failure of recapture, reassurance or strong reassurance (see Definitions 19 and 20 below). So, the reason why we stick to $\supseteq$ is a technical one, since no other relation guarantees those wanted properties.

Definition 19. Strong reassurance is the following condition: For any set of sentences Σ, if $\mathfrak{B}$ is an FDE-model of Σ, then there exists an FDE-model

$\mathfrak{A}$ of Σ with $\mathfrak{A} \prec_{\supseteq!?} \mathfrak{B}$ and $\mathfrak{A}$ is $\supseteq$!?-minimal or $\mathfrak{B}$ is already $\supseteq$!?-minimal. Reassurance is the following condition: If a set of sentences Σ is not FDE-trivial, i.e., $\Sigma \not\models_{FDE} H$ for some H in our language, then we have $\Sigma \not\models^{\supseteq!?}_{FDE} H$ for some H in our language.

Definition 20. Classical recapture is the following condition: If a set of sentences Σ is FOL consistent then we have $\Sigma \models H$ iff $\Sigma \models^{\supseteq!?}_{FDE} H$.

Remark 21. With Definition 19 we follow [6]. Note, that strong reassurance implies reassurance, but not vice versa. In the course of this article we show how Crabbé proved reassurance. Whereas, getting strong reassurance for minimal FDE is still an open problem.
If classical recapture holds, then we get modus ponens for FOL consistent premises (as intended) and therefore our strongest inference tool back.

Remark 22. Besides the right relation between models, we need to take care of the language we use. Languages with equality can lead to a failure of reassurance. An extensive discussion on that subject and under which conditions ressurance and classical recapture may hold can be found in [6]. In [7] Crabbé explains that his approach (cf. section 4) can solve the reassurance problem in such languages, if the treatment of equality is slightly changed. Nevertheless, we will omit equality for our approach of paracomplistent circumscription.

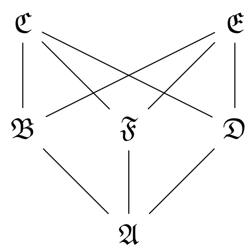
Counterexamples to reassurance, strong reassurance and classical recapture if we use $=$, $\subseteq$ or have no restrictions on the relation between the universes of the models can be found for example in [6].

Example 23. To get the general idea of how minimizing in the framework of FDE works and how we get modus ponens back, we give another short example.

Let $\Sigma = \{\forall x(P(x) \rightarrow G(x)), P(\text{Dan})\}$. Obviously we have $\Sigma \not\models_{FDE} G(\text{Dan})$, as the models below show us. The entry $\emptyset$ means that both extension and anti-extension are empty.

Model	$P(x)$	$G(x)$
$\mathfrak{A}$	Dan^+	Dan^+
$\mathfrak{B}$	Dan^+	Dan^+, Dan^-
$\mathfrak{C}$	Dan^+, Dan^-	Dan^+, Dan^-
$\mathfrak{D}$	Dan^+, Dan^-	Dan^-
$\mathfrak{E}$	Dan^+, Dan^-	$\emptyset$
$\mathfrak{F}$	Dan^+, Dan^-	Dan^+

But if we order the models after the amount of inconsistency/ incompleteness, we can see immediately that there is only one type of $\supseteq$!?-minimal model, namely $\mathfrak{A}$.



And since all $\supseteq$!?-minimal Σ-models are models of $G(\text{Dan})$ as well, we have, as wanted, $\Sigma \vDash^{\supseteq !?}_{FDE} G(\text{Dan})$.

4 Minimal entailment

In order to keep this article self-contained this section is concerned with the proofs of reassurance and classical recapture in the setting of minimal FDE for a language with relation symbols and function symbols given by Marcel Crabbé. Readers who are familiar with [7] can easily skip this section since it merely provides prerequisites that are made use of in section 5. As mentioned in section 3, minimal entailment can lead to the failure of reassurance and strong reassurance. It was Crabbé who showed in [6] and [7] how reassurance and furthermore classical recapture can be obtained in minimal entailment. In this section we will present his results but omit most of the proofs, while in section 5 Marcel Crabbé's results will be extended to a paracomplistent version of circumscription.
The first definition is concerned with a general notion of minimality. Instead of minimizing a certain kind of abnormalities, like inconsistent or incomplete formulas, the relation $\prec_{\mathcal{F}}$ allows minimizing the extension of arbitrary open formulas of $\mathcal{L}$, where not every variable is in the scope of a quantifier.

Definition 24 (Def. 1.1 in [7])**.** Let Σ be a set of First-order formulas in a language without equality, but possibly with function symbols, $\mathcal{F}$ be a set of open formulas, i.e., every $H \in \mathcal{F}$ has free variables, and $\mathfrak{A}$ an FOL model of Σ.

The $\mathcal{F}$-kernel of $\mathfrak{A}$, $\ker_{\mathcal{F}}(\mathfrak{A})$, is the set of the objects o in the universe $|\mathfrak{A}|$ such that $v_{\mathfrak{A}}(x) = o$ and $(\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^{+} H$, for some formula H of $\mathcal{F}$, some variable x occurring free in H, and some valuation $v_{\mathfrak{A}}$.

A transfer relation $\Subset_{\mathcal{F}}$ between FOL models of Σ is defined as follows:

$$\mathfrak{B} \Subset_{\mathcal{F}} \mathfrak{A} \text{ iff } \begin{cases} \ker_{\mathcal{F}}(\mathfrak{B}) \subseteq \ker_{\mathcal{F}}(\mathfrak{A}) \text{ and} \\ \text{if } (\mathfrak{B}, v_{\mathfrak{A}}) \Vdash^{+} H, \text{ then } (\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^{+} H, \text{ for all } H \text{ in } \mathcal{F}, \\ \text{and all valuations } v_{\mathfrak{A}} \text{ to } \ker_{\mathcal{F}}(\mathfrak{B}). \end{cases}$$

The relation $\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$ is defined by $\mathfrak{B} \Subset_{\mathcal{F}} \mathfrak{A}$ and $|\mathfrak{B}| \supseteq |\mathfrak{A}|$.

An FOL model $\mathfrak{A}$ of Σ is called $\mathcal{F}$-minimal iff for all FOL models $\mathfrak{B}$ of Σ if $\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$ then $\mathfrak{A} \Subset_{\mathcal{F}} \mathfrak{B}$.

Remark 25. Note, the above definition of minimality is similar to the definitions of minimality given earlier. We define a preorder $\prec_{\mathcal{F}}$ and based on this we can define $\mathcal{F}$-minimal models with respect to the strict partial ordering "$\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$ and not $\mathfrak{A} \Subset_{\mathcal{F}} \mathfrak{B}$" [7, p. 3].
Furthermore, it is important to notice that we define $\prec_{\mathcal{F}}$ between FOL models.

Definition 26 (Def. 1.3 in [7])**.** $\Sigma \vDash_{\mathcal{F}} H$ iff every $\mathcal{F}$-minimal Σ-model is a model of H.

Remark 27. Again, it is worth mentioning that the $\supseteq$-relation between the domains, i.e., $|\mathfrak{B}| \supseteq |\mathfrak{A}|$, seems a bit odd. But it was shown in [6] that $\subseteq$, $=$ or no restriction on the relation between the domains can lead to a failure of reassurance and/or classical recapture. So, even though the usage of $\supseteq$ seems to be ad hoc, in the described framework it is the only way to guarantee reassurance and classical recapture (cf. [6, p. 484]).

Proposition 28 (Prop. 1.1 in [7])**.** For every $\mathcal{F}, \Sigma$ and model $\mathfrak{A}$ of Σ, with finite $\mathcal{F}$-kernel, there is an $\mathcal{F}$-minimal-model $\mathfrak{B}$ of Σ such that $\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$.

Remark 29. To prove reassurance and classical recapture for minimal FDE, Marcel Crabbé used properties of FDE in a sophisticated way. First he showed that those properties hold, when one is dealing with so called positive formulas and then he used Lyndon's (cf. [13]) well-known notion of positive and negative occurrences of relation symbols to translate all FDE sentences into positive FOL formulas in order to prove reassurance and classical recapture for minimal FDE.

Definition 30 (p. 9 in [7])**.** Let H be a formula and P be an n-ary relation symbol. An occurrence of P in H is positive when the branch of the parse tree of H leading from this occurrence of P to H itself contains an even number of formulas $\neg F$ or $(F \to B)$, with the corresponding occurrence of P in F. The occurrence is negative when this number of formulas is odd. A formula is positive [negative] iff all occurrences of relation symbols in it are positive [negative].

Example 31. A simple example would be the following formula: $\neg(\neg P(x) \vee Q(x)) \vee \neg Q(x)$, where the occurrence of $P(x)$ is positive and both occurrences of $Q(x)$ are negative.

Definition 32 (Def. 1.2 in [7])**.** A p-trivial set of formulas is one that entails all positive formulas of its language.
A model is positively trivial iff every positive formulas is true in it.

Proposition 33 (Prop. 1.2 in [7])**.** Every non p-trivial positive set of formula has a finite non-trivial model.

Definition 34 (Def. 1.4 in [7])**.** $\mathcal{F}$ transfers triviality between models of a positive set of sentences Σ iff whenever $\mathfrak{B}$ is trivial and $\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$, then $\mathfrak{A}$ is trivial, for every model $\mathfrak{A}$ and $\mathfrak{B}$ of Σ.

Lemma 35 (Lemma 1.1 in [7])**.** A sufficient condition for $\mathcal{F}$ to transfer triviality between models of Σ is that for every atomic formula $P(x_1, \ldots, x_n)$ with pointwise distinct variables, there are positive formulas $H_1, \ldots, H_m$ in $\mathcal{F}$ such that $\Sigma \vDash (\forall x_1 \ldots \forall x_k\ (H_1 \wedge \cdots \wedge H_m) \rightarrow \forall x_1 \ldots \forall x_n\ P(x_1, \ldots, x_n))$ (or $\Sigma \vDash \forall x_1 \ldots \forall x_n\ P(x_1, \ldots, x_n)$).

Theorem 1 (Theorem 1.1 in [7])**.** If $\mathcal{F}$ transfers triviality and the positive set of sentences Σ is not p-trivial, then $\Sigma \nvDash_{\mathcal{F}} H$, for some positive H.

Remark 36. Theorem 1 shows how we are reassured with the existence of minimal models in some positive settings, since every non p-trivial positive set of sentences has minimal models (Propositions 33 and 48) which are not p-trivial as well, if $\mathcal{F}$ transfers triviality.

Note that the notion of transferring triviality is crucial here. If $\mathcal{F}$ did not transfer triviality, one could have a non p-trival set of sentences with a p-trivial minimal model. Take for instance a language which contains only one unary predicate $P(x)$, let $\Sigma = \emptyset$ and $\mathcal{F} = \{\neg P(x)\}$. Obviously, Σ is not p-trivial. Then, per Definition 34, $\mathcal{F}$ does not transfer triviality, since in all $\mathcal{F}$-minimal models $P(x)$ is true (cf. [7, p. 7]).

The next proposition shows that $\vDash_{\mathcal{F}}$ can recapture $\vDash$ in some consistent contexts.

Theorem 2 (Theorem 1.2 in [7])**.** Let $\forall\neg\mathcal{F}$ be the set of the universal closures of the negations of all formulas of $\mathcal{F}$. If $\forall\neg\mathcal{F} \cup \Sigma$ is consistent, then $\forall\neg\mathcal{F} \cup \Sigma \vDash H$ iff $\Sigma \vDash_{\mathcal{F}} H$.

Remark 37. As already mentioned the property of transferring triviality is important. Without it reassurance would fail (cf. [7, p. 7] for a counter example).

So, as we can see, the argument for reassurance works as follows:

- For every model $\mathfrak{A}$ of a set of formulas Σ with finite $\mathcal{F}$-kernel, there is an $\mathcal{F}$-minimal model $\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$.
- Every non p-trivial positive set of formulas has a finite non-trivial model.
- If $\mathcal{F}$ transfers triviality and Σ is a positive not p-trivial set of formulas, then there exists an $\mathcal{F}$-minimal not p-trivial (reassurance) model.

In the same positive setting we have in consistent environments a recapture of our consequence relation $\models$ as well. Now, if we could show that there is a positive system equivalent to FDE we are done, i.e., we have proven reassurance and classical recapture for FDE. In what follows we will give a positive translation of FDE by using Lyndon's notion of positive and negative occurrences of relation symbols (cf. [13]).

Definition 38 (p. 9 in [7])**.** We enrich our language $\mathcal{L}$ to a language $\mathcal{L}_{pos}$ by adding a new symbol $\overline{P}$ for each relational symbol P in $\mathcal{L}$. If we replace in a formula H of $\mathcal{L}$ each occurrence of an atomic formula $P(x_1, \ldots, x_n)$, with negative occurrence of P, by $\neg\overline{P}(x_1, \ldots, x_n)$, then we obtain a positive formula H^{pos} in $\mathcal{L}^{pos}$. If we replace similarly each occurrence of an atomic formula $P(x_1, \ldots, x_n)$, with positive occurrence of P, by $\neg\overline{P}(x_1, \ldots, x_n)$, then we obtain a negative formula H^{neg}.

Example 39. Let $H = P(x) \wedge \neg Q(x)$. Obviously, the occurence of $Q(x)$ is negative. Therefore, $H^{pos} = P(x) \wedge \neg\neg\overline{Q}(x)$ and similary, $H^{neg} = \neg\overline{P}(x) \wedge \neg Q(x)$.

Definition 40 (p. 10 in [7])**.** To an FDE-interpretation $\mathfrak{A}$, we associate in a natural way a positive interpretation $\mathfrak{A}^{pos}$ for the language $\mathcal{L}^{pos}$, with the same universe and interpretation of the function symbols, and put:

$$\begin{array}{rcl} P_{\mathfrak{A}^{pos}} & = & P^+_{\mathfrak{A}} \\ \overline{P}_{\mathfrak{A}^{pos}} & = & P^-_{\mathfrak{A}} \end{array}$$

for all relation symbols in the language. This means, we interpret the anti-extension of all predicates as a simple extension.

Lemma 41 (Lemma 2.1 in [7])**.** We set $v_{\mathfrak{A}}(x) = v_{\mathfrak{A}^{pos}}(x)$, truth and falsity in a positive interpretation is then as follows:

$$\begin{array}{lcl} (\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^+ H & \text{iff} & (\mathfrak{A}^{pos}, v_{\mathfrak{A}^{pos}}) \models H^{pos} \\ (\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^- H & \text{iff} & (\mathfrak{A}^{pos}, v_{\mathfrak{A}^{pos}}) \not\models H^{neg} \end{array}$$

Remark 42. H^{pos} and H^{neg} are basically the same formula, but they have a different interpretation. The positive translation H^{pos} expresses that H is true, and the negative translation H^{neg} that H is not false.

Definition 43. The definition of a positive consequence relation is straightforward:

$$\Sigma^{pos} \vDash H^{pos}$$
$$\text{iff}$$
$$\text{if } (\mathfrak{A}^{pos}, v) \vDash G^{pos} \text{ for all } G^{pos} \in \Sigma^{pos}, \text{ then } (\mathfrak{A}^{pos}, v) \vDash H^{pos}$$

Proposition 44 (Prop. 2.1 in [7])**.** From Lemma 41 follows immediately:

$$\Sigma \vDash_{FDE} H \text{ iff } \Sigma^{pos} \vDash H^{pos}$$

Remark 45. Within this positive environment, we can even construct the FOL consequence relation. We only need to guarantee that the extension of all predicates $P(x_1, \ldots, x_n)$ and $\overline{P}(x_1, \ldots, x_n)$ are disjoint and their union not empty.

Proposition 46 (Prop. 2.3 in [7])**.** Let EM (for excluded middle) be the set of all sentences $\forall x_1, \ldots \forall x_n(Px_1, \ldots, x_n \vee \neg Px_1, \ldots, x_n)$ of our language, then we have:

$$\Sigma \vDash H \text{ iff } EM^{pos}, EM^{neg}, \Sigma^{pos} \vDash H^{pos}$$

Remark 47. As clear as it is that the positive consequence relation is equivalent to the FDE consequence relation, this positive translation leads to nice properties concerning the minimization of models. If we define the set $\mathcal{F}$ properly, we can construct a consequence relation $\vDash_{\mathcal{F}}$ which is equivalent to $\vDash^{\supseteq !?}_{FDE}$ and has the properties of reassurance and classical recapture. And this is exactly what Marcel Crabbé does in [7].

Proposition 48 (Prop. 2.2 in [7])**.** Let CN (for contradictory) be the set of all formulas $P(x_1, \ldots, x_n) \wedge \neg P(x_1, \ldots, x_n)$ of our language. If we set $\mathcal{F} = CN^{pos} \cup CN^{neg}$ the following holds:

1. $\Sigma \vDash^{\supseteq !?}_{FDE} H \quad \text{iff} \quad \Sigma^{pos} \vDash_{\mathcal{F}} H^{pos}$

2. $\mathcal{F}$ transfers triviality between models of Σ^{pos}

Proof. See [7, p. 11] for 2.

Since we use a slightly different version of minimal FDE than Crabbé does we have to proof 1.

We proof $\Sigma \vDash^{\supseteq !?}_{FDE} H$ iff $\Sigma^{pos} \vDash_{\mathcal{F}} H^{pos}$ by showing that an FDE-model $\mathfrak{A}$ is a minimal Σ-model iff $\mathfrak{A}^{pos}$ is an $\mathcal{F}$-minimal Σ^{pos}-model. This follows immediately from the fact that

$$\mathfrak{A}! \subseteq \mathfrak{B}! \text{ and } \mathfrak{A}? \subseteq \mathfrak{B}? \text{ iff } \mathfrak{A}^{pos} \Subset_{\mathcal{F}} \mathfrak{B}^{pos}$$

which we prove now.

Suppose $\mathfrak{A}! \subseteq \mathfrak{B}!$ and $\mathfrak{A}? \subseteq \mathfrak{B}?$, and let v be a valuation to $\ker_{\mathcal{F}}(\mathfrak{A}^{pos})$ such that $(\mathfrak{A}^{pos}, v) \vDash H$ for some $H \in \ker_{\mathcal{F}}(\mathfrak{A}^{pos})$. We have to distinguish two cases: (i) $H = P(x_1, \ldots, x_n) \wedge \neg\neg\overline{P}(x_1, \ldots, x_n)$, (ii) $H = \neg P(x_1, \ldots, x_n) \wedge \neg\overline{P}(x_1, \ldots, x_n)$.

In case (i), $\langle P, (v(x_1), \ldots v(x_n)\rangle \in \mathfrak{A}! \subseteq \mathfrak{B}!$. Therefore, $(\mathfrak{B}^{pos}, v) \vDash P(x_1, \ldots, x_n) \wedge \neg\neg\overline{P}(x_1, \ldots, x_n)$. This also shows $\ker_{\mathcal{F}}(\mathfrak{A}^{pos}) \subseteq \ker_{\mathcal{F}}(\mathfrak{B}^{pos})$.

Cases (ii) is analogous to (i) and left to the reader. Altogether it follows that $\mathfrak{A}^{pos} \Subset_{\mathcal{F}} \mathfrak{B}^{pos}$.

For the converse, suppose that $\mathfrak{A}^{pos} \Subset_{\mathcal{F}} \mathfrak{B}^{pos}$. We distinguish two cases: (i) $\langle P, (o_1, \ldots, o_n)\rangle \in \mathfrak{A}!$, (ii) $\langle P, (o_1, \ldots, o_n)\rangle \in \mathfrak{A}?$. Furthermore, let v be a valuation such that $v(x_1) = o_1, \ldots, v(x_n) = o_n$.

In case (i) we have $(o_1, \ldots, o_n) \in P^+_{\mathfrak{A}} \cap P^-_{\mathfrak{A}}$, and it follows that $(\mathfrak{A}^{pos}, v) \vDash P(x_1, \ldots, x_n) \wedge \neg\neg\overline{P}(x_1, \ldots, x_n)$. Therefore, $(\mathfrak{B}^{pos}, v) \vDash P(x_1, \ldots, x_n) \wedge \neg\neg\overline{P}(x_1, \ldots, x_n)$ and whence $(o_1, \ldots, o_n) \in P^+_{\mathfrak{B}} \cap P^-_{\mathfrak{B}}$, i.e., $\langle P, (o_1, \ldots, o_n)\rangle \in \mathfrak{B}!$

Cases (ii) is analogous to (i) and left to the reader. Altogether it follows that $\mathfrak{A}! \subseteq \mathfrak{B}!$ and $\mathfrak{A}? \subseteq \mathfrak{B}?$, which concludes the proof. □

Remark 49. Since Crabbé's original proof of item 1. in proposition 48 is incomplete, in appendix A you'll find a corrected version of the proof based on Crabbé's definition of minimal FDE.

Theorem 3 (Theorems 2.1 and 2.2 in [7])**.** For $\Sigma^{pos} \vDash_{\mathcal{F}} H^{pos}$ and therefore $\Sigma \vDash^{\supseteq !?}_{FDE} H$ we have:

1. Reassurance
2. Classical recapture

Remark 50. Note, that strong reassurance doesn't hold with this definition of minimality. To see this just take the Example 3.1 in [6]. Nevertheless, as Christian Straßer mentioned in personal communication, it is possible to regain strong reassurance for minimal FDE by slightly changing the definition of minimality and making use of the cardinality of the set of inconsistent and incomplete objects. But, for now we will leave this topic for future research.

This section was a recapitulation of the work of Marcel Crabbé (cf. [7]). As one could see, the properties of reassurance and classical recapture in an FOL language without equality but possibly with function symbols were proven via a positive translation technique.

But since the proofs of reassurance and recapture are very general, they work not only if one wants to minimize inconsistency and incompleteness. For example, in the setting of FDE there are a number of different minimizing strategies possible (cf. [2]). One could minimize over the so called knowledge order or the truth order or even just certain relation symbols. This all depends on the choice of $\mathcal{F}$. But if this choice is suitable then we have, according to Marcel Crabbé, reassurance and recapture.

5 Paracomplistent circumscription

With the positive translation technique at hand we are almost in the position to construct a paracomplete and paraconsistent version of circumscription. There is only one more step to be taken.

In section 4 it was argued that in order to guarantee reassurance and classical recapture the relation between the universes of two models $\mathfrak{A}$ and $\mathfrak{B}$ with $\mathfrak{A} \prec \mathfrak{B}$ should be $|\mathfrak{A}| \supseteq |\mathfrak{B}|$, but the relation between the universes of two models in circumscription is $=$. So, in order to combine both circumscription and minimal FDE we need to change the $=$ of circumscription to $\supseteq$.

Note, that a definition of $\supseteq$-circumscription is not necessary if one wants to combine minimal FDE and circumscription iteratively, i.e., one first minimizes inconsistencies and incompleteness and then minimizes the abnormalities of the remaining models. But, one of the purposes of this paper is to show, that Crabbe's technique is general enough to combine different minimization techniques simultaneously.

Definition 51. Let Δ be a set of open atomic formulas and Σ be a set of wff. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two FOL models of Σ. A preorder $\prec_{\supseteq\Delta}$ on the set of all FOL models of Σ is defined as $\mathfrak{A} \prec_{\supseteq\Delta} \mathfrak{B}$ iff

1. $|\mathfrak{A}| \supseteq |\mathfrak{B}|$
2. $\Delta^+_{\mathfrak{A}} \subseteq \Delta^+_{\mathfrak{B}}$

A model $\mathfrak{A}$ of Σ is called $\supseteq\Delta$-minimal iff for all models $\mathfrak{B}$ of Σ if $\mathfrak{B} \prec_{\supseteq\Delta} \mathfrak{A}$, then $\Delta^+_{\mathfrak{A}} \subseteq \Delta^+_{\mathfrak{B}}$. Furthermore,

$$\Sigma \models^{\supseteq\Delta} H$$
$$\text{iff}$$
$$\text{all } \supseteq\Delta\text{-}FOL \text{ minimal models of } \Sigma \text{ are also models of } H$$

Remark 52. Definition 51 needs a bit more discussion. In classical circumscription (see Definition 6) it is required that two models always have the same domain and therefore, are comparable. This requirement is relaxed in $\supseteq$-circumscription for one purpose, namely to minimize inconsistency, incompleteness and abnormalities simultaneously, and to show the generality of Crabbe's result. To investigate and discuss the properties of $\supseteq$-circumscription will be left for future research. Nevertheless, we will show that classical recapture holds for $\supseteq$-circumscription.

The next proposition shows how circumscription works in a positively translated FDE setting.

Proposition 53. Let Δ be a set of open atomic formulas and $\mathcal{F} = \Delta$, then

$$\Sigma \vDash^{\supseteq\Delta} H \quad \text{iff} \quad EM^{pos}, EM^{neg}, \Sigma^{pos} \vDash_{\mathcal{F}} H^{pos}$$

Proof. As in the proof of Proposition 48, one has to show that every $\supseteq \Delta$-minimal FOL model is an $\mathcal{F}$-minimal model and vice versa, and this follows from

$$\Delta^{+}_{\mathfrak{A}} \subseteq \Delta^{+}_{\mathfrak{B}} \text{ iff } \mathfrak{A}^{pos} \Subset_{\mathcal{F}} \mathfrak{B}^{pos}$$

which will be proven now:

Suppose that $\Delta^{+}_{\mathfrak{A}} \subseteq \Delta^{+}_{\mathfrak{B}}$ and let v be a valuation to $\ker_{\mathcal{F}}(\mathfrak{A}^{pos})$, such that $(\mathfrak{A}^{pos}, v) \vDash P(x_1, \ldots, x_n)$ for some $P(x_1, \ldots, x_n) \in \mathcal{F}$. Then we have $\langle P, (v(x_1),\ \ldots, v(x_n))\rangle \in \Delta^{+}_{\mathfrak{A}} \subseteq \Delta^{+}_{\mathfrak{B}}$. But therefore we have $(\mathfrak{B}^{pos}, v) \vDash P(x_1, \ldots, x_n)$. And this also means that $\ker_{\mathcal{F}}(\mathfrak{A}^{pos}) \subseteq \ker_{\mathcal{F}}(\mathfrak{B}^{pos})$.

For the converse, suppose that $\mathfrak{A}^{pos} \Subset_{\mathcal{F}} \mathfrak{B}^{pos}$ and let $\langle P, (o_1, \ldots o_n)\rangle \in \Delta^{+}_{\mathfrak{A}}$ and v be such a valuation that $v(x_1) = o_1, \ldots, v(x_n) = o_n$. Now, we have $(o_1, \ldots, o_n) \in P^{+}_{\mathfrak{A}}$ and it follows that $(\mathfrak{A}^{pos}, v) \vDash P(x_1, \ldots, x_n)$. Therefore, $(\mathfrak{B}^{pos}, v) \vDash P(x_1, \ldots, x_n)$ and $(o_1, \ldots, o_n) \in P^{+}_{\mathfrak{B}}$, i.e., $\langle P, (o_1, \ldots o_n)\rangle \in \Delta^{+}_{\mathfrak{B}}$. □

Proposition 54. Let Δ be a set of open atomic formulas, $\Sigma \cup H$ a set of sentences and $\forall\neg\Delta$ the universal closure of the negations of the formulas of Δ. If $\Delta \cup \forall\neg\Delta$ is consistent we have

$$\Sigma \vDash^{\supseteq\Delta} H \quad \text{iff} \quad \forall\neg\Delta, \Sigma \vDash H$$

Proof. The proof of this propositions follows immediately in light of Theorem 2 and Proposition 53. □

Now we are in the position to define paracomplistent circumscription.

Definition 55. Let Δ be a set of open atomic formulas and Σ be a set of wff. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two FDE-models of Σ. A preorder $\prec_{\supseteq!?\Delta}$ is defined as $\mathfrak{A} \prec_{\supseteq!?\Delta} \mathfrak{B}$ iff

1. $|\mathfrak{A}| \supseteq |\mathfrak{B}|$
2. $\mathfrak{A}! \subseteq \mathfrak{B}!$ and $\mathfrak{A}? \subseteq \mathfrak{B}?$ and $\Delta^+_{\mathfrak{A}} \subseteq \Delta^+_{\mathfrak{B}}$

An FDE-model $\mathfrak{A}$ of Σ is called $\supseteq!?\Delta$-minimal iff for all models $\mathfrak{B}$ of Σ if $\mathfrak{B} \prec_{\supseteq!?\Delta} \mathfrak{A}$, then $\Delta^+_{\mathfrak{A}} \subseteq \Delta^+_{\mathfrak{B}}$ and $\mathfrak{A}! \subseteq \mathfrak{B}!$ and $\mathfrak{A}? \subseteq \mathfrak{B}?$. Furthermore,

$$\Sigma \vDash^{\supseteq!?\Delta}_{FDE} H$$
iff
all $\supseteq!?\Delta$-minimal models of Σ are also models of H

Lemma 56. Let $\mathcal{F} = CN^{pos} \cup CN^{neg} \cup \Delta$ for some set of open atomic formulas Δ, then

1. $\Sigma \vDash^{\supseteq!?\Delta}_{FDE} H$ iff $\Sigma^{pos} \vDash_{\mathcal{F}} H^{pos}$
2. $\mathcal{F}$ transfers triviality between models of Σ^{pos}

Proof.

1. We proof $\Sigma \vDash^{\supseteq!?\Delta}_{FDE} H$ iff $\Sigma^{pos} \vDash_{\mathcal{F}} H^{pos}$ by showing that an FDE-model $\mathfrak{A}$ is a minimal Σ-model iff $\mathfrak{A}^{pos}$ is an $\mathcal{F}$-minimal Σ^{pos}-model. This follows immediately from the fact that

$$\mathfrak{A}! \subseteq \mathfrak{B}! \text{ and } \mathfrak{A}? \subseteq \mathfrak{B}? \text{ and } \Delta^+_{\mathfrak{A}} \subseteq \Delta^+_{\mathfrak{B}} \text{ iff } \mathfrak{A}^{pos} \Subset_{\mathcal{F}} \mathfrak{B}^{pos}$$

which we prove now.

Suppose $\mathfrak{A}! \subseteq \mathfrak{B}!$ and $\mathfrak{A}? \subseteq \mathfrak{B}?$ and $\Delta^+_{\mathfrak{A}} \subseteq \Delta^+_{\mathfrak{B}}$, and let v be a valuation to $\ker_F(\mathfrak{A}^{pos})$ such that $(\mathfrak{A}^{pos}, v) \vDash H$ for some $H \in \ker_F(\mathfrak{A}^{pos})$ with free variable x_i and $v(x_i) = o_i$. We have to distinguish three cases: (i) $H = P(x_1, \ldots, x_n) \wedge \neg\neg\overline{P}(x_1, \ldots, x_n)$, (ii) $H = \neg P(x_1, \ldots, x_n) \wedge \neg\overline{P}(x_1, \ldots, x_n)$, (iii) $H = P(x_1, \ldots, x_n)$.

In case (i), $\langle P, (v(x_1), \ldots v(x_n)\rangle \in \mathfrak{A}! \subseteq \mathfrak{B}!$. Therefore, $(\mathfrak{B}^{pos}, v) \vDash P(x_1, \ldots, x_n) \wedge \neg\neg\overline{P}(x_1, \ldots, x_n)$. This also shows $\ker_F(\mathfrak{A}^{pos}) \subseteq \ker_F(\mathfrak{B}^{pos})$.

Cases (ii) and (iii) are analogous to (i) and left to the reader.

Altogether it follows that $\mathfrak{A}^{pos} \Subset_{\mathcal{F}} \mathfrak{B}^{pos}$.

For the converse, suppose that $\mathfrak{A}^{pos} \Subset_{\mathcal{F}} \mathfrak{B}^{pos}$. We distinguish three cases: (i) $\langle P, (o_1, \ldots, o_n)\rangle \in \mathfrak{A}!$, (ii) $\langle P, (o_1, \ldots, o_n)\rangle \in \mathfrak{A}?$, (iii) $\langle P, (o_1, \ldots, o_n)\rangle \in \Delta^+_{\mathfrak{A}}$. Furthermore, let v be a valuation such that $v(x_1) = o_1, \ldots, v(x_n) = o_n$.

In case (i) we have $(o_1, \ldots, o_n) \in P^+_{\mathfrak{A}} \cap P^-_{\mathfrak{A}}$, and it follows that $(\mathfrak{A}^{pos}, v) \vDash P(x_1, \ldots, x_n) \wedge \neg\neg\overline{P}(x_1, \ldots, x_n)$. Therefore, $(\mathfrak{B}^{pos}, v) \vDash P(x_1, \ldots, x_n) \wedge \neg\neg\overline{P}(x_1, \ldots, x_n)$ and whence $(o_1, \ldots, o_n) \in P^+_{\mathfrak{B}} \cap P^-_{\mathfrak{B}}$, i.e., $\langle P, (o_1, \ldots, o_n)\rangle \in \mathfrak{B}!$

Cases (ii) and (iii) are analogous to (i) and left to the reader.

Altogether it follows that $\mathfrak{A}! \subseteq \mathfrak{B}!$ and $\mathfrak{A}? \subseteq \mathfrak{B}?$ and $\Delta^+_{\mathfrak{A}} \subseteq \Delta^+_{\mathfrak{B}}$, which concludes the proof.

2. The proof is exactly as in [7, p. 11].

□

Lemma 57. Let $\mathcal{F}_1 = CN^{pos} \cup CN^{neg}$ and $\mathcal{F}_2 = \Delta$ for some set of open atomic formulas Δ, and let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. We have for $\Sigma^{pos} \vDash_{\mathcal{F}_1 \cup \mathcal{F}_2} H^{pos}$ and therefore $\Sigma \vDash^{\supseteq !?\Delta}_{FDE} H$

1. Reassurance
2. Recapture of $\vDash^{\supseteq\Delta}$ ($\supseteq$-circumscription) if the premises are FOL consistent and normal, i.e., not abnormal for every model of Σ

Proof.

1. If Σ is not FDE-trivial, then Σ^{pos} is not p-trivial and every non p-trivial theory has a non-trivial minimal model. Furthermore $\mathcal{F}$ transfers triviality.
2. If Σ is FOL consistent and normal, then it is FOL consistent with $\forall\neg\mathcal{F}_1$ and we have $\Sigma^{pos} \vDash_{\mathcal{F}} H^{pos}$ iff $\forall\neg\mathcal{F}_1, \Sigma^{pos} \vDash_{\mathcal{F}_2} H^{pos}$ iff $EM^{pos}, EM^{neg}, \Sigma^{pos} \vDash_{\mathcal{F}_2} H^{pos}$ iff $\Sigma \vDash^{\supseteq\Delta} H$

□

Remark 58. The last lemma shows that the recapture of $\vDash^{\supseteq\Delta}$ only holds for FOL consistent and normal premises, i.e., premises that are not necessarily glutty or abnormal. The motivation, why we needed to restrict the lemma to normal premises as well, can be given through an example. We would like to thank an anonymous referee of an earlier version of this article for pointing this out. Let Σ be the following set of sentences: $\Sigma = \{P(a), \neg P(a) \vee Q(a)\}$ with $\Delta = \{Q(x)\}$. It is easy to see that $\Sigma \vDash^{\supseteq!?\Delta}_{FDE} Q(a)$ doesn't hold, while $\Sigma \vDash^{\supseteq\Delta} Q(a)$ is obviously valid.

Example 59. The next and last example is an example of how we get modus ponens back in consistent subsets of Σ and that a necessarily inconsistent sentence in the model doesn't lead to explosion. It illustrates how paracomplistent circumscription works.

Let $\Sigma = \{\forall x((P(x) \wedge \neg A(x)) \rightarrow G(x)),\ P(\text{Dan}),\ Ph(\text{Dan}) \wedge \neg Ph(\text{Dan})\}$ with $\Delta = \{P(x), A(x)\}$. Obviously we have $\Sigma \vDash H$ and $\Sigma \vDash^{\supseteq\Delta}_{FOL} H$ for all H of our language, since Σ is inconsistent. Furthermore we have $\Sigma \nvDash_{FDE}$ $G(\text{Dan})$.

Models	$P(x)$	$A(x)$	$G(x)$	$Ph(x)$
$\mathfrak{A}$	Dan^+	Dan^+	Dan^+	Dan^+, Dan^-
$\mathfrak{B}$	Dan^+	Dan^+	Dan^+, Dan^-	Dan^+, Dan^-
$\mathfrak{C}$	Dan^+	Dan^+	Dan^-	Dan^+, Dan^-
$\mathfrak{D}$	Dan^+	Dan^+	$\emptyset$	Dan^+, Dan^-
$\mathfrak{E}$	Dan^+, Dan^-	Dan^+	Dan^+	Dan^+, Dan^-
$\mathfrak{F}$	Dan^+, Dan^-	Dan^+	Dan^+, Dan^-	Dan^+, Dan^-
$\mathfrak{G}$	Dan^+, Dan^-	Dan^+	Dan^-	Dan^+, Dan^-
$\mathfrak{H}$	Dan^+, Dan^-	Dan^+	$\emptyset$	Dan^+, Dan^-
$\mathfrak{I}$	Dan^+	Dan^+, Dan^-	Dan^+	Dan^+, Dan^-
$\mathfrak{J}$	Dan^+	Dan^+, Dan^-	Dan^+, Dan^-	Dan^+, Dan^-
$\mathfrak{K}$	Dan^+	Dan^+, Dan^-	Dan^-	Dan^+, Dan^-
$\mathfrak{L}$	Dan^+	Dan^+, Dan^-	$\emptyset$	Dan^+, Dan^-
$\mathfrak{M}$	Dan^+, Dan^-	Dan^+, Dan^-	Dan^+	Dan^+, Dan^-
$\mathfrak{N}$	Dan^+, Dan^-	Dan^+, Dan^-	Dan^+, Dan^-	Dan^+, Dan^-
$\mathfrak{O}$	Dan^+, Dan^-	Dan^+, Dan^-	Dan^-	Dan^+, Dan^-
$\mathfrak{P}$	Dan^+, Dan^-	Dan^+, Dan^-	$\emptyset$	Dan^+, Dan^-
$\mathfrak{Q}$	Dan^+	Dan^-	Dan^+	Dan^+, Dan^-
$\mathfrak{R}$	Dan^+	Dan^-	Dan^+, Dan^-	Dan^+, Dan^-
$\mathfrak{S}$	Dan^+	$\emptyset$	Dan^+	Dan^+, Dan^-
$\mathfrak{T}$	Dan^+	$\emptyset$	Dan^+, Dan^-	Dan^+, Dan^-

The table above shows all types of models of Σ and the only $\supseteq!?\Delta$-minimal model is $\mathfrak{Q}$, therefore $\Sigma \vDash^{\supseteq!?\Delta}_{FDE} G(\text{Dan})$.

Remark 60. It is worth mentioning that the results presented in this article can be used to obtain paraconsistent circumscription or paracomplete circumscription, since we have the following equivalences [7]:

- $\Sigma \vDash_{LP} H$ iff $EM^{pos}, \Sigma^{pos} \vDash H^{pos}$
- $\Sigma \vDash_{K_3} H$ iff $EM^{neg}, \Sigma^{pos} \vDash H^{pos}$

where $\vDash_{LP}$ is the consequence relation of LP and $\vDash_{K_3}$ the consequence relation of the strong Kleene logic.

In a comment on an earlier version of this paper it was worried that in the paracomplistent circumscription framework minimizing incompleteness seems to be fully effective for any premise set, i.e., for any premise set Σ, excluded middle is always valid. This would have posed the question why one should bother minimizing incompleteness after all. But a closer look at Example 59 shows that this can't be the case. Take for example model $\mathfrak{D}$ which is minimal with respect to inconsistencies but excluded middle doesn't hold for $G(x)$.

6 Final Remarks

The present article focused on the combination of two logical reasoning systems - minimal FDE and $\supseteq$-circumscription. As it was shown it is possible to combine both systems in a straightforward way to obtain a new reasoning system one may call paracomplistent circumscription. This system is able to deal with glutty and gappy formulas as well as it allows to reject once drawn conclusions in light of new information and satisfies reassurance and classical recapture. Paracomplistent circumscription is not able to withdraw inconsistent sentences. This means if a set of formulas contains H and $\neg H$ then those sentences will stay inconsistent. In the proposed setting it is not the job of the reasoning system to get rid of those inconsistencies. This has to be done outside of the logic.

Minimal FDE, as described here is only one way of minimizing within the setting of FDE. There are other minimizing methods that are described in [2] for example. There, the minimizing is done in a propositional setting but it is natural to ask if minimizing after the amount of information or the truth in the usual four-valued partial order a) can be done for the First-order case, b) satisfies reassurance and c) can be combined with circumscription as in this paper.

Another natural question would be that of a proof theory. = - circumscription can be done proof-theoretically with the so called circumscription axiom, which is a second order formula. But it is not clear at all if $\supseteq$-circumscription has a corresponding axiom. If so, it might be possible to give a proof theoretic account of paracomplistent circumscription in a positively translated language.

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Appendix A - Crabbé's minimal FDE and Paracomplistent Circumscription revisited

In this appendix we first want to revisit Crabbé's original version of minimal FDE, complete his proposition 2.2 based on the suggestion of the referee of an earlier version of this article and then, based on this, define another version of paracomplistent circumscription. On top of that, we address another referee's suggestion and present a further modify paracomplistent circumscription.

We start in defining minimal FDE according to Crabbé.

Definition 61. Let $\mathfrak{A}$ be an FDE-model of some set of wff Σ. We define the inconsistent and incomplete part $\mathfrak{A}!?$ of $\mathfrak{A}$ in the following way:

$$\mathfrak{A}!? = \{\langle P, (o_1, \ldots, o_n)\rangle | (o_1, \ldots, o_n) \in P^+_{\mathfrak{A}} \cap P^-_{\mathfrak{A}} \text{ or } (o_1, \ldots, o_n) \notin P^+_{\mathfrak{A}} \cup P^-_{\mathfrak{A}}\}$$

Definition 62. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two FDE-models of a set of wff Σ. A preorder $\prec_{\supseteq !?}$ on the set of all FDE-models of a set of wff Σ is defined as $\mathfrak{A} \prec_{\supseteq !?} \mathfrak{B}$ iff

1. $|\mathfrak{A}| \supseteq |\mathfrak{B}|$
2. $\mathfrak{A}!? \subseteq \mathfrak{B}!?$

A model $\mathfrak{A}$ of Σ is called $\supseteq !?$-minimal iff for all models $\mathfrak{B}$ of Σ if $\mathfrak{B} \prec_{\supseteq !?} \mathfrak{A}$, then $\mathfrak{A}!? \subseteq \mathfrak{B}!?$. Furthermore,

$$\Sigma \models^{\supseteq !?}_{FDE} H$$
$$\text{iff}$$
$$\text{all } \supseteq !?\text{-minimal models of } \Sigma \text{ are also models of } H$$

Remark 63. In Proposition 2.2 in [7], Crabbé proved $\Sigma \models^{\supseteq !?}_{FDE} H$ iff $\Sigma^{pos} \models_{\mathcal{F}} H^{pos}$, from the fact that $\mathfrak{A}^{pos} \Subset_{\mathcal{F}} \mathfrak{B}^{pos}$ iff $\mathfrak{A}!? \subseteq \mathfrak{B}!?$. However, it turned out that his proof is incomplete. He actually just proved Proposition 64. In the following we'll complete his proof based on the suggestions of an anonymous referee of an earlier version of this paper.

Proposition 64. Let $\mathcal{F} = CN^{pos} \cup CN^{neg}$, then

1. if $\mathfrak{A}!? \subseteq \mathfrak{B}!?$ then $\ker_{\mathcal{F}}(\mathfrak{A}^{pos}) \subseteq \ker_{\mathcal{F}}(\mathfrak{B}^{pos})$
2. if $\mathfrak{A}^{pos} \Subset_{\mathcal{F}} \mathfrak{B}^{pos}$ then $\mathfrak{A}!? \subseteq \mathfrak{B}!?$

Proof. See [7] and the proof of Proposition 2.2. □

Lemma 65. Let $\mathfrak{A}$ be an FDE-model of a set of sentences Σ and $\mathcal{F} = CN^{pos} \cup CN^{neg}$. Then, there exists a model $\mathfrak{B}$ of Σ such that:

1. $\mathfrak{B}!? = \{\langle P, (o_1, \ldots, o_n)\rangle | (o_1, \ldots, o_n) \in P^+_{\mathfrak{A}} \cap P^-_{\mathfrak{A}}\}$
2. $|\mathfrak{B}| = |\mathfrak{A}|$
3. $\mathfrak{B} \prec_{\supseteq !?} \mathfrak{A}$
4. $\mathfrak{B}^{pos} \prec_{\mathcal{F}} \mathfrak{A}^{pos}$

Proof. We define $\mathfrak{B}$ exactly as $\mathfrak{A}$ except for each P we set $P^+_{\mathfrak{B}} = P^+_{\mathfrak{A}} \cup \{(o_1, \ldots, o_n) \in |\mathfrak{A}|^n | (o_1, \ldots, o_n) \notin P^+_{\mathfrak{A}} \cap P^-_{\mathfrak{A}}\}$.
1.-3. then follow directly from the definition of $\mathfrak{B}$. As for 4., we need to show that $\mathfrak{B}^{pos} \Subset_{\mathcal{F}} \mathfrak{A}^{pos}$ and $|\mathfrak{B}^{pos}| \supseteq |\mathfrak{A}^{pos}|$. The latter follows immediately because of $|\mathfrak{B}^{pos}| = |\mathfrak{B}| = |\mathfrak{A}| = |\mathfrak{A}^{pos}|$.
Now, by Proposition 64 we have $\ker_{\mathcal{F}}(\mathfrak{B}^{pos}) \subseteq \ker_{\mathcal{F}}(\mathfrak{A}^{pos})$. Suppose for some $H \in \mathcal{F}$ and some valuation v to $\ker_{\mathcal{F}}(\mathfrak{B}^{pos})$, $(\mathfrak{B}^{pos}, v) \vDash H$. We have to distinguish two cases: 1. $H = P(x_1, \ldots, x_n) \wedge \neg\neg\overline{P}(x_1, \ldots, x_n)$ and 2. $H = \neg P(x_1, \ldots, x_n) \wedge \neg\overline{P}(x_1, \ldots, x_n)$.
In case 1, $(v(x_1), \ldots, v(x_n)) \in P^+_{\mathfrak{B}} \cap P^-_{\mathfrak{B}}$ and by the construction of $\mathfrak{B}$ also $(v(x_1), \ldots, v(x_n)) \in P^+_{\mathfrak{A}} \cap P^-_{\mathfrak{A}}$ and therefore $(\mathfrak{A}^{pos}, v) \vDash H$. Assume for contradiction 2., $(v(x_1), \ldots, v(x_n)) \notin P^+_{\mathfrak{B}} \cup P^-_{\mathfrak{B}}$, but we also have $(v(x_1), \ldots, v(x_n)) \notin P^+_{\mathfrak{A}} \cup P^-_{\mathfrak{A}}$. But this means by the construction of $\mathfrak{B}$, $(v(x_1), \ldots, v(x_n)) \in P^-_{\mathfrak{A}}$, which is a contradiction.
Altogether we have shown $\mathfrak{B}^{pos} \Subset_{\mathcal{F}} \mathfrak{A}^{pos}$.

We still need to show that $\mathfrak{B}$ is a model of Σ. This can be done via induction by showing that if $\mathfrak{A} \vDash^{\pm} H$ then $\mathfrak{B} \vDash^{\pm} H$. The proof is left to the reader. □

Remark 66. We are now in the position to proof the equivalence from Proposition 2.2 in [7].

Proposition 67. Let $\Sigma \cup H$ be a set of sentences and let $\mathcal{F} = CN^{pos} \cup CN^{neg}$, then $\Sigma \vDash^{\supseteq !?}_{FDE} H$ iff $\Sigma^{pos} \vDash_{\mathcal{F}} H^{pos}$.

Proof. We show that an FDE-model $\mathfrak{A}$ is a minimal model of Σ iff $\mathfrak{A}^{pos}$ is an $\mathcal{F}$-minimal model of Σ^{pos}.

Assume $\mathfrak{A}^{pos}$ is not an $\mathcal{F}$-minimal model of Σ^{pos}. But this means there is a model $\mathfrak{B}^{pos}$ of Σ^{pos} for which

1. $\mathfrak{B}^{pos} \prec_{\mathcal{F}} \mathfrak{A}^{pos}$
2. $\mathfrak{A}^{pos} \not\Subset_{\mathcal{F}} \mathfrak{B}^{pos}$

We now show that $\mathfrak{B} \prec_{\supseteq !?} \mathfrak{A}$ and $\mathfrak{A}!? \not\subseteq \mathfrak{B}!?$, which shows that $\mathfrak{A}$ is not minimal.
By 1. we have $\mathfrak{B}^{pos} \Subset_{\mathcal{F}} \mathfrak{A}^{pos}$ and $|\mathfrak{B}^{pos}| \supseteq |\mathfrak{A}^{pos}|$ and by Proposition 64 we have $\mathfrak{B}!? \subseteq \mathfrak{A}!?$. And because of $|\mathfrak{B}| = |\mathfrak{B}^{pos}|$ and $|\mathfrak{A}| = |\mathfrak{A}^{pos}|$ we have $|\mathfrak{B}| \supseteq |\mathfrak{A}|$ and therefore $\mathfrak{B} \prec_{\supseteq !?} \mathfrak{A}$.
We still have to show that $\mathfrak{A}!? \not\subseteq \mathfrak{B}!?$. By 2. either i) $\ker_{\mathcal{F}}(\mathfrak{A}^{pos}) \not\subseteq \ker_{\mathcal{F}}(\mathfrak{B}^{pos})$ or ii) there are $H \in \mathcal{F}$ and some valuation v to $\ker_{\mathcal{F}}(\mathfrak{A}^{pos})$ for which $(\mathfrak{A}^{pos}, v) \models H$ while $(\mathfrak{B}^{pos}, v) \not\models H$.
Suppose i). By Proposition 64 we have $\mathfrak{A}!? \not\subseteq \mathfrak{B}!?$.
Suppose ii). We have to distinguish two cases. $H = P(x_1, \ldots, x_n) \wedge \neg\neg\overline{P}(x_1, \ldots, x_n)$ or $H = \neg P(x_1, \ldots, x_n) \wedge \neg\overline{P}(x_1, \ldots, x_n)$.
Suppose first $H = P(x_1, \ldots, x_n) \wedge \neg\neg\overline{P}(x_1, \ldots, x_n)$. Then we have $(v(x_1), \ldots, v(x_n)) \in P^+_{\mathfrak{A}} \cap P^-_{\mathfrak{A}}$ and therefore $\langle P, (v(x_1), \ldots, v(x_n))\rangle \in \mathfrak{A}!?$, while $(v(x_1), \ldots, v(x_n)) \notin P^+_{\mathfrak{B}} \cap P^-_{\mathfrak{B}}$. Assume now for contradiction that $\langle P, (v(x_1), \ldots, v(x_n))\rangle \in \mathfrak{B}!?$. Then $(v(x_1), \ldots, v(x_n)) \notin P^+_{\mathfrak{B}} \cup P^-_{\mathfrak{B}}$ and hence $(\mathfrak{B}^{pos}, v) \models \neg P(x_1, \ldots, x_n) \wedge \neg\overline{P}(x_1, \ldots, x_n)$. But by 1. we have then $(\mathfrak{A}^{pos}, v) \models P(x_1, \ldots, x_n) \wedge \neg\overline{P}(x_1, \ldots, x_n)$ and therefore $(v(x_1), \ldots, v(x_n)) \notin P^+_{\mathfrak{A}} \cup P^-_{\mathfrak{A}}$ which is contradicting $(v(x_1), \ldots, v(x_n)) \in P^+_{\mathfrak{A}} \cap P^-_{\mathfrak{A}}$. Therefore $\mathfrak{A}!? \not\subseteq \mathfrak{B}!?$.
The case $H = \neg P(x_1, \ldots, x_n) \wedge \neg\overline{P}(x_1, \ldots, x_n)$ is analogously and left to the reader.

Suppose $\mathfrak{A}$ is not a minimal FDE-model of Σ. We distinguish two cases.

1. $\{\langle P, (o_1, \ldots, o_n)\rangle \in \mathfrak{A}!? | (o_1, \ldots, o_n) \notin P^+_{\mathfrak{A}} \cup P^-_{\mathfrak{A}}\} \neq \emptyset$
2. $\{\langle P, (o_1, \ldots, o_n)\rangle \in \mathfrak{A}!? | (o_1, \ldots, o_n) \notin P^+_{\mathfrak{A}} \cup P^-_{\mathfrak{A}}\} = \emptyset$

Suppose 1. By Lemma 65 there is a Σ-model $\mathfrak{C}$ for which a) $\mathfrak{C}^{pos} \prec_{\mathcal{F}} \mathfrak{A}^{pos}$ and b) $\mathfrak{C}!? = \{\langle P, (o_1, \ldots, o_n)\rangle \in \mathfrak{A}!? | (o_1, \ldots, o_n) \in P^+_{\mathfrak{A}} \cap P^-_{\mathfrak{A}}\}$. Now, take any $\langle P, (o_1, \ldots, o_n)\rangle \in \mathfrak{A}!?$ such that $(o_1, \ldots, o_n) \notin P^+_{\mathfrak{A}} \cup P^-_{\mathfrak{A}}$. By b) $\langle P, (o_1, \ldots, o_n)\rangle \notin \mathfrak{C}!?$. Since also $(\mathfrak{A}^{pos}, v) \models \neg P(x_1, \ldots, x_n) \wedge \neg\overline{P}(x_1, \ldots, x_n)$ where $v(x_i) = o_i$ (for $1 \leq i \leq n$), while $(\mathfrak{C}^{pos}, v) \not\models \neg P(x_1, \ldots, x_n) \wedge \neg\overline{P}(x_1, \ldots, x_n)$, we have $\mathfrak{A}^{pos} \not\Subset_{\mathcal{F}} \mathfrak{C}^{pos}$. By a) and c) $\mathfrak{A}^{pos}$ is not an $\mathcal{F}$-minimal model of Σ^{pos}.
Suppose 2. We know there is a Σ-model $\mathfrak{B}$ for which $\mathfrak{B} \prec_{\supseteq !?} \mathfrak{A}$ and $\mathfrak{A}!? \not\subseteq \mathfrak{B}!?$. Thus, $\mathfrak{B}!? \subset \mathfrak{A}!?$. By Lemma 65, there is a Σ-model $\mathfrak{C}$ for which $\mathfrak{C}^{pos} \prec_{\mathcal{F}} \mathfrak{B}^{pos}$, $\mathfrak{C} \prec_{\supseteq !?} \mathfrak{B}$, $|\mathfrak{C}| = |\mathfrak{B}|$ and $\mathfrak{C}!? = \{\langle P, (o_1, \ldots, o_n)\rangle \in \mathfrak{B}!? | (o_1, \ldots, o_n) \in P^+_{\mathfrak{B}} \cap P^-_{\mathfrak{B}}\}$. We now show that $\mathfrak{C}^{pos} \prec_{\mathcal{F}} \mathfrak{A}^{pos}$ and $\mathfrak{A}^{pos} \not\Subset_{\mathcal{F}}$

$\mathfrak{C}^{pos}$.
Since $\mathfrak{C}!? \subseteq \mathfrak{B}!? \subset \mathfrak{A}!?$, we also have $\mathfrak{C}!? \subset \mathfrak{A}!?$ and by Proposition 64, $\ker_{\mathcal{F}}(\mathfrak{C}^{pos}) \subseteq \ker_{\mathcal{F}}(\mathfrak{A}^{pos})$. Now, suppose $(\mathfrak{C}^{pos}, v) \vDash H$ for some $H \in \mathcal{F}$ and some v to $\ker_{\mathcal{F}}(\mathfrak{C}^{pos})$. Since there are no gaps in $\mathfrak{C}^{pos}$, H is of the form $P(x_1, \ldots, x_n) \wedge \neg\neg\overline{P}(x_1, \ldots, x_n)$. Therefore $\langle P, (v(x_1), \ldots, v(x_n))\rangle \subseteq \mathfrak{C}!? \subseteq \mathfrak{A}!?$, which means $(\mathfrak{A}^{pos}, v) \vDash H$, remember $\mathfrak{A}$. Since we also have $|\mathfrak{C}| = |\mathfrak{B}| \supseteq |\mathfrak{A}|$ it follows that $\mathfrak{C}^{pos} \prec_{\mathcal{F}} \mathfrak{A}^{pos}$. Now, let $\langle P, (o_1, \ldots, o_n)\rangle \in \mathfrak{A}!? \backslash \mathfrak{C}!?$. Hence, $(\mathfrak{A}^{pos}, v) \vDash P(x_1, \ldots, x_n) \wedge \neg\neg\overline{P}(x_1, \ldots, x_n)$, while
$(\mathfrak{C}^{pos}, v) \vDash P(x_1, \ldots, x_n) \wedge \neg\neg\overline{P}(x_1, \ldots, x_n)$ and therefore $\mathfrak{A}^{pos} \not\preccurlyeq_{\mathcal{F}} \mathfrak{C}^{pos}$. □

Remark 68. In an earlier version of this paper we used Definition 62 to construct paracomplistent circumscription. However, we changed our definition to the current one in order to shorten proofs and to not further complicate the combination of $\supseteq$-circumscription and minimal FDE. Nevertheless it is possible to define paracomplistent circumscription based on Definition 62. In the following we will give this definition but omit the proofs of reassurance and recapture.

Definition 69. Let Δ be a set of open atomic formulas and Σ be a set of wff. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two FDE-models of Σ. A preorder $\prec_{\supseteq!?\Delta}$ is defined as $\mathfrak{A} \prec_{\supseteq!?\Delta} \mathfrak{B}$ iff

1. $|\mathfrak{A}| \supseteq |\mathfrak{B}|$
2. $\mathfrak{A}!? \subseteq \mathfrak{B}!?$ and $\Delta^+_{\mathfrak{A}} \subseteq \Delta^+_{\mathfrak{B}}$

An FDE-model $\mathfrak{A}$ of Σ is called $\supseteq!?\Delta$-minimal iff for all models $\mathfrak{B}$ of Σ if $\mathfrak{B} \prec_{\supseteq!?\Delta} \mathfrak{A}$, then $\Delta^+_{\mathfrak{A}} \subseteq \Delta^+_{\mathfrak{B}}$ and $\mathfrak{A}! \subseteq \mathfrak{B}!$ and $\mathfrak{A}? \subseteq \mathfrak{B}?$. Furthermore,

$$\Sigma \vDash^{\supseteq!?\Delta}_{FDE} H$$
iff
all $\supseteq!?\Delta$-minimal models of Σ are also models of H

Remark 70. In Lemma 57 it was shown that recapture only holds for FOL consistent and normal contexts. As a referee of an earlier version of this article suggested it is possible to weaken this restriction by minimally modifying Definition 69. In the modified definition the minimization of gluts and gaps is prioritized over the minimization of abnormalities.

Even though it is worthwhile to investigate the following definition, since it seems to be case that Crabbé's results can be applied to this relation as well, we will leave this topic for future research.

Definition 71. $\mathfrak{A} \prec_{FDE}^{prio} \mathfrak{B}$ iff

1. $|\mathfrak{A}| \supseteq |\mathfrak{B}|$
2. $\mathfrak{A}!? \subset \mathfrak{B}!?$ or ($\mathfrak{A}!? = \mathfrak{B}!?$ and $\Delta_{\mathfrak{A}}^{+} \subseteq \Delta_{\mathfrak{B}}^{+}$)

Appendix B - Lin's *Paraconsistent Circumscription*

In 1995 ([12], [11]) Lin proposed a paraconsistent version of circumscription with LP as base logic. In what follows we will shortly describe his approach and stress some of its problematic aspects. The interesting point of his approach is that Lin also minimizes inconsistencies and abnormalities in one step. But the main difference between Lin and the approach of this article is that he doesn't distinguish between the set of abnormal objects and the set of inconsistent/ incomplete objects which leads to some unwanted properties.

Lin's approach is basically a plain combination of circumscription and minimal LP. He starts with an LP model of a set of wff Σ and minimizes a subset of the relation symbols contained in Σ after inconsistency and truth. In order to analyse this approach, we need to introduce his way of defining a glutty FOL interpretation. For every $\mathfrak{A}$ and $v_{\mathfrak{A}}$ one defines a function $\pi_{v_{\mathfrak{A}}}$ from the set of all wff to $\{0, 1/2, 1\}$ in the following way. Let H be a wff:

$$\begin{array}{lll} \pi_{v_{\mathfrak{A}}}(H) = 1 & \text{iff} & (\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^{+} H \text{ and } (\mathfrak{A}, v_{\mathfrak{A}}) \nVdash^{-} H \\ \pi_{v_{\mathfrak{A}}}(H) = 1/2 & \text{iff} & (\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^{+} H \text{ and } (\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^{-} H \\ \pi_{v_{\mathfrak{A}}}(H) = 0 & \text{iff} & (\mathfrak{A}, v_{\mathfrak{A}}) \nVdash^{+} H \text{ and } (\mathfrak{A}, v_{\mathfrak{A}}) \Vdash^{-} H \end{array}$$

As one can easily check, this interpretation gives us the LP truth-tables for $\neg, \wedge, \vee, \rightarrow$ and finally we have $\Sigma \vDash_{LP} H$, iff if $\pi_{v_{\mathfrak{A}}}(G) \in \{1, 1/2\}$ for all $G \in \Sigma$, then $\pi_{v_{\mathfrak{A}}}(H) \in \{1, 1/2\}$ for all $\pi_{v_{\mathfrak{A}}}$. Note that we don't have truth-value gaps, i.e., for every predicate the union of the extension and anti-extension exhaust the entire universe.

Lin defines his minimal semantics as follows (see Definition 10 in [12]): Let Σ be a set of wff, Δ be a set of open atomic formulas and let $\mathfrak{A}$, $\mathfrak{A}'$ be two LP-models of Σ. He then defines $\mathfrak{A}' \prec_{Lin} \mathfrak{A}$, iff

1. $|\mathfrak{A}| = |\mathfrak{A}'|$

2. for every $P(x_1, \ldots, x_n) \in \Delta$,
   if $\pi_{v_{\mathfrak{A}'}}(P(x_1, \ldots, x_n)) = 1$ then $\pi_{v_{\mathfrak{A}}}(P(x_1, \ldots, x_n)) = 1$;
   and if $\pi_{v_{\mathfrak{A}'}}(P(x_1, \ldots, x_n)) = 1/2$ then $\pi_{v_{\mathfrak{A}}}(P(x_1, \ldots, x_n)) = 1/2$, and

3. there is at least one $Q(x_1, \ldots, x_m) \in \Delta$ such that $\pi_{v_{\mathfrak{A}}}(Q(x_1, \ldots, x_m)) = 1$ but $\pi_{v_{\mathfrak{A}'}}(Q(x_1, \ldots, x_m)) \neq 1$; or there is at least one $Q \in \Delta$ such that $\pi_{v_{\mathfrak{A}}}(Q(x_1, \ldots, x_m)) = 1/2$ but $\pi_{v_{\mathfrak{A}'}}(Q(x_1, \ldots, x_m)) \neq 1/2$.

A model $\mathfrak{A}$ of Σ is said to be $\prec_{Lin}$-minimal iff there is no model $\mathfrak{A}'$ of Σ such that $\mathfrak{A}' \prec_{Lin} \mathfrak{A}$. The semantic entailment of paraconsistent circumscription LPc can then be defined as usual: Let Σ be a set of wff and H a wff, then $\Sigma \vDash_{LPc} H$ iff all $\prec_{Lin}$-minimal models of Σ are also models of H.

Lin now claims that his approach is a good account of reasoning in inconsistent and defeasible environments. But, as can be easily shown, his approach might lead to a failure of modus ponens and even the example he used doesn't work as Lin wants it to.

Let us start with an example which is structurally similar to one given by Lin. The failure of this example is due to the persistence of the values $1/2$ and 1 from smaller to greater models defined above.

Let $\Sigma = \{\forall x((P(x) \wedge \neg A(x)) \rightarrow G(x)), P(\mathrm{Dan}), Ph(\mathrm{Dan}) \wedge \neg Ph(\mathrm{Dan})\}$ with $\Delta = \{P, A\}$

Now, let $\mathfrak{A}$ be a model which assigns the value $1/2$ to all formulas under consideration. Then $\mathfrak{A}'$ (cf. table below) is minimal.

Model	$P(\mathrm{Dan})$	$A(\mathrm{Dan})$	$Ph(\mathrm{Dan})$	$G(\mathrm{Dan})$
$\mathfrak{A}$	$1/2$	$1/2$	$1/2$	$1/2$
$\mathfrak{A}'$	$1/2$	0	$1/2$	0

Therefore, $G(\mathrm{Dan})$ can not be inferred in every minimal model, as intended, even though the inconsistent formula does not even occur in the conditional. This shows that the persistence of the values $1/2$ and 1 can lead to unintended minimal models. The persistence of these values is due to item 2 of the definition of minimal models above, to be exact the word "and" is problematic. It is easy to check that the approach of the present article gives the intended minimal models.

As an anonymous referee pointed out Lin's definition of minimality might have been rushed and not check-read thoroughly and may be replaced by a more charitable reading:

2.' for every $P \in \Delta$,
if $\pi_{v_{\mathfrak{A}'}}(P(x_1, \ldots, x_n)) = 1$ then $\pi_{v_{\mathfrak{A}}}(P(x_1, \ldots, x_n)) = 1$
or $\pi_{v_{\mathfrak{A}}}(P(x_1, \ldots, x_n)) = 1/2$;
and if $\pi_{v_{\mathfrak{A}'}}(P(x_1, \ldots, x_n)) = 1/2$ then $\pi_{v_{\mathfrak{A}}}(P(x_1, \ldots, x_n)) = 1/2$

This adjusted definition seems to solve the problem of persistence at least in the example above, but the other issues we will point out seem to remain since modus ponens still might fail and because of the identity $|\mathfrak{A}| = |\mathfrak{A}'|$ we have some doubts that reassurance is satisfied.

Another issue of Lin's approach is that instead of minimizing all inconsistency in a model, he minimizes only a subset of the premise set Σ after inconsistency. But this means that every predicate $P(x)$ not contained in Δ can be evaluated with $1/2$ even though it is not necessarily inconsistent. On the other hand, one reason to construct minimal LP, as stated before, was the failure of modus monens in LP. And one would expect from a paraconsistent version of circumscription with minimal LP as core logic that the consistent part of Σ satisfies modus ponens. But since only predicates in Δ are minimized, all other predicates which may occur in a set of sentences Σ are left out. In the worst case this could lead to a failure of modus ponens. Consider for example the following $\Sigma = \{\forall x(L(x) \to A(x)),\ L(\text{Dan})\}$ with $\Delta = \{A\}$. Then, clearly, we can not infer $A(\text{Dan})$ as intended, since there are $\prec_{Lin}$-minimal models $\mathfrak{A}$ with $\pi_{v_{\mathfrak{A}}}(L(\text{Dan})) = 1/2$ and $\pi_{v_{\mathfrak{A}}}(A(\text{Dan})) = 0$. Therefore, it seems to be rather important to minimize all inconsistency in a model and not just a subset. Of course, one might argue, Lin leaves open the possibility of minimizing all predicates in $\mathcal{L}$ but this would presumably undermine the idea of circumscription.

Finally, we just note that because of the $=$-relation between the models in his definition of minimality, Lin's approach doesn't seem to satisfy reassurance and strong reassurance. To see this, one just need to apply the examples of [6] to Lin's approach. To be fair the construction of minimal models as presented in this article doesn't satisfy strong reassurance either, while reassurance is satisfied in lanaguages without equalitiy, but since Lin claimed it does satisfy both, it is a problem worth noting.

Lin's work was one of the motivations of this article. In the present article we hope we could show how to avoid some of the problematic aspects of Lin's approach when creating a paraconsistent (and paracomplete) version of circumscription.