Incompactness of the $\forall_1$ fragment of basic second order propositional relevant logic

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Abstract

In this note we provide a simple proof of the incompactness over Routley-Meyer **B**-frames of the $\forall_1$ fragment of the second order propositional relevant language. Moreover, we observe that this fragment is clearly still recursively enumerable.

Keywords: relevant logic, modal logic, propositional quantifiers, incompactness, Routley-Meyer semantics.

Introduction

We will call *basic second order propositional relevant logic* to the result of adding propositional quantifiers $\forall p$ and $\exists p$ to the standard language of relevant logic when interpreted over Routley-Meyer structures for the system **B** (cf. [7]).[1]

The idea of the present note is to show how the incompactness argument for the system KW from [3] (pp. 178-179)[2] can be adapted to establish the incompactness of the $\forall_1$ fragment (where all quantifiers are universal and

[1] In the literature, Routley-Meyer structures for the system **R** have received the most attention in this context (see, for example, [4]).

[2] Which gives virtually immediately the incompactness of second order propositional modal logic.

come at the front)[3] of the second order propositional relevant language over Routley-Meyer **B**-models.

Recall that a logic can be said to be *compact* if for any collection of sentences Γ, Γ does not have a model only if some finite $\Gamma' \subseteq \Gamma$ does not have a model either. This terminology comes from Rasiowa in [6], with the obvious topological reference. Hence, by *incompactness* we mean the failure of compactness.

So, in this article, we establish that there is a set of $\forall_1$ second order propositional relevant formulas that is finitely satifiable on the class of Routley-Meyer **B**-models even though it is not satisfiable on this class. Incidentally, this also provides an example in relevant logic of a system (with a language more expressive than the standard relevant language) which is recursively axiomatizable but incompact. Observe that the obvious expressive extensions of standard propositional relevant languages which could be hoped to be incompact, namely, infinitary extensions, cannot be recursively axiomatized since, for starters, their syntax is not arithmetizable.

We start by reviewing the Routley-Meyer semantics in §1 and establish the recursive axiomatizability of the $\forall_1$ fragment of basic second order propositional relevant logic. In §2, we prove the main theorem of the paper and a lemma also implying that the $\forall_1$ fragment of the second order propositional relevant language is more expressive than the standard propositional relevant language. Finally, in §3 we summarize the work.

1 Preliminaries

The *second order propositional relevant language* L_2 will contain a countable list `PROP` of propositional variables $p, q, r \ldots$ and the logical symbols: $\sim$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\circ$ (fusion), $\rightarrow$ (implication), $\mathbf{t}$ (the Ackermann constant), $\forall p, \exists p$ (propositional quantifiers). Formulas are constructed in the usual way:

$$\phi ::= p \mid \mathbf{t} \mid {\sim}\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \rightarrow \psi \mid\mid \phi \circ \psi \mid \forall p.\phi \mid \exists p.\phi,$$

for $p \in$ `PROP`. The $\forall_1$ fragment of L_2 will contain all formulas equivalent to formulas of the form $\forall p_0, \ldots p_n.\phi$, where ϕ is quantifierless.

[3] In classical second order logic, restriction to second order universal quantifications also suffices for incompactness: the property of well-foundedness is Π^1_1.

In this paper, a Routley-Meyer *frame* for

$$L_2$$

is a structure $\mathfrak{F} = \langle W, R, *, O \rangle$, where W is a non-empty set, $\varnothing \neq O \subseteq W$, $*$ is an operation $* : W \longrightarrow W$, and $R \subseteq W \times W \times W$ satisfies p1-p5 below. In the standard way, we will abbreviate $\exists z(Oz \wedge Rzxy)$ by $x \leqslant y$.

p1. $x \leqslant x$

p2. If $x \leqslant y$ and $Ryzv$ then $Rxzv$.

p3. If $x \leqslant y$ and $Rzyv$ then $Rzxv$.

p4. If $x \leqslant y$ and $Rzvx$ then $Rzvy$.

p5. If $x \leqslant y$ then $y^* \leqslant x^*$.

p6. If $x \leqslant y$ and $x \in O$ then $y \in O$.

The relation $\leqslant$ is a preorder. We can see this as follows. By p1, we have reflexivity. Now if $x \leqslant y$ (i.e., $\exists z(Oz \wedge Rzxy)$) and $y \leqslant z$ (i.e., $\exists v(Ov \wedge Rzyz)$), by p3, we have that $\exists v(Ov \wedge Rzxz)$, i.e., $x \leqslant z$.

A Routley-Meyer *model* for L_2 is a pair $\langle \mathfrak{F}, V \rangle$, where $V : \texttt{PROP} \longrightarrow \wp(W)$ is a function such that for any $p \in \texttt{PROP}$, $V(p)$ is upward closed under the $\leqslant$ relation, that is, $x \in V(p)$ and $x \leqslant y$ implies that $y \in V(p)$. We define satisfaction at w in M recursively as follows:

$M, w \Vdash \mathbf{t}$	iff	$w \in O$,
$M, w \Vdash p$	iff	$w \in V(p)$,
$M, w \Vdash {\sim}\phi$	iff	$M, w^* \nVdash \phi$,
$M, w \Vdash \phi \wedge \psi$	iff	$M, w \Vdash \phi$ and $M, w \Vdash \psi$,
$M, w \Vdash \phi \vee \psi$	iff	$M, w \Vdash \phi$ or $M, w \Vdash \psi$,
$M, w \Vdash \phi \rightarrow \psi$	iff	for every a, b such that $R^M wab$, if $M, a \Vdash \phi$ then $M, b \Vdash \psi$,
$M, w \Vdash \phi \circ \psi$	iff	there are a, b such that $R^M abw$, $M, a \Vdash \phi$ and $M, b \Vdash \psi$.
$M, w \Vdash \forall p.\phi$	iff	for all upwards closed subsets S of W, $M[p \mapsto S], w \Vdash \phi$, where $M[p \mapsto S]$ is the model identical to M except that is the model identical to M except that $V(p) = S$.
$M, w \Vdash \exists p.\phi$	iff	there is an upwards closed subset S of W such that $M[p \mapsto S], w \Vdash \phi$.

A formula ϕ is said to be *true* in a model M if $M, w \Vdash \phi$ for all $w \in O$. ϕ is said to be *valid* in a frame $\mathfrak{F}$ (in symbols $\mathfrak{F} \Vdash \phi$) if ϕ is true in any model M based on $\mathfrak{F}$.

Lemma 1. (Hereditary Lemma) *For any second order relevant formula ϕ, model M based on a Routley-Meyer frame, and worlds x, y of M, $x \leqslant y$ implies that $M, x \Vdash \phi$ only if $M, y \Vdash \phi$.*

Proof. By induction on formula complexity. □

Consider a monadic second order language that comes with one function symbol $*$, a constant T, a distinguished three place relation symbol R, and a unary predicate variable P for each $p \in \mathtt{PROP}$. Following the tradition in modal logic, we might call this a *correspondence language* L_2^{corr} for L. Now we can read a model M as a model for L_2^{corr} in a straightforward way: W is taken as the domain of the structure, V specifies the denotation of each of the predicates $P, Q, \ldots$, the collection O is the object assigned to the predicate O, while $*$ is the denotation of the function symbol $*$ of $\mathscr{L}_2^{corr}$ and R the denotation of the relation R of $\mathscr{L}_2^{corr}$.

Where t is a term in the correspondence language, we write $\phi^{t/x}$ for the result of replacing x with t everywhere in the formula ϕ. Let us abreviate the formula of L_2^{corr} which expresses that the value of a given predicate P is upwards closed under $\leqslant$ by $Up_{\leqslant}(P)$. As expected, it is easy to specify a translation from the formulas of the relevant language into formulas of first order logic with *one free variable* as follows:

$$
\begin{array}{rcl}
T_x(\mathbf{t}) & = & Ox \\
T_x(p) & = & Px \\
T_x(\sim\phi) & = & \neg T_x(\phi)^{x^*/x} \\
T_x(\phi \wedge \psi) & = & T_x(\phi) \wedge T_x(\psi) \\
T_x(\phi \vee \psi) & = & T_x(\phi) \vee T_x(\psi) \\
T_x(\phi \rightarrow \psi) & = & \forall y, z(Rxyz \wedge T_x(\phi)^{y/x} \supset T_x(\psi)^{z/x}) \\
T_x(\phi \circ \psi) & = & \exists y, z(Ryzx \wedge T_x(\phi)^{y/x} \wedge T_x(\psi)^{z/x}). \\
T_x(\forall p.\phi) & = & \forall P(Up_{\leqslant}(P) \supset T_x(\phi)) \\
T_x(\exists p.\phi) & = & \exists P(Up_{\leqslant}(P) \wedge T_x(\phi))
\end{array}
$$

The symbols $\neg$ and $\supset$ represent, respectively, boolean negation in classical logic and material implication either in classical or relevant logic (which should not be confused with $\sim$ and $\rightarrow$).

Next we prove a proposition to the effect that our proposed translation is adequate. While $\Vdash$ stands for satisfaction as defined for relevant languages, $\models$ will be the usual Tarskian satisfaction relation from classical logic.

Proposition 2. *For any w, $M, w \Vdash \phi$ if and only if $M \models T_x(\phi)[w]$.*

The next result is easily established using Proposition 2.

Proposition 3. *For any relevant formula $\phi(p_1, \ldots, p_n)$, Routley-Meyer frame $\mathfrak{F}$ and world w of $\mathfrak{F}$, the following holds:*

> $\mathfrak{F} \Vdash \phi$ *iff* $\mathfrak{F} \models \forall P_1, \ldots, P_n(Up_{\leqslant}(P_1) \wedge \cdots \wedge Up_{\leqslant}(P_n) \supset \forall w(Ow \supset T_x(\phi)^{w/x}))$.

The $\forall_1$ fragment of the second order propositional relevant language is more expressive over **B**-models than the relevant language without propositional quantifiers. For the latter is less expressive than first order logic while the former can express some non-first order concepts (cf. Lemma 5 and [1]).

Proposition 4. *The set of $\forall_1$-validities of basic second order propositional relevant logic is recursively axiomatizable.*

Proof. First observe that a formula ϕ of the form $\forall p_{k+1}, \ldots, p_m\psi(p_0, \ldots, p_k)$ is an $\forall_1$-validity iff $\forall P_0, \ldots, P_k, P_{k+1}, \ldots, P_m(\bigwedge_{i\leqslant m} Up_{\leqslant}(P_i) \supset \forall w(Ow \supset T_x(\psi)^{T/x}))$ is a logical consequence (in the sense of classical logic) of σ, where σ is the conjunction of p1-p4. The symbols $P_0, \ldots, P_m$ do not occur in σ, so ϕ is an $\forall_1$-validity iff $\bigwedge_{i\leqslant m} Up_{\leqslant}(P_i) \supset \forall w(Ow \supset T_x(\psi)^{T/x})$ is a logical consequence of σ. By the recursive enumerability of the set of validities of first order logic, the collection of all $\forall_1$-validities of basic second order propositional relevant logic is also recursively enumerable.

Now we use Craig's trick (see Theorem 1 in [5]). Let $f(0), f(1), f(2), \ldots$ be one such enumeration of the $\forall_1$-validities. For any second order propositional relevant formula ϕ, define recursively $\phi^0 =_{df} \phi$ and $\phi^{n+1} =_{df} \phi^n \wedge \phi$. Then $\Delta = \{f(k)^k : k < \omega\}$ is a recursive set of second order propositional relevant formulas. Moreover, since $\phi \vdash_{\mathbf{B}} \phi^k$ and $\phi^k \vdash_{\mathbf{B}} \phi$ for any k where $\vdash_{\mathbf{B}}$ is the deducibility relation of the system **B**, we see that the collection Δ of formulas is a recursive axiomatization of the set of all $\forall_1$-validities.

□

2 Incompactness

In this section we establish the main result of the paper (Theorem 6) by paralleling an argument from [3] in the context of relevant logic.

Put $R^{\#}xy =_{df} \exists z(Rxyz \vee Rxzy)$ and $\blacksquare p =_{df} (p \vee {\sim}p \to p) \wedge ({\sim}p \to p \wedge {\sim}p)$. Observe that for any frame $\mathfrak{F}$, and world $x \in W$, there is $T \in O$ such that $R^{\#}Tx$ holds. In any frame $\mathfrak{F}$ where $\forall x(x^* \leqslant x \wedge x \leqslant x^*)$, using the Hereditary Lemma, we see that a formula of the form ${\sim}\phi \vee \psi$ behaves essentially as a material implication in a classical language at the level of models based on $\mathfrak{F}$. Also, for any valuation V in any such frame $\mathfrak{F}$, $\langle \mathfrak{F}, V\rangle, w \Vdash \blacksquare p$ iff for all x, y such that $Rwxy$, $\langle \mathfrak{F}, V\rangle, x \Vdash p$ and $\langle \mathfrak{F}, V\rangle, y \Vdash p$ iff for all x such that $R^{\#}wx$, $\langle \mathfrak{F}, V\rangle, x \Vdash p$.

Now, for any frame $\mathfrak{F}$ and valuation V on it, $\langle \mathfrak{F}, V\rangle \Vdash \forall p, q((p \wedge {\sim}p \to q) \wedge (q \to p \vee {\sim}p))$ iff $\mathfrak{F} \models \forall x(x^* \leqslant x \wedge x \leqslant x^*)$. Let $\blacklozenge =_{df} {\sim}\blacksquare{\sim}$, then on a frame $\mathfrak{F}$ where $\forall x(x^* \leqslant x \wedge x \leqslant x^*)$ holds, for any valuation V, $\langle \mathfrak{F}, V\rangle, w \Vdash \blacklozenge p$ iff there some x such that $R^{\#}wx$ and $\langle \mathfrak{F}, V\rangle, x \Vdash p$.

Lemma 5. $\langle \mathfrak{F}, V\rangle \Vdash \forall p, q((p \wedge {\sim}p \to q) \wedge (q \to p \vee {\sim}p) \wedge ((\blacksquare(\blacksquare p \supset p) \wedge p) \supset \blacksquare p))$ *iff (i)* $\mathfrak{F} \models \forall x(x^* \leqslant x \wedge x \leqslant x^*)$ *and (ii) there is* $T \in O$ *s.t. there is no infinite sequence of worlds* $s_0, s_1, s_2 \ldots$ *such that* $T = s_0 \nleqslant s_i (0<i<\omega)$ *and* $R^{\#}s_0s_1, R^{\#}s_1s_2, R^{\#}s_2s_3, \ldots$

Proof. Let $\mathfrak{F}$ be an arbitrary Routley-Meyer frame. We have that if (i) holds, $\forall p(\blacksquare(\blacksquare p \supset p) \wedge p \supset \blacksquare p)$ implies (ii). For suppose (ii) fails, then for every $T \in O$ there is an infinite sequence of worlds $T = s_0 \nleqslant s_1, s_2 \ldots$ such that $R^{\#}s_0s_1, R^{\#}s_1s_2, R^{\#}s_2s_3, \ldots$ Now take any valuation V based on $\mathfrak{F}$ such that $V(p) = \{w : w \nleqslant s_i, 0<i<\omega\}$. By transitivity of $\leqslant$, $V(p)$ is upwards closed under $\leqslant$. For each s_i, $s_i \leqslant s_i$, so $\langle \mathfrak{F}, V\rangle, s_i \nVdash p$. Hence, $\langle \mathfrak{F}, V\rangle, T \nVdash \blacksquare p$. Also, by assumption, $T \nleqslant s_i$ $(0<i<\omega)$, which mean that $\langle \mathfrak{F}, V\rangle, T \Vdash p$. Now let $R^{\#}Tv$ and suppose that $\langle \mathfrak{F}, V\rangle, v \Vdash \blacksquare p$ but $\langle \mathfrak{F}, V\rangle, v \nVdash p$. The latter means that $v \leqslant s_i$ for some $0<i<\omega$, however since $\langle \mathfrak{F}, V\rangle, s_{i+1} \nVdash p$ and $R^{\#}s_is_{i+1}$, it must be that $\langle \mathfrak{F}, V\rangle, s_i \nVdash \blacksquare p$, and by the Hereditary Lemma, $\langle \mathfrak{F}, V\rangle, v \nVdash \blacksquare p$. Hence, $\langle \mathfrak{F}, V\rangle, T \Vdash \blacksquare(\blacksquare p \supset p)$. This concludes the left to right direction of the proposition.

For the converse suppose $\langle \mathfrak{F}, V\rangle, T \nVdash \blacksquare(\blacksquare p \supset p) \wedge p \supset \blacksquare p$. If (i) holds, one can build the desired sequence to falsify (ii) by taking x such that $R^{\#}Tx$ while $\langle \mathfrak{F}, V\rangle, x \nVdash p$ and applying $\langle \mathfrak{F}, V\rangle, T \Vdash \blacksquare(\blacksquare p \supset p)$ in conjunction with the observation that $\mathfrak{F} \models \forall x(R^{\#}Tx)$.

□

Theorem 6. *The $\forall_1$ fragment of basic second order propositional relevant logic is incompact.*

Proof. This time consider the set

$$\Theta = \{\forall p, q((p \wedge {\sim}p \to q) \wedge (q \to p \vee {\sim}p) \wedge ((\blacksquare(\blacksquare p \supset p) \wedge p) \supset \blacksquare p))\} \cup \{{\sim}p_i : i < \omega\} \cup \{\blacklozenge p_0\} \cup \{\blacksquare(p_i \supset \blacklozenge p_{i+1}) : i < \omega\}.$$

First we note that this set is unsatisfiable. For if there were a model M such that Θ holds at every $T \in O$, M would contain a sequence of the sort forbidden by $\forall p((\blacksquare(\blacksquare p \supset p) \wedge p) \supset \blacksquare p)$ according to Lemma 5. To see this note that since $M, T \Vdash \blacklozenge p_0$ there is y such that $R^{\#}Ty$ and $M, y \Vdash p_0$. Obviously, $T \nleqslant y$ by the Hereditary Lemma and the fact that $M, T \Vdash {\sim}p_0$. Put $s_1 = y$. Having obtained the $n+1$ element of the chain, s_{n+1} (and guaranteeing that $M, s_n \Vdash p_n$ by construction), we get s_{n+2} as follows. Recall that for every world x, for some $T' \in O$, $R^{\#}T'x$. Since $M, s_n \Vdash p_n$ and for every $T \in O$, $M, T \Vdash \blacksquare(p_n \supset \blacklozenge p_{n+1})$, then $M, s_{n+1} \Vdash \blacklozenge p_{n+2}$, that is, there is z such that $R^{\#}s_{n+1}z$ and $M, z \Vdash p_{n+2}$. We simply let s_{n+2} be z. Again, $T \nleqslant z$ by the Hereditary Lemma and the fact that $M, T \Vdash {\sim}p_{n+2}$.

Now we show that Θ is finitely satisfiable. Suppose $\Theta_0 \subset \Theta$ is finite. For each $n{>}0$, let $\mathfrak{F}_n$ be the frame where $W_n = \{k : k \leqslant n\}$, $R_n = \{\langle 0, i, i\rangle : i \leqslant n\} \cup \{\langle j, j+1, j+1\rangle : j{<}n\}$, O has only one memeber, T, which is simply the number 0, and $*_n$ is the identity. Now let m be the biggest natural number such that $\blacksquare(p_m \supset \blacklozenge p_{m+1}) \in \Theta_0$. Then consider a valuation V on the domain of the frame $\mathfrak{F}_{m+2}$ such that $V(p_i)$ is an arbitrary upwards closed subset of W_{m+2} for $i > m+1$, while $V(p_i) = \{i+1\}$ (which is always upwards closed in $\mathfrak{F}_{m+2}$) for $i < m+2$. It is not difficult to see, using Lemma 5, that then Θ_0 is satisfied at T in $\langle \mathfrak{F}_{m+2}, V\rangle$.

□

3 Conclusion

We showed that, on the class of Routley-Meyer **B**-frames, there is a set of $\forall_1$ second order propositional relevant formulas which is finitely satisfiable but not satisfiable. This fragment of L_2 is rather powerful, indeed, for it can express some non-first order concepts. However, it is still recursively axiomatizable. As a referee points out, the problem of whether this incompactness carries over to the alternative semantics from [2] is still open.

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