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Solutions to Some Open Problems from Slaney

Branden Fitelson
Department of Philosophy & Religion, Northeastern University.

Abstract

In response to a paper by Harris & Fitelson [1], Slaney [6] states several open questions concerning possible strategies for proving distributivity in a wide class of positive sentential logics. In this note, I provide answers to all of Slaney's open questions. The result is a better understanding of the class of positive logics in which distributivity holds.

1 Introduction

Harris & Fitelson [1] used Otter to prove distributivity in $Ł_{\aleph_0}$ and other non-classical sentential logics. Their proofs involved axiomatizations in terms of implication ($\rightarrow$) and negation ($\neg$). Slaney [6] showed how to prove these results in the positive fragments of these logics, which involve only implication ($\rightarrow$), conjunction ($\wedge$), and disjunction ($\vee$). Slaney also provided a much more general framework for thinking about distributivity in a wide class of positive logics. This led him to state several open questions regarding strategies for establishing distributivity in this broad class of non-classical (positive) logics. In this note, I will provide answers to all of Slaney's open questions. All of these results were obtained using (various) automated reasoning tools.[1]

[1] I used a combination of `prover9` [3], `Otter` [4], `E` [5], and `Vampire` [2] to solve Slaney's open problems. All proofs are presented in the Appendix, in `Otter` format.

2 Slaney's Three (Background) Positive Logics

Slaney [6] presents a large class of (positive) logics, which involve various combinations of the following axioms and rules (*i.e.*, axiom and rule *schemata*).[2]

(AxK)	$\vdash A \to (B \to A)$
(AxB)	$\vdash (B \to C) \to ((A \to B) \to (A \to C))$
(AxŁ)	$\vdash (A \to (B \to B)) \to (B \to (A \to A))$
(AxTO)	$\vdash ((A \to B) \to (B \to A)) \to (B \to A)$
(AxC)	$\vdash (A \to (B \to C)) \to (B \to (A \to C))$
(AxI)	$\vdash A \to A$
(AxB′)	$\vdash (A \to B) \to ((B \to C) \to (A \to C))$
(Det)	From $\vdash A \to B$ and $\vdash A$, infer $\vdash B$
(Ax∧E1)	$\vdash (A \wedge B) \to A$
(Ax∧E2)	$\vdash (A \wedge B) \to B$
(Ax∧I)	$\vdash ((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
(Ax∨I1)	$\vdash A \to (A \vee B)$
(Ax∨I2)	$\vdash B \to (A \vee B)$
(Ax∨E)	$\vdash ((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
(Adj)	From $\vdash A$ and $\vdash B$, infer $\vdash A \wedge B$

Specifically, Slaney's open questions involve the following three (background) positive logics.

1. TW^+[AxŁ, AxTO], the pure implicational fragment of which ($\mathrm{TW}^\to$) is given by the axioms AxB, AxI, AxB′, AxŁ, and AxTO, and the rule Det. The full logic TW^+[AxŁ, AxTO] is then obtained by adding all of the axioms and rules for conjunction and disjunction to this implicational base. In other words, TW^+[AxŁ, AxTO] is given by: AxB, AxI, AxB′, AxŁ, AxTO, Det, Ax∧E1, Ax∧E2, Ax∧I, Ax∨I1, Ax∨I2, Ax∨E, and Adj.

2. $\mathrm{BCK}^\to$[AxŁ], which consists of the axioms AxK, AxB, AxC, and AxŁ, and the rule Det.

[2]Here, I follow Slaney's [6] notation and nomenclature, which differs slightly from that of Harris & Fitelson [1].

3. $TW^{\rightarrow}$[AxŁ], which consists of the axioms AxB, AxI, AxB′, and AxŁ, and the rule Det.

3 Four Other Principles Implicated in Slaney's Open Questions

In addition to these three background positive logics, Slaney's open questions also involve the following four additional axioms/theorems and rules:

(Dist)	$\vdash (A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))$
(RTO)	$\vdash (A \rightarrow B) \vee (B \rightarrow A)$
(IO)	$\vdash ((A \rightarrow B) \rightarrow B)) \rightarrow (A \vee B)$
(Resid)	$\vdash (A \circ B) \rightarrow C \Longleftrightarrow \vdash A \rightarrow (B \rightarrow C)$[3]

4 Slaney's (Six) Open Questions and Their Solutions

Slaney's first four open questions involve the background positive logic TW^{+}[AxŁ, AxTO]. The first two of these open questions are as follows.

1. Is (Dist) provable in TW^{+}[AxŁ, AxTO]?

2. Is (RTO) provable in TW^{+}[AxŁ, AxTO]?

Slaney [6, *p*. 65] notes that affirmative answers to both questions (1) and (2) are forthcoming, *if* it is possible to prove (IO) in TW^{+}[AxŁ, AxTO]. Our first theorem therefore implies affirmative answers to both (1) and (2).[4]

Theorem 1. *(IO) is provable in TW^{+}[AxŁ, AxTO].*

Slaney's next two open questions regarding TW^{+}[AxŁ, AxTO] involve the addition of a fusion operator '$\circ$' to TW^{+}[AxŁ, AxTO], *via* the (Resid) rule.

[3]The meaning of "$p \Longleftrightarrow q$" is "From p, infer q; and, from q, infer p." Thus, (Resid) adds a new "fusion" connective '$\circ$,' which obeys the two-way rule of inference in question.

[4]See the Appendix for `Otter` proofs of all theorems reported in this paper.

3. Is the addition of fusion a conservative extension of the positive logic TW^+[AxŁ, AxTO]? That is, does the addition of (Resid) to TW^+[AxŁ, AxTO] imply *no new theorems* involving *only* $\langle \rightarrow, \wedge, \vee \rangle$?

4. If the answer to (3) is *negative* (*i.e.*, if new $\langle \rightarrow, \wedge, \vee \rangle$-theorems *are* derivable upon adding (Resid) to TW^+[AxŁ, AxTO]), then does the addition of (Resid) to TW^+[AxŁ, AxTO] allow us to prove *both* (AxK) *and* (AxC)?

Our second theorem implies both a negative answer to (3) and a positive answer to (4).

Theorem 2. *(AxK) and (AxC) are provable in TW^+[AxŁ, AxTO] + (Resid).*

Slaney's fifth open question involves the background positive logic $BCK^{\rightarrow}$[AxŁ].

5. Is the addition of fusion, with its two-way rule (Resid), enough to generate a(nother) negation-free proof of (AxTO) from $BCK^{\rightarrow}$[AxŁ]? In other words, is (AxTO) provable in $BCK^{\rightarrow}$[AxŁ] + (Resid)?

Our third theorem implies an affirmative answer to (5).

Theorem 3. *(AxTO) is provable in $BCK^{\rightarrow}$[AxŁ] + (Resid).*

That brings us to Slaney's sixth (and final) open question (implicitly asked on page 66), which involves his third background positive logic $TW^{\rightarrow}$[AxŁ].

6. Is (AxK) provable in $TW^{\rightarrow}$[AxŁ] + (Resid)?

Our fourth (and final) theorem implies an affirmative answer to (6).

Theorem 4. *(AxK) is provable in $TW^{\rightarrow}$[AxŁ] + (Resid).*

Appendix: Proofs of Theorems

In this Appendix, I provide `Otter` proofs of our four theorems. Instead of using infix notation involving $\langle \rightarrow, \wedge, \vee, \circ \rangle$, I will use prefix notation involving $\langle \texttt{i}, \texttt{and}, \texttt{or}, \texttt{f} \rangle$. That is to say, we will adopt the following `Otter` notation:

$$\vdash A \rightarrow B \mapsto \texttt{p(i(A,B))}$$

$A \wedge B \mapsto \texttt{and}(\texttt{A},\texttt{B})$

$A \vee B \mapsto \texttt{or}(\texttt{A},\texttt{B})$

$A \circ B \mapsto \texttt{f}(\texttt{A},\texttt{B})$

See Harris & Fitelson [1] for further explanation of how our `Otter` proof objects are to be interpreted (and related to more traditional presentations of proofs in sentential logics). The proofs presented here are the shortest/simplest proofs I was able to find using `Otter`.

`Otter` Proof of Theorem 1.[5]

```
Length of proof is 36.  Level of proof is 14.

---------------- PROOF ----------------

38 [] -p(i(A,B))| -p(A)|p(B)   # label(Det).
40 [] p(i(i(A,B),i(i(B,C),i(A,C))))   # label(AxBp).
41 [] p(i(i(i(X,Y),Y),i(i(Y,X),X)))   # label(AxL).
42 [] p(i(i(i(X,Y),i(Y,X)),i(Y,X)))   # label(AxTO).
43 [] p(i(X,or(X,Y)))   # label(AxorI1).
44 [] p(i(Y,or(X,Y)))   # label(AxorI2).
51 [hyper,38,40,40] p(i(i(i(i(A,B),i(C,B)),D),i(i(C,A),D))).
52 [hyper,38,40,41] p(i(i(i(i(A,B),B),C),i(i(i(B,A),A),C))).
53 [hyper,38,40,42] p(i(i(i(A,B),C),i(i(i(B,A),i(A,B)),C))).
54 [hyper,38,40,43] p(i(i(or(A,B),C),i(A,C))).
55 [hyper,38,40,44] p(i(i(or(A,B),C),i(B,C))).
56 [hyper,38,51,51] p(i(i(A,i(B,C)),i(i(D,B),i(A,i(D,C))))).
57 [hyper,38,51,42] p(i(i(A,A),i(A,A))).
58 [hyper,38,51,52] p(i(i(A,i(B,C)),i(i(i(C,B),B),i(A,C)))).
59 [hyper,38,52,53] p(i(i(i(A,B),B),i(i(i(A,B),i(B,A)),A))).
60 [hyper,38,40,54] p(i(i(i(A,B),C),i(i(or(A,D),B),C))).
61 [hyper,38,51,56] p(i(i(A,B),i(i(C,A),i(i(B,D),i(C,D))))).
62 [hyper,38,56,52] p(i(i(A,i(i(B,C),C)),i(i(i(i(C,B),B),D),i(A,D)))).
63 [hyper,38,42,57] p(i(A,A)).
64 [hyper,38,51,58] p(i(i(A,B),i(i(i(C,A),A),i(i(B,C),C)))).
65 [hyper,38,58,56] p(i(i(i(i(A,i(B,C)),i(B,D)),i(B,D)),i(i(A,i(D,C)),i(A,i(B,C))))).
66 [hyper,38,56,59] p(i(i(A,i(i(B,C),i(C,B))),i(i(i(B,C),C),i(A,B)))).
67 [hyper,38,62,60] p(i(i(i(i(A,or(B,C)),or(B,C)),D),i(i(i(B,A),A),D))).
68 [hyper,38,62,56] p(i(i(i(i(i(A,i(B,C)),B),B),D),i(i(A,i(i(A,i(B,C)),C)),D))).
69 [hyper,38,56,64] p(i(i(A,i(i(B,C),C)),i(i(C,D),i(A,i(i(D,B),B))))).
70 [hyper,38,64,55] p(i(i(i(A,i(or(B,C),D)),i(or(B,C),D)),i(i(i(C,D),A),A))).
```

[5]In fact, this `Otter` proof establishes something *stronger* than Theorem 1. It shows that (IO) is derivable from {Det, AxB′, AXŁ, AxTO, Ax∨I1, Ax∨I2}. An `Otter` input file which verifies this proof is available from `http://fitelson.org/slaney_theorem_1.in`.

```
71 [hyper,38,56,66] p(i(i(A,i(i(B,C),C)),i(i(D,i(i(B,C),i(C,B))),i(A,i(D,B))))).
72 [hyper,38,68,63] p(i(i(A,i(i(A,i(B,C)),C)),i(i(i(A,i(B,C)),B),B))).
73 [hyper,38,51,71] p(i(i(i(A,B),C),i(i(D,i(i(A,B),i(B,A))),i(i(C,B),i(D,A))))).
74 [hyper,38,42,72] p(i(i(i(A,i(A,A)),A),A)).
75 [hyper,38,69,73] p(i(i(i(i(A,B),i(B,A)),C),i(i(i(A,B),A),i(i(C,B),B)))).
76 [hyper,38,61,74] p(i(i(A,i(i(B,i(B,B)),B)),i(i(B,C),i(A,C)))).
77 [hyper,38,75,42] p(i(i(i(A,B),A),i(i(i(B,A),B),B))).
78 [hyper,38,40,76] p(i(i(i(i(A,B),i(C,B)),D),i(i(C,i(i(A,i(A,A)),A)),D))).
79 [hyper,38,51,77] p(i(i(A,i(A,B)),i(i(i(B,i(A,B)),B),B))).
80 [hyper,38,78,42] p(i(i(A,i(i(A,i(A,A)),A)),i(A,A))).
81 [hyper,38,58,79] p(i(i(i(A,i(i(A,i(B,A)),A)),i(i(A,i(B,A)),A)),i(i(B,i(B,A)),A))).
82 [hyper,38,65,81] p(i(i(A,i(A,A)),i(A,i(i(A,i(A,A)),A)))).
83 [hyper,38,40,82] p(i(i(i(A,i(i(A,i(A,A)),A)),B),i(i(A,i(A,A)),B))).
84 [hyper,38,83,80] p(i(i(A,i(A,A)),i(A,A))).
85 [hyper,38,70,84] p(i(i(i(A,or(B,A)),or(B,A)),or(B,A))).
86 [hyper,38,67,85] p(i(i(i(A,B),B),or(A,B))).

------------ end of proof -------------
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Otter Proof of Theorem 2.[6]

```
Length of proof is 74.  Level of proof is 24.

---------------- PROOF ----------------

75 [] -p(i(A,B))| -p(A)|p(B)   # label(Det).
76 [] -p(i(f(A,B),C))|p(i(A,i(B,C)))   # label(Resid1).
77 [] p(i(f(A,B),C))| -p(i(A,i(B,C)))   # label(Resid2).
79 [] p(i(i(A,B),i(i(B,C),i(A,C))))   # label(AxBp).
80 [] p(i(i(i(X,Y),Y),i(i(Y,X),X)))   # label(AxL).
81 [] p(i(i(i(X,Y),i(Y,X)),i(Y,X)))   # label(AxTO).
88 [hyper,77,79] p(i(f(i(A,B),i(B,C)),i(A,C))).
89 [hyper,75,79,79] p(i(i(i(i(A,B),i(C,B)),D),i(i(C,A),D))).
90 [hyper,75,79,80] p(i(i(i(i(A,B),B),C),i(i(i(B,A),A),C))).
91 [hyper,75,79,81] p(i(i(i(A,B),C),i(i(i(B,A),i(A,B)),C))).
92 [hyper,75,79,88] p(i(i(i(A,B),C),i(f(i(A,D),i(D,B)),C))).
93 [hyper,75,89,89] p(i(i(A,i(B,C)),i(i(D,B),i(A,i(D,C))))).
94 [hyper,75,89,81] p(i(i(A,A),i(A,A))).
95 [hyper,75,89,90] p(i(i(A,i(B,C)),i(i(i(C,B),B),i(A,C)))).
96 [hyper,75,79,90] p(i(i(i(i(i(A,B),B),C),D),i(i(i(i(B,A),A),C),D))).
97 [hyper,75,90,89] p(i(i(i(i(A,B),i(C,B)),i(C,B)),i(i(A,C),i(A,B)))).
98 [hyper,75,89,91] p(i(i(A,B),i(i(i(C,B),i(B,C)),i(A,C)))).
99 [hyper,75,92,81] p(i(f(i(i(A,B),C),i(C,i(B,A))),i(B,A))).
100 [hyper,75,92,80] p(i(f(i(i(A,B),C),i(C,B)),i(i(B,A),A))).
101 [hyper,75,93,91] p(i(i(A,i(i(B,C),i(C,B))),i(i(i(C,B),D),i(A,D)))).
```

[6] In fact, this Otter proof establishes something *stronger* than Theorem 2. It shows that (AxC) and (AxK) are both derivable from {Det, Resid, AxB′, AxL, AxTO}. An Otter input file which verifies this proof is available from http://fitelson.org/slaney_theorem_2.in.

```
102 [hyper,75,93,90] p(i(i(A,i(i(B,C),C)),i(i(i(i(C,B),B),D),i(A,D)))).
103 [hyper,75,81,94] p(i(A,A)).
104 [hyper,75,93,95] p(i(i(A,i(i(B,C),C)),i(i(D,i(C,B)),i(A,i(D,B))))).
105 [hyper,75,79,95] p(i(i(i(i(i(A,B),B),i(C,A)),D),i(i(C,i(B,A)),D))).
106 [hyper,75,95,93] p(i(i(i(i(A,i(B,C)),i(B,D)),i(B,D)),i(i(A,i(D,C)),i(A,i(B,C))))).
107 [hyper,75,93,98] p(i(i(A,i(i(B,C),i(C,B))),i(i(D,C),i(A,i(D,B))))).
108 [hyper,75,79,98] p(i(i(i(i(i(A,B),i(B,A)),i(C,A)),D),i(i(C,B),D))).
109 [hyper,76,99] p(i(i(i(A,B),C),i(i(C,i(B,A)),i(B,A)))).
110 [hyper,76,100] p(i(i(i(A,B),C),i(i(C,B),i(i(B,A),A)))).
111 [hyper,76,103] p(i(A,i(B,f(A,B)))).
112 [hyper,75,91,104] p(i(i(i(i(i(A,B),B),C),i(C,i(i(A,B),B))),i(i(D,i(B,A)),i(C,i(D,A))))).
113 [hyper,75,105,106] p(i(i(A,i(i(A,B),B)),i(i(B,i(B,B)),i(B,i(A,B))))).
114 [hyper,75,102,109] p(i(i(i(i(i(A,B),C),C),D),i(i(i(B,A),C),D))).
115 [hyper,75,90,109] p(i(i(i(A,B),B),i(i(A,i(A,B)),i(A,B)))).
116 [hyper,75,89,109] p(i(i(A,B),i(i(i(A,C),i(C,B)),i(C,B)))).
117 [hyper,75,110,79] p(i(i(i(i(A,B),i(C,B)),A),i(i(A,C),C))).
118 [hyper,75,79,111] p(i(i(i(A,f(B,A)),C),i(B,C))).
119 [hyper,75,114,108] p(i(i(i(i(A,B),i(B,A)),i(C,B)),i(i(C,A),i(C,B)))).
120 [hyper,75,114,97] p(i(i(i(A,B),i(C,A)),i(i(B,C),i(B,A)))).
121 [hyper,75,102,115] p(i(i(i(i(i(A,B),A),A),C),i(i(i(A,B),B),C))).
122 [hyper,75,102,116] p(i(i(i(i(i(A,B),i(C,A)),i(C,A)),D),i(i(C,B),D))).
123 [hyper,75,118,79] p(i(A,i(i(f(A,B),C),i(B,C)))).
124 [hyper,75,108,119] p(i(i(A,B),i(i(A,B),i(A,B)))).
125 [hyper,75,89,120] p(i(i(A,B),i(i(B,A),i(B,B)))).
126 [hyper,75,120,109] p(i(i(A,i(A,i(B,B))),i(A,i(B,B)))).
127 [hyper,75,121,117] p(i(i(i(i(A,B),B),B),i(i(i(A,B),A),A))).
128 [hyper,75,93,124] p(i(i(A,i(B,C)),i(i(B,C),i(A,i(B,C))))).
129 [hyper,75,90,125] p(i(i(i(A,B),B),i(i(A,i(B,A)),i(A,A)))).
130 [hyper,75,122,127] p(i(i(A,i(A,B)),i(i(i(B,i(A,B)),B),B))).
131 [hyper,75,105,127] p(i(i(A,i(i(A,B),B)),i(i(i(B,i(A,B)),B),B))).
132 [hyper,75,96,127] p(i(i(i(i(A,B),B),A),i(i(i(B,A),B),B))).
133 [hyper,75,89,128] p(i(i(A,B),i(i(A,C),i(i(B,C),i(A,C))))).
134 [hyper,75,129,126] p(i(i(A,i(i(A,i(B,B)),A)),i(A,A))).
135 [hyper,75,131,130] p(i(i(i(A,i(i(A,i(A,A)),A)),A),A)).
136 [hyper,75,93,132] p(i(i(A,i(i(B,C),B)),i(i(i(i(C,B),B),C),i(A,B)))).
137 [hyper,75,118,133] p(i(A,i(i(B,C),i(i(f(A,B),C),i(B,C))))).
138 [hyper,75,123,134] p(i(i(f(i(i(A,i(i(A,i(B,B)),A)),i(A,A)),C),D),i(C,D))).
139 [hyper,75,92,135] p(i(f(i(i(A,i(i(A,i(A,A)),A)),B),i(B,A)),A)).
140 [hyper,75,136,88] p(i(i(i(i(A,B),B),A),i(f(i(i(B,A),C),i(C,B)),B))).
141 [hyper,75,79,137] p(i(i(i(i(A,B),i(i(f(C,A),B),i(A,B))),D),i(C,D))).
142 [hyper,75,138,139] p(i(i(i(A,A),A),A)).
143 [hyper,75,140,142] p(i(f(i(i(A,A),B),i(B,A)),A)).
144 [hyper,76,143] p(i(i(i(A,A),B),i(i(B,A),A))).
145 [hyper,75,93,144] p(i(i(A,i(B,C)),i(i(i(C,C),B),i(A,C)))).
146 [hyper,75,89,144] p(i(i(A,B),i(i(i(A,B),B),B))).
147 [hyper,75,93,145] p(i(i(A,i(i(B,B),C)),i(i(D,i(C,B)),i(A,i(D,B))))).
148 [hyper,75,145,146] p(i(i(i(A,A),i(i(B,A),A)),i(i(B,A),A))).
149 [hyper,75,97,148] p(i(i(A,i(B,A)),i(A,A))).
150 [hyper,75,141,149] p(i(A,i(i(B,C),i(B,C)))).
152 [hyper,75,107,150] p(i(i(A,B),i(C,i(A,B)))).
153 [hyper,75,101,150] p(i(i(i(A,A),B),i(C,B))).
155 [hyper,75,147,152] p(i(i(A,i(i(B,C),D)),i(i(B,C),i(A,D)))).
157 [hyper,75,79,152] p(i(i(i(A,i(B,C)),D),i(i(B,C),D))).
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160 [hyper,75,153,153] p(i(A,i(B,i(C,C)))).
163 [hyper,75,155,95] p(i(i(i(A,B),B),i(i(C,i(B,A)),i(C,A)))).
170 [hyper,75,163,160] p(i(i(A,i(i(B,i(C,C)),D)),i(A,D))).
173 [hyper,75,170,113] p(i(i(A,i(i(A,B),B)),i(B,i(A,B)))).
174 [hyper,75,157,173] p(i(i(i(A,B),B),i(B,i(A,B)))).
176 [hyper,75,112,174] p(i(i(A,i(B,C)),i(B,i(A,C)))).
177 [hyper,75,81,174] p(i(A,i(B,A))).

------------ end of proof -------------
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Otter Proof of Theorem 3.[7]

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Length of proof is 28.  Level of proof is 13.

---------------- PROOF ----------------

29 [] -p(i(A,B))| -p(A)|p(B)   # label(Det).
30 [] -p(i(f(A,B),C))|p(i(A,i(B,C)))   # label(Resid1).
31 [] p(i(f(A,B),C))| -p(i(A,i(B,C)))   # label(Resid2).
33 [] p(i(i(A,B),i(i(B,C),i(A,C))))   # label(AxBp).
34 [] p(i(i(i(A,B),B),i(i(B,A),A)))   # label(AxL).
35 [] p(i(i(A,i(B,C)),i(B,i(A,C))))   # label(AxC).
36 [hyper,29,33,33] p(i(i(i(i(A,B),i(C,B)),D),i(i(C,A),D))).
37 [hyper,29,35,35] p(i(A,i(i(B,i(A,C)),i(B,C)))).
38 [hyper,29,33,35] p(i(i(i(A,i(B,C)),D),i(i(B,i(A,C)),D))).
39 [hyper,29,35,34] p(i(i(A,B),i(i(i(B,A),A),B))).
40 [hyper,29,35,33] p(i(i(A,B),i(i(C,A),i(C,B)))).
41 [hyper,29,36,36] p(i(i(A,i(B,C)),i(i(D,B),i(A,i(D,C))))).
42 [hyper,29,33,37] p(i(i(i(i(A,i(B,C)),i(A,C)),D),i(B,D))).
43 [hyper,29,38,35] p(i(i(A,i(B,C)),i(A,i(B,C)))).
44 [hyper,29,38,33] p(i(i(A,i(B,C)),i(i(i(A,C),D),i(B,D)))).
45 [hyper,29,41,39] p(i(i(A,i(i(B,C),C)),i(i(C,B),i(A,B)))).
47 [hyper,29,42,35] p(i(A,i(B,i(i(B,i(A,C)),C)))).
48 [hyper,29,35,43] p(i(A,i(i(A,i(B,C)),i(B,C)))).
49 [hyper,29,44,43] p(i(i(i(i(A,i(B,C)),i(B,C)),D),i(A,D))).
50 [hyper,31,47] p(i(f(A,B),i(i(B,i(A,C)),C))).
51 [hyper,29,45,48] p(i(i(i(A,B),C),i(C,C))).
52 [hyper,29,44,50] p(i(i(i(f(A,B),C),D),i(i(B,i(A,C)),D))).
53 [hyper,29,51,49] p(i(i(A,B),i(A,B))).
54 [hyper,29,44,53] p(i(i(i(i(A,B),B),C),i(A,C))).
56 [hyper,29,54,51] p(i(A,i(B,B))).
57 [hyper,29,56,56] p(i(A,A)).
58 [hyper,29,35,56] p(i(A,i(B,A))).
59 [hyper,30,57] p(i(A,i(B,f(A,B)))).
60 [hyper,31,58] p(i(f(A,B),A)).
61 [hyper,29,41,59] p(i(i(A,B),i(C,i(A,f(C,B))))).
```

[7]In fact, this Otter proof establishes something *stronger* than Theorem 3. It shows that (AxTO) is derivable from {Det, Resid, AxB′, AxC, AxL}. An Otter input file which verifies this proof is available from `http://fitelson.org/slaney_theorem_3.in`.

```
62 [hyper,29,40,60] p(i(i(A,f(B,C)),i(A,B))).
63 [hyper,29,61,62] p(i(A,i(i(B,f(C,D)),f(A,i(B,C))))).
64 [hyper,29,45,63] p(i(i(f(A,i(B,A)),B),i(A,B))).
65 [hyper,29,52,64] p(i(i(i(A,B),i(B,A)),i(B,A))).

------------ end of proof -------------
```

Otter Proof of Theorem 4.[8]

```
Length of proof is 52.  Level of proof is 26.

---------------- PROOF ----------------

53 [] -p(i(A,B))| -p(A)|p(B)   # label(Det).
54 [] -p(i(f(A,B),C))|p(i(A,i(B,C)))   # label(Resid1).
55 [] p(i(f(A,B),C))| -p(i(A,i(B,C)))   # label(Resid2).
57 [] p(i(A,A))   # label(AxI).
58 [] p(i(i(A,B),i(i(B,C),i(A,C))))   # label(AxBp).
59 [] p(i(i(i(A,B),B),i(i(B,A),A)))   # label(AxL).
66 [hyper,55,57] p(i(f(i(A,B),A),B)).
67 [hyper,54,57] p(i(A,i(B,f(A,B)))).
68 [hyper,55,58] p(i(f(i(A,B),i(B,C)),i(A,C))).
69 [hyper,53,58,58] p(i(i(i(i(A,B),i(C,B)),D),i(i(C,A),D))).
70 [hyper,53,58,66] p(i(i(A,B),i(f(i(C,A),C),B))).
71 [hyper,53,58,67] p(i(i(i(A,f(B,A)),C),i(B,C))).
72 [hyper,53,58,68] p(i(i(i(A,B),C),i(f(i(A,D),i(D,B)),C))).
73 [hyper,53,69,69] p(i(i(A,i(B,C)),i(i(D,B),i(A,i(D,C))))).
74 [hyper,53,70,66] p(i(f(i(A,f(i(B,C),B)),A),C)).
75 [hyper,53,70,59] p(i(f(i(A,i(i(B,C),C)),A),i(i(C,B),B))).
76 [hyper,53,69,71] p(i(i(A,B),i(C,i(A,f(C,B))))).
77 [hyper,53,69,73] p(i(i(A,B),i(i(C,A),i(i(B,D),i(C,D))))).
78 [hyper,53,73,59] p(i(i(A,i(B,C)),i(i(i(C,B),B),i(A,C)))).
79 [hyper,54,74] p(i(i(A,f(i(B,C),B)),i(A,C))).
80 [hyper,54,75] p(i(i(A,i(i(B,C),C)),i(A,i(i(C,B),B)))).
81 [hyper,53,73,78] p(i(i(A,i(i(B,C),C)),i(i(D,i(C,B)),i(A,i(D,B))))).
82 [hyper,53,69,78] p(i(i(A,B),i(i(i(C,A),A),i(i(B,C),C)))).
83 [hyper,53,58,78] p(i(i(i(i(i(A,B),B),i(C,A)),D),i(i(C,i(B,A)),D))).
84 [hyper,53,78,77] p(i(i(i(i(i(A,B),i(C,B)),i(C,D)),i(C,D)),i(i(D,A),i(i(A,B),i(C,B))))).
85 [hyper,53,78,73] p(i(i(i(i(A,i(B,C)),i(B,D)),i(B,D)),i(i(A,i(D,C)),i(A,i(B,C))))).
86 [hyper,53,76,79] p(i(A,i(i(B,f(i(C,D),C)),f(A,i(B,D))))).
87 [hyper,53,58,80] p(i(i(i(A,i(i(B,C),C)),D),i(i(A,i(i(C,B),B)),D))).
88 [hyper,53,71,82] p(i(A,i(i(i(B,C),C),i(i(f(A,C),B),B)))).
89 [hyper,53,83,85] p(i(i(A,i(i(A,B),B)),i(i(B,i(B,B)),i(B,i(A,B))))).
90 [hyper,53,81,86] p(i(i(A,i(f(i(i(B,C),C),i(B,C)),B)),i(i(i(B,C),C),i(A,B)))).
91 [hyper,53,73,88] p(i(i(A,i(i(B,C),C)),i(D,i(A,i(i(f(D,C),B),B))))).
```

[8] In fact, this `Otter` proof establishes something *stronger* than Theorem 4. It shows that (AxK) is derivable from {Det, Resid, AxI, AxB′, AxŁ}. An `Otter` input file which verifies this proof is available from `http://fitelson.org/slaney_theorem_4.in`.

```
92 [hyper,53,87,89] p(i(i(A,i(i(B,A),A)),i(i(B,i(B,B)),i(B,i(A,B))))).
93 [hyper,53,90,72] p(i(i(i(A,A),A),i(i(i(A,A),A),A),A))).
94 [hyper,53,87,91] p(i(i(A,i(i(B,C),C)),i(D,i(A,i(i(f(D,B),C),C))))).
95 [hyper,53,89,93] p(i(i(A,i(A,A)),i(A,i(i(i(A,A),A),A)))).
96 [hyper,53,94,93] p(i(A,i(i(i(B,B),B),i(i(f(A,i(i(B,B),B)),B),B)))).
97 [hyper,53,95,58] p(i(i(A,A),i(i(i(i(A,A),i(A,A)),i(A,A)),i(A,A)))).
98 [hyper,55,96] p(i(f(A,i(i(B,B),B)),i(i(f(A,i(i(B,B),B)),B),B))).
99 [hyper,53,97,57] p(i(i(i(i(A,A),i(A,A)),i(A,A)),i(A,A))).
100 [hyper,53,89,98] p(i(i(A,i(A,A)),i(A,i(f(B,i(i(A,A),A)),A)))).
102 [hyper,53,59,99] p(i(i(i(A,A),i(i(A,A),i(A,A))),i(i(A,A),i(A,A)))).
103 [hyper,53,100,58] p(i(i(A,A),i(f(B,i(i(i(A,A),i(A,A)),i(A,A))),i(A,A)))).
104 [hyper,53,103,57] p(i(f(A,i(i(i(B,B),i(B,B)),i(B,B))),i(B,B))).
105 [hyper,54,104] p(i(A,i(i(i(i(B,B),i(B,B)),i(B,B)),i(B,B)))).
107 [hyper,53,58,105] p(i(i(i(i(i(i(A,A),i(A,A)),i(A,A)),i(A,A)),B),i(C,B))).
108 [hyper,53,107,84] p(i(A,i(i(B,B),i(i(B,B),i(B,B))))).
109 [hyper,53,58,108] p(i(i(i(i(A,A),i(i(A,A),i(A,A))),B),i(C,B))).
110 [hyper,53,109,102] p(i(A,i(i(B,B),i(B,B)))).
112 [hyper,53,73,110] p(i(i(A,i(B,B)),i(C,i(A,i(B,B))))).
116 [hyper,53,92,112] p(i(i(A,i(A,A)),i(A,i(i(B,i(C,C)),A)))).
122 [hyper,53,116,110] p(i(i(A,A),i(i(B,i(C,C)),i(A,A)))).
125 [hyper,53,73,122] p(i(i(A,i(B,i(C,C))),i(i(D,D),i(A,i(D,D))))).
131 [hyper,53,125,110] p(i(i(A,A),i(B,i(A,A)))).
137 [hyper,53,131,57] p(i(A,i(B,B))).
142 [hyper,53,116,137] p(i(A,i(i(B,i(C,C)),A))).
155 [hyper,53,73,142] p(i(i(A,i(B,i(C,C))),i(D,i(A,D)))).
173 [hyper,53,155,137] p(i(A,i(B,A))).

------------ end of proof -------------
```

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