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ω-Inconsistency without Cuts and Nonstandard Models*

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Abstract

This paper concerns the relationship between transitivity of entailment, ω-inconsistency and nonstandard models of arithmetic. First, it provides a cut-free sequent calculus for the non-transitive logic of truth STT based on Robinson Arithmetic and shows that this logic is ω-inconsistent. It then identifies the conditions in McGee (1985) for an ω-inconsistent logic as quantified standard deontic logic, presents a cut-free labelled sequent calculus for quantified standard deontic logic based on Robinson Arithmetic where the deontic modality is treated as a predicate, proves ω-inconsistency and shows thus, *pace* Cobreros et al. (2013), that the result in McGee (1985) does not rely on transitivity. Finally, it also explains why the ω-inconsistent logics of truth in question do not require nonstandard models of arithmetic.

Keywords: Theories of truth, non-transitive logics, cut-free sequent calculus, ω-inconsistency, semantic paradoxes, nonstandard models of arithmetic

1 Introduction

McGee (1985) shows that a logic of truth satisfying certain conditions is ω-inconsistent, conditions which for example are satisfied by the logic of truth familiar as FS introduced by Friedman and Sheard (1987) and discussed in for example Halbach (1994), Leitgeb (2001) and Halbach (2011). In a recent comparison between FS and the non-transitive logic of truth STT, Cobreros et al. (2013) make the following statement:

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> FS is ω-inconsistent, and so can have no standard models. [STT], on the other hand, is shown to have standard models by the Kripke construction. In this regard, it is worth noting that [STT_{PA}] which contains the compositional principles, PA, and a transparent truth predicate, seems to more than satisfy the conditions for the 'negative result' in [McGee (1985)], showing that any system meeting weaker conditions than these must be ω-inconsistent. (It is this result that shows FS to be ω-inconsistent.) Nonetheless, the result does not apply to [STT_{PA}], as McGee's argument depends on assuming transitivity.(Cobreros et al. 2013, p.860)

One possible reading[1] of the last claim is that STT based on Peano Arithmetic is not ω-inconsistent. I show in the next section that this is not the case by providing a derivation of ω-inconsistency in a cut-free sequent calculus for STT based on Robinson Arithmetic. Since STT based on an arithmetical theory seems to more than satisfy McGee's conditions, I proceed in section 3 to identify McGee's conditions as quantified standard deontic logic with the modality treated as predicate rather than operator and provide a derivation of ω-inconsistency in a cut-free sequent calculus for quantified standard deontic logic based on Robinson Arithmetic where, of course, the modality is treated as predicate rather than operator. In the final section I elaborate on consequences of these results for the received view that ω-inconsistent logics of truth require nonstandard models of arithmetic.

2 ω-inconsistency with STT based on Robinson Arithmetic

2.1 Introducing the language and the system

A logic is ω-inconsistent if both a formula $\neg\forall x A x$ and $A\overline{n}$ for each $n \in \omega$ are theorems of that logic, and our aim in this section is thus to show that we can derive this for the logic we obtain by defining STT on Robinson Arithmetic.

A sequent calculus for STT is typically obtained simply by augmenting a cut-free two-sided sequent calculus for classical logic with a transparent truth-predicate in such a way that the logic remains reflexive, monotonic and contractive. Instead, the logic is non-transitive and it is fair to say that the rule

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \ (\text{CUT})$$

implies transitivity in such a setting, and that a derivation in a standard

[1] And the intended meaning as acknowledged by David Ripley.

two-sided sequent calculus relies on transitivity if the rule

$$\frac{\Rightarrow A \qquad A \Rightarrow B}{\Rightarrow B}\ \text{(Tr)}$$

is derivable from the rules applied in the derivation.[2] Notice that a cut-free two-sided sequent calculus might thus define a transitive logic even if none of the derivations rely on transitivity, as we for example can assume to be the case with the logic we define below in section 3. The logic we define in this subsection however, will be non-transitive.

Our aim in this subsection is to define a cut-free sequent calculus for STT based on Robinson Arithmetic.[3] We shall focus on the language $\mathcal{L}_T^{A+}$, a language for first-order arithmetic with the connectives $\forall$, $\vee$, $\neg$, an equality-predicate $=$, a truth-predicate T and various function-symbols, in particular $\underset{\cdot}{f}$ and $\underset{\cdot}{T}$. We thus adopt the convention to use a dot under a letter to signify a function-symbol except in the case of the function symbols for successor, addition and multiplication, and furthermore also the convention to let $\ulcorner A \urcorner$ represent the Gödel-code of a formula A while $\overline{\ulcorner A \urcorner}$ is the numeral of that number in our language but as usual omit the overline unless there is danger of confusion.

As usual, a rule in a sequent calculus consists of one or more premise-sequents and one conclusion-sequent where the premise-sequents contain one or more active formulae in addition to contexts such as Γ and Δ whereas the conclusion-sequent contains one or more principal formulae in addition to contexts. Following Negri and von Plato (2001), we shall treat sequents as pairs of multisets of formulae but nonetheless rely on rules and initial sequents that make the rules of weakening and contraction admissible. First of all, we adopt thus every instance of $\Gamma, P \Rightarrow P, \Delta$ as initial sequents where Γ and Δ are multisets of formulae and P is an atomic $\mathcal{L}_T^{A+}$-formula. For $\vee$, $\neg$ and $\forall$ we use the following additive two-premise and multiplicative one-premise rules:

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ (\vee\text{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ (\vee\text{R})$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma, \neg A \Rightarrow \Delta}\ (\neg\text{L}) \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}\ (\neg\text{R})$$

[2]The connection between transitivity and CUT is a artifact of the two-sidedness of the calculus and the reading of $\Rightarrow$ as "entails". See for example (Ripley 2012) for a three-sided sequent calculus for STT with a cut-rule and thus a calculus in which a cut-rule does not imply transitivity.

[3]The use of Robinson Arithmetic and thus Gödel-coding to generate names of formulae is itself a novelty in the literature on proof theory for STT. Instead, it is typical to use meta-linguistic coding explicitly as in Ripley (2012) or implicitly as in Ripley (2013a) when defining STT.

$$\frac{\forall x A(x), A(a), \Gamma \Rightarrow \Delta}{\forall x A(x), \Gamma \Rightarrow \Delta}\ (\forall\text{L}_C) \qquad \frac{\Gamma \Rightarrow \Delta, A(y)}{\Gamma \Rightarrow \Delta, \forall x A}\ (\forall\text{R})$$

where y in ($\forall$R) does not occur in Γ,Δ or $\forall xA$, i.e. is an *eigenvariable*. We define equality $=$ with the following rules:[4]

$$\frac{a = a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (=\text{ID})$$

$$\frac{A(a), a = b, A(b), \Gamma \Rightarrow \Delta}{a = b, A(a), \Gamma \Rightarrow \Delta}\ (=rep\text{L}_C) \qquad \frac{a = b, \Gamma \Rightarrow A(a), A(b), \Delta}{a = b, \Gamma \Rightarrow A(b), \Delta}\ (=rep\text{R}_C)$$

This gives us a cut-free sequent calculus for first-order classical logic with equality, and a standard completeness proof will thus show that CUT is admissible. A cut-elimination proof along the lines of Negri and von Plato (2001, p.132) can be provided if we remove ($=rep\text{R}_C$) and restrict ($=rep\text{L}_C$) to atomic formulae only.[5]

To obtain a sequent calculus for Robinson Arithmetic the standard approach is to augment a sequent calculus like ours with CUT and the following list of initial sequents from Takeuti (1987) where a and b are arbitrary terms, and s represents the successor function:

$$\Gamma, sa = 0 \Rightarrow \Delta \qquad \Gamma, sa = sb \Rightarrow a = b, \Delta$$

$$\Gamma \Rightarrow a + 0 = a, \Delta \qquad \Gamma \Rightarrow a + sb = s(a + b), \Delta$$

$$\Gamma \Rightarrow a \times 0 = 0, \Delta \qquad \Gamma \Rightarrow a \times sb = (a \times b) + a, \Delta$$

$$\Gamma \Rightarrow a = 0, \exists y(a = sy), \Delta$$

Now, since STT requires that CUT is not admissible across the board, we must either restrict CUT to arithmetical formulae or replace CUT and the initial sequents for Robinson Arithmetic with suitable rules that allow us to derive the axioms and theorems of Robinson Arithmetic without CUT. I choose the latter approach because it is more in the spirit of a cut-free approach.[6] To obtain suitable rules we shall employ a method introduced by Negri and von Plato (1998) and extended by Negri (2003) which consists

[4]Variants of ($=rep\text{L}_C$) and ($=rep\text{R}_C$) without the occurrences of $a = b$ and $A(b)$ as active formulae in the premise-sequent are used by Restall (2013) and together with ($=$ID) by Boolos et al. (2007, p.170) and Ripley (2015).

[5]The inclusion of every formula in both rules and the right substitution rule are added to deal with issues that arise when not every instance of CUT is admissible because of the transparent truth-predicate.

[6]As suggested by an anonymous referee, one might also argue that the former approach is incompatible with the idea that CUT is legitimate for non-problematic sentences involving the truth-predicate.

roughly in transforming axioms involving atomic formulae to rules of a certain format that allows for cut-elimination. The idea is that axioms of the form $P_1 \wedge ... \wedge P_m \rightarrow Q_1 \vee ... \vee Q_n$ are transformed into rules of the following scheme:

$$\frac{P_1, ..., P_m, Q_1, \Gamma \Rightarrow \Delta \qquad ... \qquad P_1, ..., P_m, Q_n, \Gamma \Rightarrow \Delta}{P_1, ..., P_m, \Gamma \Rightarrow \Delta}$$

Applying it on the above initial sequents which involve only atomic formulae we obtain the following rules:

$$\frac{}{\Gamma, 0 = sa \Rightarrow \Delta}\ (\text{Q1}) \qquad \frac{\Gamma, a = b, sa = sb \Rightarrow \Delta}{\Gamma, sa = sb \Rightarrow \Delta}\ (\text{Q2})$$

$$\frac{\Gamma, a + 0 = a \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\text{Q4}) \qquad \frac{\Gamma, a + sb = s(a + b) \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\text{Q5})$$

$$\frac{\Gamma, a \times 0 = 0 \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\text{Q6}) \qquad \frac{\Gamma, a \times sb = (a \times b) + a \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\text{Q7})$$

Things are slightly more complicated with regard to the additional initial sequent required to obtain Robinson Arithmetic, namely $\Gamma \Rightarrow a = 0 \vee \exists y(a = sy), \Delta$. The problem is that the scheme does not apply because of the existential quantifier. Following Negri (2003) we must instead ensure that y is an *eigenvariable*: it does not occur free in the conclusion-sequent of the rule corresponding to the axiom. We obtain the following rule where y does not occur free in Γ or Δ:

$$\frac{\Gamma, a = 0 \Rightarrow \Delta \qquad \Gamma, a = sy \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\text{Q3})$$

Augmenting our sequent calculus for first-order classical logic with equality with the rules Q1-Q7 results thus in a cut-free sequent calculus for Robinson Arithmetic. Indeed, removing ($=rep\mathrm{R}_C$) and restricting ($=rep\mathrm{L}_C$) to atomic formulae will permit a proof of cut-elimination along the lines of (Negri and von Plato 2001, p.132) and (Negri and von Plato 2011, p.142).

To illustrate how to derive theorems with our sequent calculus for Robinson Arithmetic, consider for example the following derivation of 1+1=2:

$$\cfrac{\cfrac{\cfrac{s0 + 0 = s0, s0 + s0 = ss0 \Rightarrow s0 + s0 = ss0}{s0 + 0 = s0, s0 + s0 = s(s0 + 0) \Rightarrow s0 + s0 = ss0}\ (=rep\mathrm{L}_C)}{s0 + s0 = s(s0 + 0) \Rightarrow s0 + s0 = ss0}\ (\text{Q4})}{\Rightarrow s0 + s0 = ss0}\ (\text{Q5})$$

Notice that the initial sequent contains the formula we wish to prove and one instance of the axiom corresponding to (Q4).

In addition to rules for the standard arithmetical vocabulary we shall also introduce rules to define the function employed in McGee (1985) to prove ω-inconsistency. McGee (1985) observes that there is a binary primitive recursive function f that takes as arguments a natural number n and the Gödel-code of a sentence A, and yields a sentence $T\underset{\cdot}{T}\ldots\underset{\cdot}{T}\ulcorner A\urcorner$ with n truth-predications of A where $\underset{\cdot}{T}$ is a function-symbol representing a function f_T such that $f_T(\ulcorner A\urcorner) = \ulcorner T\ulcorner A\urcorner\urcorner$. f is representable in arithmetic and there is thus an arithmetical formula φ such that $\forall y\varphi(\overline{n}, \overline{\ulcorner A\urcorner}, y) \leftrightarrow y = \overline{c}$ is provable if and only if $f(n, \ulcorner A\urcorner) = c$ where c is of the form $\underset{\cdot}{T}\ldots\underset{\cdot}{T}\ulcorner A\urcorner$ with n occurrences of $\underset{\cdot}{T}$. However, it is more convenient to introduce a primitive function-symbol $\underset{\cdot}{f}$ to our language and let it represent the function f by defining it in such a way that for example the following sequents are derivable:

- $\Rightarrow \underset{\cdot}{f}(0, \ulcorner\phi\urcorner) = \ulcorner\phi\urcorner$
- $\Rightarrow \underset{\cdot}{f}(1, \ulcorner\phi\urcorner) = \underset{\cdot}{T}\ulcorner\phi\urcorner$
- $\Rightarrow \underset{\cdot}{f}(2, \ulcorner\phi\urcorner) = \underset{\cdot}{T}\underset{\cdot}{T}\ulcorner\phi\urcorner$

Or in general,

- $\underset{\cdot}{f}(sa, \ulcorner\phi\urcorner) = \underset{\cdot}{T}\underset{\cdot}{f}(a, \ulcorner\phi\urcorner)$

Using the above format for sequent calculus rules we now obtain the following rules for $\underset{\cdot}{f}$:

$$\frac{\Gamma, \underset{\cdot}{f}(0, \ulcorner A\urcorner) = \ulcorner A\urcorner \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (O1) \qquad \frac{\Gamma, \underset{\cdot}{f}(sa, \ulcorner A\urcorner) = \underset{\cdot}{T}\underset{\cdot}{f}(a, \ulcorner A\urcorner) \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (O2)$$

We also add the following rule for $\underset{\cdot}{T}$:

$$\frac{\underset{\cdot}{T}a = \ulcorner Ta\urcorner, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\underset{\cdot}{T})$$

To illustrate (O1) and (O2), we prove $\Rightarrow \underset{\cdot}{f}(1, \ulcorner A\urcorner) = \underset{\cdot}{T}\ulcorner A\urcorner$:

$$\cfrac{\cfrac{\cfrac{\underset{\cdot}{f}(0, \ulcorner A\urcorner) = \ulcorner A\urcorner, \underset{\cdot}{f}(s0, \ulcorner A\urcorner) = \underset{\cdot}{T}\ulcorner A\urcorner \Rightarrow \underset{\cdot}{f}(s0, \ulcorner A\urcorner) = \underset{\cdot}{T}\ulcorner A\urcorner}{\underset{\cdot}{f}(0, \ulcorner A\urcorner) = \ulcorner A\urcorner, \underset{\cdot}{f}(s0, \ulcorner A\urcorner) = \underset{\cdot}{T}\underset{\cdot}{f}(0, \ulcorner A\urcorner) \Rightarrow \underset{\cdot}{f}(s0, \ulcorner A\urcorner) = \underset{\cdot}{T}\ulcorner A\urcorner}\ (=rep_{L_C})}{\underset{\cdot}{f}(s0, \ulcorner A\urcorner) = \underset{\cdot}{T}\underset{\cdot}{f}(0, \ulcorner A\urcorner) \Rightarrow \underset{\cdot}{f}(s0, \ulcorner A\urcorner) = \underset{\cdot}{T}\ulcorner A\urcorner}\ (O1)}{\Rightarrow \underset{\cdot}{f}(s0, \ulcorner A\urcorner) = \underset{\cdot}{T}\ulcorner A\urcorner}\ (O2)$$

Again, we employ the rules and substitute equalities.

It is left to present the rules for the truth-predicate. We shall use the following perhaps slightly surprising rules:

$$\frac{\Gamma, A, T\ulcorner A\urcorner \Rightarrow \Delta}{\Gamma, T\ulcorner A\urcorner \Rightarrow \Delta}\ (T\text{L}_C) \qquad \frac{\Gamma \Rightarrow A, T\ulcorner A\urcorner, \Delta}{\Gamma \Rightarrow T\ulcorner A\urcorner, \Delta}\ (T\text{R}_C)$$

The extra occurrence of $T\ulcorner A\urcorner$ as active formula in the premise-sequents has the same function as $\forall x A(x)$ in ($\forall\text{L}_C$), namely to ensure admissibility of contraction by effectively absorbing extra occurrences of paradoxical sentences. For an illustration, see theorem 2.6 below.

We are finally in a position to define our sequent calculus for STT based on Robinson Arithmetic.

Definition 2.1. *Let* $\text{SC}_{\text{STT}_{\text{Q+}}}$ *be a sequent calculus system for sequents of the form* $\Gamma \Rightarrow \Delta$ *with* Γ *and* Δ *being multisets of* $\mathcal{L}_T^{A+}$*-formulas. The system consists of the initial sequents of the form*

$$\Gamma, P \Rightarrow P, \Delta$$

where P *is an atomic* $\mathcal{L}_T^{A+}$*-formula, and the rules* ($\neg$L), ($\neg$R), ($\vee$L), ($\vee$R), ($\forall\text{L}_C$), ($\forall$R), (=ID), (=*rep*R_C) *and* (=*rep*L_C), ($Q1$)-($Q7$) *the corresponding rules for each extra function-symbol added to* $\mathcal{L}_T^{A+}$*, in particular* ($O1$), ($O2$) *and* ($\underset{\cdot}{T}$)*, and finally* ($T\text{L}_C$) *and* ($T\text{R}_C$)*.*

To observe a few lemmas, we first adopt the definition of the height of a derivation by Negri and von Plato (2001, p.30):

Definition 2.2. *The height of a derivation in* $\text{SC}_{\text{STT}_{\text{Q+}}}$ *is the greatest number of successive applications of rules in it, where initial sequents and zero-premise rules have height 0.*

Lemma 2.3. *The rules of weakening are admissible in* $\text{SC}_{\text{STT}_{\text{Q+}}}$*:*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ (\text{WL}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}\ (\text{WR})$$

Proof. The proof is by induction on the height of a derivation and proceeds along the lines of Negri and von Plato (2001, p.31). □

Lemma 2.4. *Every one- and two-premise rule in* $\text{SC}_{\text{STT}_{\text{Q+}}}$ *is height-preserving invertible: if there is a derivation with at most height* n *of the conclusion-sequent then there is a derivation with at most height* n *of the premise-sequent(s).*

Proof. We distinguish between the trivial and the non-trivial cases. Inversion is trivial in cases where the principal formulae are also active formulae in the premise-sequent(s). The other cases are established by induction on the height of a derivation and proceeds along the lines of Negri and von Plato (2001, p.32,p.49,p.71). □

Lemma 2.5. *The rules of contraction are admissible in* $\textsc{sc}_{\textsc{stt}_{Q+}}$*:*

$$\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\,(\textsc{cl}) \qquad \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}\,(\textsc{cr})$$

Proof. By induction on the height of a derivation along the lines of Negri and von Plato (2001). □

We now observe the following:

Theorem 2.6. $\textsc{sc}_{\textsc{stt}_{Q+}}$ *is inconsistent. There is a formula* A *such that the sequents* $\Rightarrow A$ *and* $\Rightarrow \neg A$ *are derivable.*

Proof. Robinson Arithmetic proves the diagonal lemma according to which for every formula ϕ with one free variable x there is a formula ψ such that $\psi \leftrightarrow \phi\ulcorner\psi\urcorner$ is a theorem.[7] It follows that the sequent $\Rightarrow \lambda \leftrightarrow \neg T\ulcorner\lambda\urcorner$ is derivable in $\textsc{sc}_{\textsc{stt}_{Q+}}$ which through the invertibility of the rules for $\neg$ and $\vee$ amounts to that the sequents $\Rightarrow \lambda, T\ulcorner\lambda\urcorner$ and $\lambda, T\ulcorner\lambda\urcorner \Rightarrow$ are derivable. We can now proceed as follows:

$$\frac{\Rightarrow \lambda, T\ulcorner\lambda\urcorner}{\Rightarrow T\ulcorner\lambda\urcorner}\,(T\textsc{r}_C) \qquad \cfrac{\cfrac{\lambda, T\ulcorner\lambda\urcorner \Rightarrow}{T\ulcorner\lambda\urcorner \Rightarrow}\,(T\textsc{l}_C)}{\Rightarrow \neg T\ulcorner\lambda\urcorner}\,(\neg\textsc{r})$$

□

It is important to notice that neither the formula $T\ulcorner\lambda\urcorner$ nor the formula λ are arithmetical in the sense that they consists only of arithmetical vocabulary. Instead, they both contain a predicate T which, in virtue of being transparent, is not definable in arithmetic as established by Tarski (1983).

This concludes our presentation of the sequent calculus and anyone familiar with sequent calculus for STT should be in position to see that $\textsc{sc}_{\textsc{stt}_{Q+}}$ delivers STT based on Robinson Arithmetic. The sceptical reader can confer with Ripley (2013b) and Ripley (2013a). We have after all merely augmented a cut-free sequent calculus for classical logic with a transparent truth-predicate while ensuring that the logic remains reflexive, contractive and monotonic.

[7] Cf. (Smith 2007).

2.2 Proving ω-inconsistency without cuts

To provide a cut-free derivation of ω-inconsistency in $\textsc{sc}_{\textsc{stt}_{\textsc{q}+}}$ a quick dissection of the formulae in the diagonal lemma is required.

The diagonal lemma states roughly that for every formula $\phi(x)$ with one free variable there is a formula ψ such that $\phi\ulcorner\psi\urcorner \leftrightarrow \psi$ is derivable. The key to the diagonal lemma is the observation that the function f_{diag} is recursively enumerable, a function that, when applied to the Gödel-code of some formula $\phi(x)$ with just x free, returns the Gödel-code of the diagonalization of $\phi(x)$ defined as $\phi(\ulcorner\phi(x)\urcorner))$, that is, $f_{\text{diag}}(\ulcorner\phi(x)\urcorner) = \ulcorner\phi(\ulcorner\phi(x)\urcorner))\urcorner$. Following for example Meadows (2015), we let the formula ψ be a formula of the form $\exists x(D(\ulcorner\chi(x)\urcorner, x) \wedge \phi(x))$ where $\chi(x)$ is $\exists y(D(x,y) \wedge \phi(y))$ and D is a Σ^0_1-formula capturing the function f_{diag} in the sense that $\textsc{sc}_{\textsc{stt}_{\textsc{q}+}} \vdash\Rightarrow \forall y(D(\overline{n}, y) \leftrightarrow y = \overline{m})$ if and only if $f_{diag}(n) = m$.

In line with the above clarifications and Halbach (2011) we let μ be the formula $\exists x(D(\ulcorner\chi(x)\urcorner, x) \wedge \neg\forall yT\underset{\cdot}{f}(y,x))$ where $\chi(x)$ is $\exists y(D(x,y) \wedge \neg\forall zT\underset{\cdot}{f}(z,y))$. Our relevant instance of the diagonal lemma becomes thus $\mu \leftrightarrow \neg\forall yT\underset{\cdot}{f}(y,\ulcorner\mu\urcorner)$.

We now provide a few lemmas.

Lemma 2.7. *The following rule is admissible in* $\textsc{sc}_{\textsc{stt}_{\textsc{q}+}}$ *iff* $f_{diag}(n) = m$*:*

$$\frac{\Gamma, a = \overline{m} \Rightarrow \Delta}{\Gamma, D(\overline{n}, a) \Rightarrow \Delta}$$

Proof. The admissibility of CUT for arithmetical formulae and the usual rules for the (defined) biconditional guarantees that if $\Rightarrow A \leftrightarrow B$, $A \Rightarrow A$ and $B \Rightarrow B$ then, if $\Gamma, A \Rightarrow \Delta$ then $\Gamma, B \Rightarrow \Delta$. First, we obtain $B \Rightarrow A$ from $\Rightarrow A \leftrightarrow B$ and $A \leftrightarrow B, B \Rightarrow A$ with CUT and then furthermore $\Gamma, B \Rightarrow \Delta$ from $\Gamma, A \Rightarrow \Delta$ and $B \Rightarrow A$, again with CUT. Analogous for the other direction. □

Lemma 2.8. *There is a derivation of* $\mu, \forall yT\underset{\cdot}{f}(y,\ulcorner\mu\urcorner) \Rightarrow$ *in* $\textsc{sc}_{\textsc{stt}_{\textsc{q}+}}$.

Proof. This is guaranteed by the invertibility of ¬R. It can also be derived directly as follows

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{a = \ulcorner\mu\urcorner, \forall yT\underset{\cdot}{f}(y,\ulcorner\mu\urcorner) \Rightarrow \forall yT\underset{\cdot}{f}(y,\ulcorner\mu\urcorner)}{a = \ulcorner\mu\urcorner, \neg\forall yT\underset{\cdot}{f}(y,\ulcorner\mu\urcorner), \forall yT\underset{\cdot}{f}(y,\ulcorner\mu\urcorner) \Rightarrow}\ (\neg\text{L})}{a = \ulcorner\mu\urcorner, \neg\forall yT\underset{\cdot}{f}(y,a), \forall yT\underset{\cdot}{f}(y,\ulcorner\mu\urcorner) \Rightarrow}\ (= rep\text{L}_C)}{D(\ulcorner\chi(x)\urcorner, a), \neg\forall yT\underset{\cdot}{f}(y,a), \forall yT\underset{\cdot}{f}(y,\ulcorner\mu\urcorner) \Rightarrow}\ \text{Lemma 2.7}}{D(\ulcorner\chi(x)\urcorner, a) \wedge \neg\forall yT\underset{\cdot}{f}(y,a), \forall yT\underset{\cdot}{f}(y,\ulcorner\mu\urcorner) \Rightarrow}\ (\text{L}\wedge)}{\exists x(D(\ulcorner\chi(x)\urcorner, x) \wedge \neg\forall yT\underset{\cdot}{f}(y,x)), \forall yT\underset{\cdot}{f}(y,\ulcorner\mu\urcorner) \Rightarrow}\ (L\exists)$$

□

Lemma 2.9. *The following rule is admissible in* $\textsc{sc}_{\textsc{stt}_{Q+}}$*:*

$$\frac{\Gamma \Rightarrow \neg\forall y T \underset{\cdot}{f}(y, \ulcorner\mu\urcorner), \Delta}{\Gamma \Rightarrow \mu, \Delta}$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{D(\ulcorner\chi(x)\urcorner, \ulcorner\mu\urcorner) \Rightarrow D(\ulcorner\chi(x)\urcorner, \ulcorner\mu\urcorner)}{\ulcorner\mu\urcorner = \ulcorner\mu\urcorner \Rightarrow D(\ulcorner\chi(x)\urcorner, \ulcorner\mu\urcorner)}\ \text{Lemma 2.7}}{\Rightarrow D(\ulcorner\chi(x)\urcorner, \ulcorner\mu\urcorner)}\ (= \textsc{id}) \qquad \Gamma \Rightarrow \neg\forall y T \underset{\cdot}{f}(y, \ulcorner\mu\urcorner), \Delta}{\Gamma \Rightarrow D(\ulcorner\chi(x)\urcorner, \ulcorner\mu\urcorner) \wedge \neg\forall y T \underset{\cdot}{f}(y, \ulcorner\mu\urcorner), \Delta}\ (\wedge\textsc{r})}{\Gamma \Rightarrow \exists x(D(\ulcorner\chi(x)\urcorner, x) \wedge \neg\forall y T \underset{\cdot}{f}(y, x)), \Delta}\ (\exists\textsc{r})$$

□

And now ω-inconsistency.

Theorem 2.10. $\textsc{sc}_{\textsc{stt}_{Q+}}$ *is ω-inconsistent: There is a formula A such that* $\textsc{sc}_{\textsc{stt}_{Q+}} \vdash\Rightarrow \neg\forall x Ax$ *and* $\textsc{sc}_{\textsc{stt}_{Q+}} \vdash\Rightarrow A(\overline{n})$ *for every natural number n.*

Proof. First, we derive $\Rightarrow \neg\forall x Ax$ from the following instance of lemma 2.8 with $\ulcorner\mu\urcorner = \underset{\cdot}{f}(0, \ulcorner\mu\urcorner)$ and $T\ulcorner\mu\urcorner$ weakened in:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\ulcorner\mu\urcorner = \underset{\cdot}{f}(0, \ulcorner\mu\urcorner), T\ulcorner\mu\urcorner, \mu, \forall y T \underset{\cdot}{f}(y, \ulcorner\mu\urcorner) \Rightarrow}{\ulcorner\mu\urcorner = \underset{\cdot}{f}(0, \ulcorner\mu\urcorner), T\ulcorner\mu\urcorner, \forall y T \underset{\cdot}{f}(y, \ulcorner\mu\urcorner) \Rightarrow}\ (T\textsc{l}_C)}{\ulcorner\mu\urcorner = \underset{\cdot}{f}(0, \ulcorner\mu\urcorner), T\underset{\cdot}{f}(0, \ulcorner\mu\urcorner), \forall y T \underset{\cdot}{f}(y, \ulcorner\mu\urcorner) \Rightarrow}\ (= rep\textsc{l}_C)}{\ulcorner\mu\urcorner = \underset{\cdot}{f}(0, \ulcorner\mu\urcorner), \forall y T \underset{\cdot}{f}(y, \ulcorner\mu\urcorner) \Rightarrow}\ (\forall\textsc{l}_C)}{\forall y T \underset{\cdot}{f}(y, \ulcorner\mu\urcorner) \Rightarrow}\ (O1)}{\Rightarrow \neg\forall y T \underset{\cdot}{f}(y, \ulcorner\mu\urcorner)}\ (\neg\textsc{r})$$

To derive $\Rightarrow T\underset{\cdot}{f}(0, \ulcorner\mu\urcorner)$, we continue from $\Rightarrow \neg\forall y T \underset{\cdot}{f}(y, \ulcorner\mu\urcorner)$ with $\ulcorner\mu\urcorner = \underset{\cdot}{f}(0, \ulcorner\mu\urcorner)$ and $T\ulcorner\mu\urcorner$ weakened in:

$$\dfrac{\dfrac{\dfrac{\dfrac{\ulcorner\mu\urcorner = \underset{\cdot}{f}(0, \ulcorner\mu\urcorner) \Rightarrow \neg\forall y T \underset{\cdot}{f}(y, \ulcorner\mu\urcorner), T\ulcorner\mu\urcorner}{\ulcorner\mu\urcorner = \underset{\cdot}{f}(0, \ulcorner\mu\urcorner) \Rightarrow \mu, T\ulcorner\mu\urcorner}\ \text{Lemma 2.9}}{\ulcorner\mu\urcorner = \underset{\cdot}{f}(0, \ulcorner\mu\urcorner) \Rightarrow T\ulcorner\mu\urcorner}\ (T\textsc{r})}{\ulcorner\mu\urcorner = \underset{\cdot}{f}(0, \ulcorner\mu\urcorner) \Rightarrow T\underset{\cdot}{f}(0, \ulcorner\mu\urcorner)}\ (= rep\textsc{r}_C)}{\Rightarrow T\underset{\cdot}{f}(0, \ulcorner\mu\urcorner)}\ (\text{O1})$$

To derive $\Rightarrow T\underset{\cdot}{f}(\overline{n}, \ulcorner\mu\urcorner)$ for every n we establish that the following rule is admissible:

$$\frac{\Gamma \Rightarrow T\underset{\cdot}{f}(a, \ulcorner\mu\urcorner), \Delta}{\Gamma \Rightarrow T\underset{\cdot}{f}(sa, \ulcorner\mu\urcorner), \Delta}$$

The proof is straight-forward where we weaken in formulae when they are relevant for readability:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow T\underset{\cdot}{f}(a, \ulcorner\mu\urcorner), T\ulcorner T\underset{\cdot}{f}(a, \ulcorner\mu\urcorner)\urcorner, \Delta}{\Gamma \Rightarrow T\ulcorner T\underset{\cdot}{f}(a, \ulcorner\mu\urcorner)\urcorner, \Delta}\;(T\mathrm{R}^G_C)}{\Gamma, \ulcorner T\underset{\cdot}{f}(a, \ulcorner\mu\urcorner)\urcorner = \underset{\cdot}{T}\underset{\cdot}{f}(a, \ulcorner\mu\urcorner) \Rightarrow T\underset{\cdot}{T}\underset{\cdot}{f}(a, \ulcorner\mu\urcorner), \Delta}\;(=rep\mathrm{R}_C)}{\Gamma \Rightarrow T\underset{\cdot}{T}\underset{\cdot}{f}(a, \ulcorner\mu\urcorner), \Delta}\;(\underset{\cdot}{T})}{\Gamma, \underset{\cdot}{T}\underset{\cdot}{f}(a, \ulcorner\mu\urcorner) = \underset{\cdot}{f}(sn, \ulcorner\mu\urcorner) \Rightarrow T\underset{\cdot}{f}(sn, \ulcorner\mu\urcorner), \Delta}\;(=rep\mathrm{R}_C)}{\Gamma \Rightarrow T\underset{\cdot}{f}(sa, \ulcorner\mu\urcorner), \Delta}\;(\mathrm{O2})$$

We can now proceed to apply this rule on the sequent $\Rightarrow T\underset{\cdot}{f}(0, \ulcorner\mu\urcorner)$ from above to conclude $\Rightarrow T\underset{\cdot}{f}(s0, \ulcorner\mu\urcorner)$ and furthermore $\Rightarrow T\underset{\cdot}{f}(ss0, \ulcorner\mu\urcorner)$ and so on:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow T\underset{\cdot}{f}(ss0, \ulcorner\mu\urcorner)}{\Rightarrow T\underset{\cdot}{f}(sss0, \ulcorner\mu\urcorner)}}{\Rightarrow T\underset{\cdot}{f}(ssss0, \ulcorner\mu\urcorner)}}{\Rightarrow T\underset{\cdot}{f}(sssss0, \ulcorner\mu\urcorner)}}{\dfrac{\Rightarrow T\underset{\cdot}{f}(ssssss0, \ulcorner\mu\urcorner)}{\vdots}}$$

This completes the proof. $\square$

An obvious feature of an ω-inconsistent theory is that it is just an ω-rule away from being simply inconsistent.[8] Indeed, if we add the following rule,

$$\frac{\Rightarrow A(\overline{n}) \text{ (for each } n \in \omega)}{\Rightarrow \forall x A(x)}$$

we can derive $\Rightarrow \forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner)$ and we would have another inconsistency. However, since the logic is already inconsistent, the fact that an ω-rule introduces an inconsistency cannot be anything more than a curiosity.

On the other hand, one might think that the result in this section suggests that there is a general relationship between inconsistency and ω-inconsistency,

[8]See for example (Leitgeb 2001) and (Barrio 2010).

and in particular that a non-trivial inconsistent theory satisfying certain conditions will be ω-inconsistent. An anonymous referee pointed for example out that the following derivations are available in $\textsc{sc}_{\textsc{stt}_{\textsc{q}+}}$:

$$\dfrac{\dfrac{\vdots}{\Rightarrow T\ulcorner\lambda\urcorner} \qquad \dfrac{\vdots}{\Rightarrow n = n}}{\Rightarrow T\ulcorner\lambda\urcorner \wedge n = n} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{T\ulcorner\lambda\urcorner, \overline{0} = \overline{0} \Rightarrow}}{T\ulcorner\lambda\urcorner \wedge \overline{0} = \overline{0} \Rightarrow}}{\forall x(T\ulcorner\lambda\urcorner \wedge x = x) \Rightarrow}}{\Rightarrow \neg\forall x(T\ulcorner\lambda\urcorner \wedge x = x)}$$

Indeed, as soon as a logic is inconsistent and we have our standard rules for equality, conjunction and weakening, and can introduce a negation to the right, we have ω-inconsistency. We didn't even require functions definable in a theory of arithmetic as in the case above. However, it is precisely because we didn't use any arithmetical vocabulary that I think we should be wary of the theoretical significance of such derivations; they merely piggy-back on the inconsistency as opposed to the derivations for theorem 2.10.

Now, that doesn't mean that there is not an intimate relationship between ω-inconsistency and inconsistency, and in particular that we can expect non-trivial inconsistent theories of truth to become ω-inconsistent when we add arithmetical machinery. It is for example easy to show that augmenting LP with transparent arithmetical truth results in ω-inconsistency. I will however not explore this relationship any further in this paper, but rather turn my attention to a consistent but ω-inconsistent theory of truth.

3 ω-inconsistency with DT based on Robinson Arithmetic

3.1 A sequent calculus for McGee's conditions

Cobreros et al. (2013) claimed that the result in McGee (1985) relies on transitivity, and while we have shown that there is a cut-free derivation of ω-inconsistency in STT based on an arithmetical theory, it might still be the case that the original result requires transitivity. I shall now show that we have good reasons to think that this is not the case. Our strategy will be to identify McGee's conditions as quantified standard deontic logic, provide a cut-free sequent calculus for quantified standard deontic logic where the modality is treated as a predicate and finally show that the resulting logic is ω-inconsistent.

McGee (1985) shows that a logic of truth satisfying the following conditions is ω-inconsistent:[9]

[9] I deviate slightly from the conditions as stated in (McGee 1985) by adopting the formulation of (a), (b) and (c) in (Leitgeb 2001) for presentational purposes.

(1) Closed under first-order classical logic with equality.

(2) Contains the axioms of Robinson Arithmetic and axioms for f.

(3) Contains $T\ulcorner A\urcorner$ as theorem if it contains A as theorem.

(4) Contains every instance of the following schemas:

(4a) $T\ulcorner\neg A\urcorner \to \neg T\ulcorner A\urcorner$

(4b) $T\ulcorner A \to B\urcorner \to T\ulcorner A\urcorner \to T\ulcorner B\urcorner$

(4c) $\forall x T\ulcorner A\dot{x}\urcorner \to (T\ulcorner\forall x A x\urcorner)$

$\ulcorner A\dot{a}\urcorner$ in (4c) is short-hand for $\underset{\cdot}{f_s}(\overline{\ulcorner A\urcorner}, a)$ where $\underset{\cdot}{f_s}$ is a function such that $f_s(\ulcorner A\urcorner, a) = \ulcorner A(z/a)\urcorner$, that is, the function which returns the Gödel-code of the formula obtained by substituting a for z in a formula A with exactly one free variable z. The language contains thus a function-symbol $\underset{\cdot}{f_s}$ which is defined with the appropriate rules, namely such that $\underset{\cdot}{f_s}(\overline{\ulcorner A\urcorner}, a) = \overline{\ulcorner A(z/a)\urcorner}$ becomes a theorem for every formula A and term a.

A quick inspection of these principles reveals that they are essentially the axioms of the quantified standard deontic logic as obtained on frame semantics with constant domain and serial accessibility-relation.[10] Such frames will with a standard definition of the $\Box$ validate the principles corresponding to (3), (4a) and (4b) as formulated with $\Box$. (4a) is equivalent to $T\ulcorner A\urcorner \to \neg T\ulcorner\neg A\urcorner$ in classical logic which corresponds to the principle which ensures that obligations are consistent. If the frames are defined for a first-order language (still with an operator $\Box$) with constant domain, then those frames will also validate (4c) for $\Box$ which is known as *the Barcan formula*. It follows that if we can formulate a cut-free sequent calculus for quantified deontic logic in which the modality is treated as a predicate rather than an operator, that calculus, if augmented with rules for Robinson Arithmetic, should be equivalent to McGee's conditions and we should be able to prove that it is ω-inconsistent. Since we will continue using a language containing a truth-predicate, we might as well call the resulting logic *Deontic Truth* and pretend that it is based on a conception of truth according to which truth should be conceived as a deontic modality that satisfies the conditions associated with quantified standard deontic logic. As we shall see below, it is not actually too far-fetched and comes with certain virtues.

Now, the conditions (3), (4a) and (4b) are derivable in a cut-free sequent calculus for classical logic closed under the following rule where $T\ulcorner\Gamma\urcorner$ abbreviates $T\ulcorner\gamma\urcorner$ for each $\gamma \in \Gamma$ and which permits the special case in which there

[10]See for example (McNamara 2010).

is no active formula on the right-hand-side:[11]

$$\frac{\Gamma \Rightarrow A}{T\ulcorner\Gamma\urcorner \Rightarrow T\ulcorner A\urcorner}\ \text{(T-dist)}$$

The fact that our *Deontic Truth* is supposed to satisfy (T-dist) illustrates a virtue with such a conception of truth, namely how it expresses that entailment is truth-preserving.

(T-dist) is however a rather impractical rule. Not only is it not "compositional" by following the left- and right-introduction rule format and cannot accommodate admissibility of weakening in an elegant way. It is also not sufficient to derive (4c). While we can derive its converse with (T-dist), (4c) requires either a specific rule for quantifiers such as those presented by Halbach (2011, p.70) or a more suitable calculus, for example one based on hypersequents or labelled sequents. I shall employ a labelled sequent calculus so as to easily ensure that we are working with nothing over and above quantified standard deontic logic.[12]

The basic idea with a labelled sequent calculus is to internalize in the sequent calculus the machinery employed in frame semantics by adding labels that "represent" the points of evaluation at which the formulae are assigned a value. Sequents are thus not pairs of multisets of formulae but rather multisets of label-formulae pairs of the form $i : A$ where i is a label and A is a formula. In addition to expressing at which point of evaluation a formula is true at, we should also incorporate facts about the accessibility-relation. Following Negri (2005), we incorporate such features of the frame semantics in a sequent calculus by permitting also pairs of the form iRj where i and j are labels. The labelled sequent calculus for quantified modal logic discussed by Negri and von Plato (2011) involves also pairs of the form $t \in D(i)$ and extra rules for such pairs to mimic features such as variable or constant domains, increasing or decreasing domains and the empty domain. We can ignore those features of quantified modal logic because we will think of ω as our domain.

A labelled calculus comes thus with a countable set of labels. While Negri (2005) considers the extra resources as part of the language, one can also, as suggested by Restall (2006), consider them as structural features of sequents. I prefer the second option but nothing hinges on that here.

[11]We are thus combining the standard sequent calculus rule to obtain K and the extra rule one typically adds to obtain KD. The rule to obtain K requires an active formula in the succedent and the extra rule one typically adds to obtain KD requires no active formula in the succedent. See (Negri 2011).

[12]The alternative to the use of labelled sequent calculus to obtain a cut-free calculus for *Deontic Truth* would be to employ tree-hypersequents as introduced by Poggiolesi (2011).

In addition to initial sequents of the form $\Gamma, i : P \Rightarrow i : P, \Delta$ where Γ and Δ are multisets of pairs of either form and P is an atomic formula, we add the following rules for T where j does not occur in Γ, Δ and $i : T\ulcorner A \urcorner$ in ($T\textsc{r}_L$):

$$\frac{iRj, i : T\ulcorner A\urcorner, j : A, \Gamma \Rightarrow \Delta}{iRj, i : T\ulcorner A\urcorner, \Gamma \Rightarrow \Delta}\ (T\textsc{l}^L_C) \qquad \frac{iRj, \Gamma \Rightarrow \Delta, j : A}{\Gamma \Rightarrow \Delta, i : T\ulcorner A\urcorner}\ (T\textsc{r}^L)$$

For the other connectives, arithmetical vocabulary and equality we simply relativize the above rules to a label i in the natural way:[13]

$$\frac{i : A, \Gamma \Rightarrow \Delta \qquad i : B, \Gamma \Rightarrow \Delta}{i : A \vee B, \Gamma \Rightarrow \Delta}\ (\vee\textsc{l}^L) \qquad \frac{\Gamma \Rightarrow \Delta, i : A, i : B}{\Gamma \Rightarrow \Delta, i : A \vee B}\ (\vee\textsc{r}^L)$$

$$\frac{\Gamma \Rightarrow \Delta, i : A}{\Gamma, i : \neg A \Rightarrow \Delta}\ (\neg\textsc{l}^L) \qquad \frac{i : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow i : \neg A, \Delta}\ (\neg\textsc{r}^L)$$

$$\frac{i : \forall x A(x), i : A(a), \Gamma \Rightarrow \Delta}{i : \forall x A(x), \Gamma \Rightarrow \Delta}\ (\forall\textsc{l}^L_C) \qquad \frac{\Gamma \Rightarrow \Delta, i : A(y)}{\Gamma \Rightarrow \Delta, i : \forall x A}\ (\forall\textsc{r}^L)$$

$$\frac{i : a = a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (=\textsc{id}^L)$$

$$\frac{i : A(a), i : a = b, i : A(b), \Gamma \Rightarrow \Delta}{i : a = b, i : A(a), \Gamma \Rightarrow \Delta}\ (=rep\textsc{l}^L_C) \qquad \frac{i : a = b, \Gamma \Rightarrow i : A(a), i : A(b), \Delta}{i : a = b, \Gamma \Rightarrow A(b), \Delta}\ (=rep\textsc{r}^L_C)$$

$$\frac{}{\Gamma, i : 0 = sa \Rightarrow \Delta}\ (\mathrm{Q1}^L) \qquad \frac{\Gamma, i : a = b, i : sa = sb \Rightarrow \Delta}{\Gamma, i : sa = sb \Rightarrow \Delta}\ (\mathrm{Q2}^L)$$

$$\frac{\Gamma, j : a = 0 \Rightarrow \Delta \qquad \Gamma, j : a = sy \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\mathrm{Q3}^L)$$

$$\frac{\Gamma, i : a + 0 = a \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\mathrm{Q4}^L) \qquad \frac{\Gamma, i : a + sb = s(a + b) \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\mathrm{Q5}^L)$$

$$\frac{\Gamma, i : a \times 0 = 0 \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\mathrm{Q6}^L) \qquad \frac{\Gamma, i : a \times sb = (a \times b) + a \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\mathrm{Q7}^L)$$

$$\frac{\Gamma, i : \underset{\cdot}{f}(0, \ulcorner A\urcorner) = \ulcorner A\urcorner \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\mathrm{O1}^L) \qquad \frac{\Gamma, i : \underset{\cdot}{f}(sa, \ulcorner A\urcorner) = \underset{\cdot}{T}\underset{\cdot}{f}(a, \ulcorner A\urcorner) \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\mathrm{O2}^L)$$

$$\frac{i : \underset{\cdot}{f}_s(\ulcorner A\urcorner, a) = \ulcorner A(z/a)\urcorner, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\dot{n}^L) \qquad \frac{i : \underset{\cdot}{T}a = \ulcorner Ta\urcorner, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\underset{\cdot}{T}^L)$$

where y in ($\forall$R) does not occur in Γ,Δ or $\forall x A$, and where j in (Q3) does not occur in Γ or Δ.

[13] See for example (Negri 2005, p.512).

We can now derive (4c) as follows, omitting the various double occurrences of formulae required to ensure admissibility of contraction:

$$\begin{array}{rl}
iRj, j : A(a) \Rightarrow j : A(a) & \\ \hline
iRj, i : \ulcorner A(a)\urcorner = \ulcorner A\dot{a}\urcorner, i : T\ulcorner A(a)\urcorner \Rightarrow j : A(a) & (T\mathrm{L}^L_C) \\ \hline
iRj, i : \ulcorner A(a)\urcorner = \ulcorner A\dot{a}\urcorner, i : T\ulcorner A\dot{a}\urcorner \Rightarrow j : A(a) & (=rep\mathrm{L}^L_C) \\ \hline
iRj, i : T\ulcorner A\dot{a}\urcorner \Rightarrow j : A(a) & (\dot{n}^L) \\ \hline
iRj, i : \forall x T\ulcorner A\dot{x}\urcorner \Rightarrow j : A(a) & (\forall\mathrm{L}^L) \\ \hline
iRj, i : \forall x T\ulcorner A\dot{x}\urcorner \Rightarrow j : \forall x A(x) & (\forall\mathrm{R}^L) \\ \hline
i : \forall x T\ulcorner A\dot{x}\urcorner \Rightarrow i : T\ulcorner \forall x A(x)\urcorner & (T\mathrm{R}^L)
\end{array}$$

In addition to rules for our vocabulary, we also require a rule for R that makes it serial. Negri (2005) observes that seriality is mimicked by the following rule:

$$\dfrac{iRj, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{Ser}$$

where j does not occur in Γ or Δ. To illustrate the rule, consider the following derivation of (2):

$$\begin{array}{rl}
iRj, j : A \Rightarrow j : A & \\ \hline
iRj, j : A, j : \neg A \Rightarrow & (\neg\mathrm{L}^L) \\ \hline
iRj, j : A, i : T\ulcorner \neg A\urcorner \Rightarrow & (T\mathrm{L}^L_C) \\ \hline
iRj, i : T\ulcorner A\urcorner, i : T\ulcorner \neg A\urcorner \Rightarrow & (T\mathrm{L}^L_C) \\ \hline
i : T\ulcorner A\urcorner, i : T\ulcorner \neg A\urcorner \Rightarrow & \mathrm{Ser} \\ \hline
i : T\ulcorner \neg A\urcorner \Rightarrow i : \neg T\ulcorner A\urcorner & (\neg\mathrm{R}^L)
\end{array}$$

In this derivation, iRj is removed when j no longer occurs anywhere else in the sequent.

We can now define our labelled sequent calculus.

Definition 3.1. *Let* $\mathrm{SC}_{\mathrm{DT}_{\mathrm{Q}+}}$ *be a sequent calculus system for sequents of the form* $\Gamma \Rightarrow \Delta$ *with* Γ *and* Δ *being multisets of pairs of the form* $i : A$ *where* i *is a label and* A *a* $\mathcal{L}^{A+}_T$*-formula and pairs of the form* jRi *where* j *and* i *are labels and there is a label* i *for each* $n \in \omega$*. The system consists of the initial sequents of the form*

$$\Gamma, i : P \Rightarrow i : P, \Delta$$

where P *is an atomic* $\mathcal{L}^{A+}_T$*-formula, and the rules Ser,* $(\neg\mathrm{L}^L)$, $(\neg\mathrm{R}^L)$, $(\vee\mathrm{L}^L)$, $(\vee\mathrm{R}^L)$, $(\forall\mathrm{L}^L_C)$, $(\forall\mathrm{R}^L)$, $(=\mathrm{ID}^L)$, $(=rep\mathrm{R}^L_C)$ *and* $(=rep\mathrm{L}^L_C)$*, the rules* $(Q1^L)$*-*$(Q7^L)$*, the corresponding rules for each extra function-symbol added to* $\mathcal{L}^{A+}_T$*, in particular* $(O1^L)$*,* $(O2^L)$ *and* $(\underset{\cdot}{T}^L)$*, and finally* $(T\mathrm{L}^L_C)$ *and* $(T\mathrm{R}^L)$*.*

We do not add ($\dot{n}^L$) because it is not required to derive an ω-inconsistency. It was required above to derive (4c) because we needed a device to push the connective inside the scope of the truth-predicate to derive universal generalizations involving the truth-predicate. Since f_s is a recursively enumerable function, there is anyway a formula capturing it in our system and a corresponding variant of (4c) is still derivable in the system.

Notice that a cut-rule in the case of a labelled sequent calculus like this would amount to the rule

$$\frac{\Gamma \Rightarrow \Delta, i : A \qquad i : A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \text{ (CUT}^L\text{)}$$

and that a derivation in the system would rely on transitivity if it involves rules from which the rule

$$\frac{\Rightarrow i : A \qquad i : A \Rightarrow i : B}{\Rightarrow i : B} \text{ (Tr}^L\text{)}$$

is derivable.

As in the case of $\text{SC}_{\text{STT}_{\text{Q}+}}$ we observe the following lemmas:

Lemma 3.2. *The rules of weakening are admissible in* $\text{SC}_{\text{DT}_{\text{Q}+}}$*:*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, i : A \Rightarrow \Delta} \text{ (WL}^L\text{)} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow i : A, \Delta} \text{ (WR}^L\text{)}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, iRj \Rightarrow \Delta} \text{ (WL}^L_R\text{)} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow iRj, \Delta} \text{ (WR}^L_R\text{)}$$

Proof. The proof is by induction on the height of a derivation and proceeds along the lines of Negri (2005, p.518). □

Lemma 3.3. *Every one- and two-premise rule in* $\text{SC}_{\text{DT}_{\text{Q}+}}$ *is height-preserving invertible: if there is a derivation with at most height n of the conclusion-sequent then there is a derivation with at most height n of the premise-sequent(s).*

Proof. By induction on the height of a derivation and proceeds along the lines of Negri (2005, p.520-521). □

Lemma 3.4. *The rules of contraction are admissible in* $\text{SC}_{\text{DT}_{\text{Q}+}}$*:*

$$\frac{\Gamma, i : A, i : A \Rightarrow \Delta}{\Gamma, i : A \Rightarrow \Delta} \text{ (CL}^L\text{)} \qquad \frac{\Gamma \Rightarrow i : A, i : A, \Delta}{\Gamma \Rightarrow i : A, \Delta} \text{ (CR}^L\text{)}$$

$$\frac{\Gamma, iRj, iRj \Rightarrow \Delta}{\Gamma, iRj \Rightarrow \Delta} \text{ (CL}^L_R\text{)} \qquad \frac{\Gamma \Rightarrow iRj, iRj, \Delta}{\Gamma \Rightarrow iRj, \Delta} \text{ (CR}^L_R\text{)}$$

Proof. By induction on the height of a derivation along the lines of Negri (2005, p.522-523). □

With that concluding our brief discussion of the rules, it is left to clarify that we define a logic using a labelled sequent calculus of this form by stipulating that a set of formulae Γ entails a set of formulae Δ relative to this labelled sequent calculus if and only if a sequent is derivable which contains nothing but $i : \gamma$ for every $\gamma \in \Gamma$ in antecedent position and nothing but $i : \delta$ for every $\delta \in \Delta$ in succedent position where i is an arbitrary label.

I shall not prove that the logic we define with our labelled sequent calculus is equivalent to McGee's conditions. Instead, I merely point out that we have good reasons to assume that our sequent calculus proves nothing over and above McGee's conditions because Negri and von Plato (2011) prove that a calculus with the above rules we used for the predicate T are, when used to define an operator $\Box$, together with Ser for R, sound and complete with regard to serial frames.

3.2 Proving ω-inconsistency without cuts again

We now proceed to establish ω-inconsistency. The first thing to note is that the lemmas we employed in the previous section to derive ω-inconsistency hold also in the current setting.

Lemma 3.5. *The following rule is admissible in* $\text{SC}_{\text{DT}_{\text{Q}+}}$ *iff* $f_{diag}(n) = m$*:*

$$\frac{\Gamma, i : a = \overline{m} \Rightarrow \Delta}{\Gamma, i : D(\overline{n}, a) \Rightarrow \Delta}$$

Lemma 3.6. *The following rules are admissible in* $\text{SC}_{\text{DT}_{\text{Q}+}}$*:*

$$\frac{\Gamma \Rightarrow i : \neg\forall y T \underset{\cdot}{f}(y, \ulcorner\mu\urcorner), \Delta}{\Gamma \Rightarrow i : \mu, \Delta}\ (a) \qquad \frac{\Gamma, i : \neg\forall y T \underset{\cdot}{f}(y, \ulcorner\mu\urcorner) \Rightarrow \Delta}{\Gamma, i : \mu \Rightarrow \Delta}\ (b)$$

Proof. With the proof of (a) being more or less the same as the proof of Lemma 2.9, I prove here only (b).

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Gamma, i : \neg\forall y T \underset{\cdot}{f}(y, \ulcorner\mu\urcorner) \Rightarrow \Delta}{\Gamma, i : a = \ulcorner\mu\urcorner, i : \neg\forall y T \underset{\cdot}{f}(y, \ulcorner\mu\urcorner) \Rightarrow \Delta}}{\Gamma, i : a = \ulcorner\mu\urcorner, i : \neg\forall y T \underset{\cdot}{f}(y, a) \Rightarrow \Delta}\ (= rep\text{L}^L_C)}{\Gamma, i : D(\ulcorner\chi(x)\urcorner, a), i : \neg\forall y T \underset{\cdot}{f}(y, a) \Rightarrow \Delta}\ \text{Lemma 3.5}}{\Gamma, i : D(\ulcorner\chi(x)\urcorner, a) \wedge \neg\forall y T \underset{\cdot}{f}(y, a) \Rightarrow \Delta}\ (\wedge\text{L}^L)}{\Gamma, i : \exists x(D(\ulcorner\chi(x)\urcorner, x) \wedge \neg\forall y T \underset{\cdot}{f}(y, x)) \Rightarrow \Delta}\ (\exists\text{L}^L)$$

□

Theorem 3.7. $\text{SC}_{\text{DT}_{\text{Q+}}}$ *is* ω*-inconsistent: There is a formula* A *such that* $\text{SC}_{\text{DT}_{\text{Q+}}} \vdash\Rightarrow i : \neg\forall x Ax$ *and* $\text{SC}_{\text{DT}_{\text{Q+}}} \vdash\Rightarrow i : A(\overline{n})$ *for every natural number* n.

Proof. We first establish that there is a formula A such that $\text{SC}_{\text{DT}_{\text{Q+}}} \vdash\Rightarrow i : \neg\forall x Ax$ with following derivation in which we omit the introduction and elimination of equality-premises for presentational purposes.

$$\begin{array}{rl}
iRj,\ j : T\underset{\cdot}{f}(a, \ulcorner\mu\urcorner) \Rightarrow j : T\underset{\cdot}{f}(a, \ulcorner\mu\urcorner) & \\ \hline
iRj,\ i : T\underset{\cdot}{T}\underset{\cdot}{f}(a, \ulcorner\mu\urcorner) \Rightarrow j : T\underset{\cdot}{f}(a, \ulcorner\mu\urcorner) & (T\text{L}^L_C/\underset{\cdot}{T}^L) \\ \hline
iRj,\ i : T\underset{\cdot}{f}(Sa, \ulcorner\mu\urcorner) \Rightarrow j : T\underset{\cdot}{f}(a, \ulcorner\mu\urcorner) & (=rep\text{R}^L_C)/(\text{O2}^L) \\ \hline
iRj,\ i : \forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner) \Rightarrow j : T\underset{\cdot}{f}(a, \ulcorner\mu\urcorner) & (\forall\text{L}^L) \\ \hline
iRj,\ i : \forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner) \Rightarrow j : \forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner) & (\forall\text{R}^L) \\ \hline
iRj,\ i : \forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner), j : \mu \Rightarrow & \text{Lemma 3.6} \\ \hline
iRj,\ i : \forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner), i : T\ulcorner\mu\urcorner \Rightarrow & (T\text{L}^L_C) \\ \hline
iRj,\ i : \forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner), i : T\underset{\cdot}{f}(0, \ulcorner\mu\urcorner) \Rightarrow & (=rep\text{R}^L_C)/(\text{O1}^L) \\ \hline
iRj,\ i : \forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner) \Rightarrow & (\forall\text{L}^L) \\ \hline
i : \forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner) \Rightarrow & \text{Ser} \\ \hline
\Rightarrow i : \neg\forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner) & (\neg\text{R}^L)
\end{array}$$

To also establish that $\text{SC}_{\text{DT}_{\text{Q+}}} \vdash\Rightarrow i : A(\overline{n})$ for every natural number n we proceed as follows. First we note that the above derivation goes through with the sequent $iRj, jRj',\ j' : T\underset{\cdot}{f}(a, \ulcorner\mu\urcorner) \Rightarrow j' : T\underset{\cdot}{f}(a, \ulcorner\mu\urcorner)$ as initial sequent to derive the sequent $iRj \Rightarrow j : \neg\forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner)$. We can then proceed as follows:

$$\begin{array}{rl}
iRj \Rightarrow j : \neg\forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner) & \\ \hline
iRj \Rightarrow j : \mu & \text{Lemma 3.6} \\ \hline
\Rightarrow i : T\ulcorner\mu\urcorner & ((T\text{R}_L)) \\ \hline
\Rightarrow i : T\underset{\cdot}{f}(0, \ulcorner\mu\urcorner) & (=rep\text{R}_C)/(\text{O1})
\end{array}$$

This is the first of the of sequents required for ω-inconsistency. We now observe that the following rule is admissible:

$$\frac{\Gamma, iRj \Rightarrow j : T\underset{\cdot}{f}(a, \ulcorner\mu\urcorner), \Delta}{\Gamma \Rightarrow i : T\underset{\cdot}{f}(sa, \ulcorner\mu\urcorner), \Delta}$$

The derivation is straight-forward:

$$
\begin{array}{rl}
iRj, \Gamma \Rightarrow j : T\underset{\cdot}{f}(a, \ulcorner \mu \urcorner), \Delta & \\
\hline
\Gamma \Rightarrow i : T\ulcorner T\underset{\cdot}{f}(a, \ulcorner \mu \urcorner)\urcorner, \Delta & (T\mathrm{R}_L) \\
\hline
i : \ulcorner T\underset{\cdot}{f}(a, \ulcorner \mu \urcorner)\urcorner = \underset{\cdot}{T}\underset{\cdot}{f}(a, \ulcorner \mu \urcorner), \Gamma \Rightarrow i : T\underset{\cdot}{T}\underset{\cdot}{f}(a, \ulcorner \mu \urcorner), \Delta & (=rep\mathrm{R}_C) \\
\hline
\Gamma \Rightarrow i : T\underset{\cdot}{T}\underset{\cdot}{f}(a, \ulcorner \mu \urcorner), \Delta & (\underset{\cdot}{T}) \\
\hline
i : \underset{\cdot}{T}\underset{\cdot}{f}(a, \ulcorner \mu \urcorner) = \underset{\cdot}{f}(sa, \ulcorner \mu \urcorner), \Gamma \Rightarrow i : T\underset{\cdot}{f}(sa, \ulcorner \mu \urcorner), \Delta & (=rep\mathrm{R}_C) \\
\hline
\Gamma \Rightarrow i : T\underset{\cdot}{f}(sa, \ulcorner \mu \urcorner), \Delta & (\mathrm{O2})
\end{array}
$$

This suffices to establish $\textsc{sc}_{\textsc{dt}_{\mathrm{Q}+}} \vdash\Rightarrow i : A(\overline{n})$ for every natural number n. Indeed, to establish $\textsc{sc}_{\textsc{dt}_{\mathrm{Q}+}} \vdash\Rightarrow T\underset{\cdot}{f}(s0, \ulcorner \mu \urcorner)$ we pick some new label j'' and start from $iRj, jRj', j'Rj'', \ j'' : T\underset{\cdot}{f}(a, \ulcorner \mu \urcorner) \Rightarrow j'' : T\underset{\cdot}{f}(a, \ulcorner \mu \urcorner)$ to first derive $iRj, jRj' \Rightarrow j' : \neg\forall x T\underset{\cdot}{f}(x, \ulcorner \mu \urcorner)$ followed by $iRj \Rightarrow j : T\underset{\cdot}{f}(0, \ulcorner \mu \urcorner)$ and finally $\Rightarrow i : T\underset{\cdot}{f}(s0, \ulcorner \mu \urcorner)$. The road to infinity is paved with tedious labour. □

Notice how the above proof did not involve any applications of a cut-rule formulated for labelled sequent calculus. To avoid any confusion, it is nonetheless important to point out that we have no reasons to think that the logic we obtain with $\textsc{sc}_{\textsc{dt}_{\mathrm{Q}+}}$ is non-transitive. Indeed, the logic is most likely transitive precisely in the virtue of the format of the rules. We just didn't assume it to prove ω-inconsistency.

One might suspect that our domain of labels must contain nonstandard numbers. However, to see that it suffices with ω, consider how we can substitute 0 for i, 1 for j , 2 for j' and 3 for j'' in the derivations in the proof of theorem 3.7. In this way we prove $\Rightarrow 0 : T\underset{\cdot}{f}(\overline{n}, \ulcorner \mu \urcorner)$ for every natural number n. Clearly we could do the same for each natural number as label without being forced to pick some number outside ω because we can always just pick the next number to get a new iteration of T for each label.

4 ω-inconsistency and nonstandard models

We have now seen that both STT based on Robinson Arithmetic and the logic of truth we called DT based on Robinson Arithmetic which arguably corresponds to McGee's original conditions are ω-inconsistent, and that we could show all this without relying on transitivity.

Now, the problem with ω-inconsistent logics is said to be that they cannot have a standard model of arithmetic, that is, a model that has as its domain the set of all and only the natural numbers. Instead, their models are not isomorphic to the standard model. Such models are known as nonstandard models and are basically models that contain some successor numbers that are, so to speak, beyond the standard natural numbers. Some authors are not that bothered by ω-inconsistency, in particular in connection to FS because FS nevertheless tells us that provable sentences of arithmetic are true and does not prove any false arithmetical sentences, examples being Halbach and Horsten (2008) and Sheard (2001). Others, for example Leitgeb (2007) and Barrio (2010), argue that ω-inconsistent logics of truth disrupt our ontological commitments:

> A theory of truth should not exclude this standard interpretation, for otherwise the theory could not be understood as speaking about the very objects that it was designed to refer to. Put differently: a theory of truth does not only have to be consistent (of course it has to be!), it also should not mess up its intended ontological commitments.(Leitgeb 2007, p.280)

> No monadic predicate [..] will express legitimate truth if its introduction to the language of arithmetic produces in turn a dramatic deviation in the theory's intended ontology: in order to be able to express the concept of arithmetic truth, [an ω-inconsistent theory of truth] has to abandon the possibility of speaking about standard natural numbers.(Barrio 2010, p.384)

If a logic of truth based on an arithmetical theory requires a nonstandard model, how can we still say that we have a theory of arithmetical truth? Instead, such a truth-predicate is merely a nonstandard truth-predicate and cannot be said to align with our philosophical intuitions concerning arithmetical truth.

However, STT "is shown to have standard models by the Kripke construction"(Cobreros et al. 2013, p.860). How can a proof theory which clearly is STT based on Robinson Arithmetic be ω-inconsistent while we nonetheless can use standard models of arithmetic to define STT using a Kripke construction? To understand how this is possible, we will have a quick look at STT defined on models of arithmetic.

Models of arithmetic for STT are obtained in two steps, closely following Halbach and Horsten (2006, p.680), however with the twist that we make explicit that our models are trivalent. First, we define Strong Kleene models inductively as follows:

Definition 4.1. *Let a* SKT_ω *model for* $\mathcal{L}_T^{A+}$ *be a triple* $\langle \mathbb{N}, \mathcal{T}, \mathcal{F} \rangle$ *where*

- $\mathbb{N}$ *is the standard model of arithmetic.*
- $\mathcal{T}, \mathcal{F} \subseteq \omega$

- $\mathcal{T} \cap \mathcal{F} = \emptyset$

The sets $\mathcal{T}$ and $\mathcal{F}$ are the extension and anti-extension of the truth-predicate.

Definition 4.2. *Let V be a function from a* SKT_ω*-model and the sentences of $\mathcal{L}_T^{A+}$ to $\{1, \frac{1}{2}, 0\}$ such that*

- *if $P(a_1, \ldots, a_n)$ is an atomic formula not of the form $T(a)$,*
  - $V_M(P(a_1, \ldots, a_n)) = 1$ *iff* $P(a_1, \ldots, a_n)$ *is true in* $\mathbb{N}$
  - $V_M(P(a_1, \ldots, a_n)) = 0$ *iff* $P(a_1, \ldots, a_n)$ *is false in* $\mathbb{N}$
- $V_M(\neg A) = 1 - V_M(A)$
- $V_M(A \vee B) = \max(V_M(A), V_M(B))$
- $V_M(\forall x A(x)) = \min(\{n \in \omega \mid V_M(A_x(\overline{n})\})$
- $V_M(T((\overline{a})) = 1$ *if and only if* $a \in \mathcal{T}$
- $V_M(T(\overline{a})) = 0$ *if and only if* $a \in \mathcal{F}$

Now, some Vs might not be transparent for some SKT_ω-models since there can be formulae A such that $V_M(A) \neq V_M(T\ulcorner A \urcorner)$. To define the set of transparent SKT_ω-models, we define the function $\Phi(\mathcal{T}, \mathcal{F}) : \mathcal{P}(\omega)^2 \to \mathcal{P}(\omega)^2$ such that

$$\Phi(\mathcal{T}, \mathcal{F}) = \langle \{\ulcorner A \urcorner \mid V_M(A) = 1\}, \{\ulcorner A \urcorner \mid V_M(A) = 0\} \rangle$$

where M is some SKT_ω-model. Φ is a jump-function for a fixed point construction which is discussed in detail by for example Halbach (2011). To obtain transparent models, we focus on those SKT_ω-models which are such that $\Phi(\mathcal{T}, \mathcal{F}) = \langle \mathcal{T}, \mathcal{F} \rangle$ because those models are such that $V_M(A) = V_M(T\ulcorner A \urcorner)$.[14]

Definition 4.3. *Let a* SKT_ω^F*-model be a* SKT_ω*-model s.t.* $\Phi(\mathcal{T}, \mathcal{F}) = \langle \mathcal{T}, \mathcal{F} \rangle$.

Definition 4.4. $\Gamma \vDash_{\text{STT}}^{\mathbb{N}} \Delta$ *if and only if every* SKT_ω^F*-model M is such that if* $\forall \gamma \in \Gamma, V_M(\gamma) = 1$ *then* $\exists \delta \in \Delta, V_M(\delta) \in \{1, \frac{1}{2}\}$

The definition of entailment is of course along the lines of Ripley (2012) and based on the idea that every model is such that if the premisses are *strictly satisfied* (assigned 1) then at least one of the conclusions is *tolerantly satisfied* (assigned 1 or $\frac{1}{2}$).

[14]Cf. (Halbach and Horsten 2006, p.681).

Theorem 4.5. *$\textsc{sc}_{\textsc{stt}_{\textsc{q}+}}$ is sound with regard to $\textsc{skt}^F_\omega$-models: If $\textsc{sc}_{\textsc{stt}_{\textsc{q}+}} \vdash \Gamma \Rightarrow \Delta$ then $\Gamma \vDash^{\mathbb{N}}_{\textsc{stt}} \Delta$*

Proof. The proof proceeds by induction on the height of a derivation and is left as exercise for the reader. The only interesting bit is that the substitution rules for $=$ requires a subinduction on the complexity of a formula because we have not defined $=$ with a clause for V_M, but rather lumped it together with the arithmetical vocabulary. □

To establish ω-inconsistency we first note the following version of the diagonal lemma:[15]

Lemma 4.6. *For every formula $\phi(x)$ with one free variable x there is a formula ψ such that every $\textsc{skt}^F\omega$-model is such that $V_M(\phi(\ulcorner\psi\urcorner)) = V_M(\psi)$.*

Proof. Let ψ be the formula $\exists x(D(\ulcorner\chi(x)\urcorner, x) \wedge \phi(x))$ as above. The key is now to observe that $V_M(D(\ulcorner\chi(x)\urcorner, \ulcorner\psi\urcorner)) = 1$ due to it being an arithmetical truth, and then proceed by reductio. Details are left for the reader. □

This suffices to establish that both μ and $\neg\forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner)$ are assigned $\frac{1}{2}$ on every $\textsc{skt}^F_\omega$-model and thus that $\vDash^{\mathbb{N}}_{\textsc{stt}} \neg\forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner)$ and $\vDash^{\mathbb{N}}_{\textsc{stt}} \forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner)$:

Theorem 4.7. *$\vDash^{\mathbb{N}}_{\textsc{stt}} \neg\forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner)$ and $\vDash^{\mathbb{N}}_{\textsc{stt}} \forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner)$*

Proof. It is sufficient to show that every $\textsc{skt}^F_\omega$-model is such that $V_M(\neg\forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner)) = \frac{1}{2}$. I show here that it cannot be assigned 1. Analogous reasoning will show that it cannot also be assigned 0. Let M be an arbitrary $\textsc{skt}^F_\omega$-model. Assume that $V_M(\neg\forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner)) = 1$. It follows that $V_M(\forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner)) = 0$ and thus that there is a number n such that $V_M(T\underset{\cdot}{f}(\overline{n}, \ulcorner\mu\urcorner)) = 0$. However, by lemma 4.6, it follows that $V_M(\mu) = 1$ and thus $V_M(T\ulcorner\mu\urcorner) = 1$. By the definition of f, we know that $V_M(\ulcorner\mu\urcorner = \underset{\cdot}{f}(0, \ulcorner\mu\urcorner)) = 1$ and thus $V_M(T\underset{\cdot}{f}(0, \ulcorner\mu\urcorner)) = 1$. By transparency of truth, it also follows that any number of truth-predications on μ is true and thus $V_M(T\underset{\cdot}{f}(\overline{n}, \ulcorner\mu\urcorner)) = 1$ for every $n \in \omega$. Since ω is our whole domain, it follows that $V_M(\forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner)) = 1$ which contradicts that there is a number n such that $V_M(T\underset{\cdot}{f}(\overline{n}, \ulcorner\mu\urcorner)) = 0$. □

In the case of the function f, inconsistency entails ω-inconsistency:

Theorem 4.8. *$\vDash^{\mathbb{N}}_{\textsc{stt}}$ is ω-inconsistent.*

[15] Lemma 4.6 is superficially similar to what is known in the literature as the strong diagonal lemma. See for example Heck (2007). However, the strong diagonal lemma establishes that for every formula $\phi(x)$ with one free variable x there is a term t such that $t = \ulcorner\phi(x)\urcorner$ where the term t is defined using a primitive function-term for the diagonalization-function.

Proof. Since we know from theorem 4.7 that $\vDash^{\mathbb{N}}_{\text{STT}} \neg\forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner)$ it suffices to prove that for every $n \in \omega$, $\vDash^{\mathbb{N}}_{\text{STT}} T\underset{\cdot}{f}(\overline{n}, \ulcorner\mu\urcorner)$. Let M' be an arbitrary SKT^{F}_{ω}-model. It is thus the case that $V_{M'}(\forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner))) = \frac{1}{2}$. Assuming that there is an $a \in \omega$ such that $V_{M'}(T\underset{\cdot}{f}(\overline{a}, \ulcorner\mu\urcorner)) = 0$ will contradict the former and it follows thus that for every $n \in \omega$, $\vDash^{\mathbb{N}}_{\text{STT}} T\underset{\cdot}{f}(\overline{n}, \ulcorner\mu\urcorner)$. □

So, not only are we using standard models of arithmetic, we can also show that the logic we defined on those models is ω-inconsistent.

To see how this is possible, it suffices to observe that the models are constructed in such a way that none of the numbers returned by $f(n, \ulcorner\mu\urcorner)$ for each n are in $\mathcal{T} \cup \mathcal{F}$. It follows that $Tf(n, \ulcorner\mu\urcorner)$ for each n is assigned $\frac{1}{2}$, and thus that $\forall x f(x, \ulcorner\mu\urcorner)$ is assigned $\frac{1}{2}$. By the clause for negation, it is also the case that $\neg\forall x f(x, \ulcorner\mu\urcorner)$ is assigned $\frac{1}{2}$. The definition of entailment is now such that those sentences become valid.[16] If we had chosen a different definition of entailment, and in particular one which is such that formulae assigned $\frac{1}{2}$ in each model do not come out as valid, then the logic would not be ω-inconsistent. One such definition is obtained by requiring that every model is such that if all the premisses are assigned either $\frac{1}{2}$ or 1 then at least one of the conclusions is assigned 1. This would deliver a non-reflexive logic as opposed to a non-transitive logic.[17]

While we can employ the non-classicality of the models to deal with the ω-inconsistency without invoking nonstandard models, we have good reasons to think that matters are going to be slightly more complicated in the case of *Deontic Truth* and its ω-inconsistency. In particular it seems reasonable to assume that the obvious frame semantics for it would require nonstandard "worlds" since they would have classical bivalent connectives but be such that both $\neg\forall x T\underset{\cdot}{f}(x, \ulcorner\mu\urcorner)$ and $T\underset{\cdot}{f}(\overline{n}, \ulcorner\mu\urcorner)$ for each $n \in \omega$ are true.

On the other hand, we do not have to search for long to find a similar logic of truth that is ω-inconsistent but which models must not be thought of as being nonstandard, namely the good old FS. To provide a model for FS (or more precisely the logic of truth they refer to as 'D'), Friedman and

[16] A curious reader might wonder whether something similar can be observed with regard to the ω-inconsistency of infinitely valued Łukasiewicz logic pointed out by Restall (1992), and furthermore discussed by for example Bacon (2013). This is not unreasonable to assume, but the devil is in the details, and that question is left for another occasion.

[17] A suitable sequent calculus for such a non-reflexive logic is obtained by restricting the initial sequents in definition 2.1 to equalities, i.e. formulae of the form $s = t$, a move that suffices to block the above derivations of inconsistency and ω-inconsistency in theorem 2.6 and 2.10. For a discussion of this approach to paradoxes, see French (2016). For a sequent calculus and an alternative semantics for a language based on distinguished names rather than arithmetic to generate paradoxical sentences, see Fjellstad (2016), in particular sections 4 and B.

Sheard (1987) proceed as follows: First they define by induction a sequence of models of the form $\langle \mathbb{N}, \mathcal{T} \rangle$ where $\mathbb{N}$ is our standard model of arithmetic and $\mathcal{T}$ the extension of the truth-predicate by letting $\mathcal{M}_0$ be $\langle \mathbb{N}, \emptyset \rangle$ and $\mathcal{M}_{n+1}$ be $\langle \mathbb{N}, \{\ulcorner A \urcorner \mid \mathcal{M}_n \vDash A\} \rangle$ where $\vDash$ is a suitable satisfaction-relation for classical logic. This amounts in effect to defining a finite revision sequence along the lines of Gupta and Belnap (1993). They then define a set Th_∞ to be $\{A \mid \exists k \forall n > k \; \mathcal{M}_n \vDash A\}$.

Now, the last step should not be understood as collecting the sentences into one model as in the case of a Kripke construction but rather as defining the set of formulae that are satisfied by that sequence of models. Even if the set of formulae satisfied by the sequence is ω-inconsistent in the same way as the set of formulae that are tolerantly satisfied by $\textsc{skt}^F_\omega$-models is ω-inconsistent, none of the models in our inductively defined sequence must be nonstandard models of arithmetic since the sequence does not have a *final* or *last* model $\mathcal{M}_{\mathring{a}}$ such that $\mathcal{M}_{\mathring{a}} \vDash \neg \forall x T \dot{f}(x, \ulcorner \mu \urcorner)$ and $\mathcal{M}_{\mathring{a}} \vDash T \dot{f}(\overline{n}, \ulcorner \mu \urcorner)$ for every natural number n. In other words, as long as we do not look for a model for FS with *one* bivalent valuation which for example seems to be the aim of Halbach (2011, pp. 162-175), but rather consider the logic as satisfied by a sequence of valuations, it seems reasonable to conclude that FS does not require nonstandard models.

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