

First degree formulas in quantified S5

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Abstract

This note provides a proof that the formula $\Box\exists x[Fx \wedge \neg\Box Fx]$ is not equivalent to any first degree formula in the context of the quantified version of the modal logic S5. This solves a problem posed by Max Cresswell.

1 Introduction

It is a well known result that in the purely propositional version of the modal logic S5, every formula is equivalent to a first degree formula. A proof of this result can be found in the text by Hughes and Cresswell [8, p. 101]. This is one of the oldest results in modal logic, and traces its ancestry to Wajsberg [10] and Carnap [2].

Max Cresswell, in a talk given in May 2015 [3] posed the question of whether the result still holds true for the quantificational version of S5. He conjectured that it does not, and that the formula $\Box\exists x[Fx \wedge \neg\Box Fx]$ is a counter-example. The aim of this note is to show that his conjecture is correct.

We employ the usual vocabulary of modal predicate logic. In the first two sections, we consider a language with predicates of any arity, but in Sections 3 and 4, we specialize to the language with only a single one-place predicate F.

Definition 1.1 *The modal degree of a formula of modal predicate logic is defined inductively as follows:*

1. *An atomic formula has degree 0;*
2. *$\neg A$ has the same degree as A ;*
3. *The degree of $A \wedge B$ is the maximum of the degrees of A and B ;*
4. *$\exists xA$ has the same degree as A ;*
5. *If A has degree n , then $\Box A$ has degree $n + 1$.*

To simplify matters, we assume a constant domain for all worlds in a given model. A model has the form $\mathfrak{M} = \langle V, R, \mathcal{A}, \Phi \rangle$, where V is a non-empty set of possible worlds, R is an accessibility relation and \mathcal{A} is a non-empty set of individuals that constitutes the domain for all worlds in V . The function Φ provides the interpretation for the primitive predicates relative to a world, so that if P is a k -place predicate, and v is a world in V , $\Phi(P, v) \subseteq \mathcal{A}^k$.

We shall employ the vector notation for sequences \vec{a} of elements from a given set. The notation \vec{a}_i is used to denote the i th element in the sequence \vec{a} ; thus a sequence of length m can be written as $\vec{a}_1, \dots, \vec{a}_m$. If L is a language of quantified modal logic, and $\mathfrak{M} = \langle V, R, \mathcal{A}, \Phi \rangle$ a model of L , then we use the notation $L[\mathfrak{M}]$ for the language resulting from L by adding names for all the elements of the domain \mathcal{A} . We write $B[\vec{x}]$ for a formula B whose free variables are contained in the sequence \vec{x} . If \vec{a} is a sequence of the same length as \vec{x} , then $B[\vec{a}]$ denotes B with the elements of \vec{a} substituted as constants for the corresponding variables in \vec{x} .

Definition 1.2 *Let L be a language, and $\mathfrak{M} = \langle V, R, \mathcal{A}, \Phi \rangle$ a model for L . The forcing relation \Vdash between $v \in V$ and sentences of $L[\mathfrak{M}]$ is defined as follows:*

1. $v \Vdash P\vec{a} \Leftrightarrow \vec{a} \in \Phi(P, v)$, for $P\vec{a}$ an atomic sentence of $L[\mathfrak{M}]$;
2. $v \Vdash B \wedge C \Leftrightarrow (v \Vdash B \wedge v \Vdash C)$;
3. $v \Vdash \neg B \Leftrightarrow v \not\Vdash B$;
4. $v \Vdash \Box B \Leftrightarrow (\forall u \in V)(vRu \Rightarrow u \Vdash B)$;
5. $v \Vdash \exists x B \Leftrightarrow (\exists a \in \mathcal{A})(v \Vdash B[a])$.

2 Bisimulation

In this section, we define a notion of bisimulation appropriate to quantified modal logic; it is a graded version of the concept defined by van Benthem [9]. In the case of propositional modal logic, the notion seems to have first appeared in a paper of Kit Fine [5, p. 33], who calls it “model equivalence,” and describes it as the modal analogue of the back-and-forth method introduced in first-order model theory by Roland Fraïssé [6] (later described in terms of a game by Ehrenfeucht [4]). Detailed expositions of the notion (in the purely propositional case) can be found in the chapters by Blackburn and van Benthem [1] and Goranko and Otto [7] in the *Handbook of Modal Logic*.

The method of Fraïssé and Ehrenfeucht is widely employed in finite model theory, since it is one of the few techniques in general first-order model theory that can be transferred to the finite case. The graded version of their method can be used to distinguish formulas of differing quantifier rank.

Similarly, in propositional modal logic, the graded version of bisimulation can be used to distinguish formulas of differing modal degree. Van Benthem [9] defined the ungraded version for modal predicate logic, which does not distinguish between modal degrees. The graded version defined here appears to be new in the literature of modal logic; consequently, we give a fairly general definition in this section, though we only require a much simpler version for the main application.

Let $\mathfrak{A} = \langle V, R, \mathcal{A}, \Phi \rangle$ and $\mathfrak{B} = \langle W, S, \mathcal{B}, \Psi \rangle$ be two models for constant domain quantified modal logic.

Definition 2.1 *A graded bisimulation between \mathfrak{A} and \mathfrak{B} is a family of relations $\{\cong_k \mid k \in \omega\}$ satisfying the conditions:*

1. $\cong_k \subseteq \bigcup_{l \in \omega} (V \times \mathcal{A}^l) \times (W \times \mathcal{B}^l)$;
2. $(v, \vec{a} \cong_0 w, \vec{b}) \Rightarrow (v \Vdash P\vec{a} \Leftrightarrow w \Vdash P\vec{b})$, for P a predicate letter;
3. $[(t, \vec{a} \cong_{k+1} u, \vec{b}) \wedge uSw] \Rightarrow (\exists v \in V)[tRv \wedge (v, \vec{a} \cong_k w, \vec{b})]$;
4. $[(t, \vec{a} \cong_{k+1} u, \vec{b}) \wedge tRv] \Rightarrow (\exists w \in W)[uSw \wedge (v, \vec{a} \cong_k w, \vec{b})]$;
5. $[(t, \vec{a} \cong_k u, \vec{b}) \wedge b \in \mathcal{B}] \Rightarrow (\exists a \in \mathcal{A})(t, \vec{a}, a \cong_k u, \vec{b}, b)$;
6. $[(t, \vec{a} \cong_k u, \vec{b}) \wedge a \in \mathcal{A}] \Rightarrow (\exists b \in \mathcal{B})(t, \vec{a}, a \cong_k u, \vec{b}, b)$.

The third and fourth conditions above are the back-and-forth requirements designed to take care of the modal operators, while the fifth and sixth deal with the quantifiers. Note that some (or even all) of the relations \cong_k may be empty.

Lemma 2.2 *Let $\mathfrak{A} = \langle V, R, \mathcal{A}, \Phi \rangle$ and $\mathfrak{B} = \langle W, S, \mathcal{B}, \Psi \rangle$ be two models for constant domain quantified modal logic. If $\{\cong_k \mid k \in \omega\}$ is a graded bisimulation between \mathfrak{A} and \mathfrak{B} , $v \in V$, $w \in W$, and $A[\vec{x}]$ is a formula of modal predicate logic of modal degree at most k , then*

$$(v, \vec{a} \cong_k w, \vec{b}) \Rightarrow (v \Vdash A[\vec{a}] \Leftrightarrow w \Vdash A[\vec{b}]).$$

Proof. By induction on the complexity of $A[\vec{x}]$. For atomic formulas, the lemma holds by the second condition in the definition of bisimulation. Now assume that the lemma holds for formulas of modal degree $j < k$. For the truth-functional connectives, the induction step is straightforward.

Let $\Box A[\vec{x}]$ be a formula of modal degree $j + 1$. Assume in addition that $(v, \vec{a} \cong_{j+1} w, \vec{b})$ and $v \Vdash \Box A[\vec{a}]$. If wSu then by the third condition, there is a t in V so that $(t, \vec{a} \cong_j u, \vec{b})$ and vRt , hence $t \Vdash A[\vec{a}]$. By the induction hypothesis, $u \Vdash A[\vec{b}]$, showing that $w \Vdash \Box A[\vec{b}]$. The converse implication follows symmetrically, by the fourth condition.

Let $\exists yA[\vec{x}, y]$ be a formula of modal degree $j \leq k$. Assume in addition that $(v, \vec{a} \cong_j w, \vec{b})$ and $v \Vdash \exists yA[\vec{a}, y]$. Then there is an element $a \in \mathcal{A}$ so that $v \Vdash A[\vec{a}, a]$. By the sixth condition, there is a $b \in \mathcal{B}$ so that $(v, \vec{a}, a \cong_j w, \vec{b}, b)$ so that by induction hypothesis, $w \Vdash A[\vec{b}, b]$, hence $w \Vdash \exists yA[\vec{b}, y]$. The converse implication follows by a symmetrical argument. \square

3 Two models

In this section, we define two models that we use to prove the main result; both are models for quantified **S5** with countably many worlds and countably many individuals. The first model \mathfrak{A} validates $\Box\exists x[Fx \wedge \neg\Box Fx]$, while the second model \mathfrak{B} falsifies it. However, there is a world v_1 in \mathfrak{A} and a world w_1 in \mathfrak{B} such that $v_1 \cong_1 w_1$, showing that the formula $\Box\exists x[Fx \wedge \neg\Box Fx]$ is not equivalent to any first degree formula. We now describe the two models: the accessibility relation is omitted because it is the universal relation in both cases.

The first model $\mathfrak{A} = \langle V, \mathcal{A}, \Phi \rangle$ is based on a countably infinite set of worlds $V = \{v_1, v_2, v_3, \dots\}$ and an infinite set of individuals $\mathcal{A} = \{a_0, a_1, a_2, \dots\}$. The assignment Φ is given by $\Phi(F, v_i) = \{a_0, a_i\}$.

The second model $\mathfrak{B} = \langle W, \mathcal{B}, \Psi \rangle$ is isomorphic to the first model, with the exception of an added world w_0 designed to invalidate $\Box\exists x[Fx \wedge \neg\Box Fx]$. Thus we have $W = \{w_0, w_1, w_2, \dots\}$ and $\mathcal{B} = \{b_0, b_1, b_2, \dots\}$. The assignment Ψ is given by $\Psi(F, w_i) = \{b_0, b_i\}$ for $i > 0$, and $\Psi(F, w_0) = \{b_0\}$. The function I defined on \mathcal{A} by $I(a_i) = b_i$ is an isomorphism between \mathfrak{A} and the submodel of \mathfrak{B} defined on $W \setminus \{w_0\}$.

\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	
w_{10}	1	0	0	0	0	0	0	0	0	0	1	...	
w_9	1	0	0	0	0	0	0	0	0	0	1	0	...
w_8	1	0	0	0	0	0	0	0	0	1	0	0	...
w_7	1	0	0	0	0	0	0	0	1	0	0	0	...
w_6	1	0	0	0	0	0	1	0	0	0	0	0	...
w_5	1	0	0	0	0	1	0	0	0	0	0	0	...
w_4	1	0	0	0	1	0	0	0	0	0	0	0	...
w_3	1	0	0	1	0	0	0	0	0	0	0	0	...
w_2	1	0	1	0	0	0	0	0	0	0	0	0	...
w_1	1	1	0	0	0	0	0	0	0	0	0	0	...
w_0	1	0	0	0	0	0	0	0	0	0	0	0	...
	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	...	

The accompanying table represents the incidence matrix of the model \mathfrak{B} . An entry of 1 in the w_i, b_j place means that $w_i \Vdash Fb_j$. The table for \mathfrak{A} is the

same after the deletion of the row for w_0 , and appropriate relabeling of the worlds and individuals.

We define a family of relations \cong_k between \mathfrak{A} and \mathfrak{B} as follows. For $v_i \in V$ and $w_j \in W$, $j > 0$, we define

$$v_i, \vec{a} \cong_0 w_j, \vec{b} \Leftrightarrow [i = j \wedge \vec{b} = I(\vec{a})].$$

For $v_i \in V$ and $w_0 \in W$, we define

$$v_i, \vec{a} \cong_0 w_0, \vec{b} \Leftrightarrow \forall m \leq l(v_i \Vdash F\vec{a}_m \Leftrightarrow w_0 \Vdash F\vec{b}_m),$$

where l is the length of the sequences \vec{a} and \vec{b} .

The relation \cong_1 is defined on the pair of worlds v_1 and w_1 by setting

$$v_1, \vec{a} \cong_1 w_1, \vec{b} \Leftrightarrow \vec{b} = I(\vec{a}).$$

For $k > 1$, the relation \cong_k is empty.

Lemma 3.1 *The family \cong_k is a graded bisimulation between \mathfrak{A} and \mathfrak{B} .*

Proof. The second condition in Definition 2.1 holds by construction. It remains to prove that the third to sixth conditions hold for the relation \cong_1 .

We start with the third condition. For any $\vec{a} \in \mathcal{A}^l$, we have $v_1, \vec{a} \cong_1 w_1, I(\vec{a})$. Let w_i be any world in W , with $i > 0$. Then by construction, the world v_i satisfies the relation $v_i, \vec{a} \cong_0 w_i, I(\vec{a})$. Now consider the extra world w_0 ; this is the more difficult case. Choose v_j in V so that the subscript j is larger than the subscript of any element \vec{a}_i in \vec{a} . Then the relation $v_j, \vec{a} \cong_0 w_0, I(\vec{a})$ holds. The fourth condition is proved by a symmetrical (though easier) argument.

Now for the fifth and sixth conditions. They are easily seen to hold for the cases involving $v_1, \vec{a} \cong_1 w_1, I(\vec{a})$. Now assume $v_i, \vec{a} \cong_0 w_i, I(\vec{a})$, where $i > 0$, and that $a_j \in A$; then $v_i, \vec{a}, a_j \cong_0 w_i, I(\vec{a}), b_j$ by construction. If $v_i, \vec{a} \cong_0 w_0, \vec{b}$ and $a_j \in A$, then two cases arise. If $a_j \notin \Phi(F, v_i)$, then we have $v_i, \vec{a}, a_j \cong_0 w_0, \vec{b}, b_j$. In the second case, where $a_j \in \Phi(F, v_i)$, we have $v_i, \vec{a}, a_j \cong_0 w_0, \vec{b}, b_0$. This proves the sixth condition; the fifth condition is proved symmetrically. \square

Theorem 3.2 *In the context of quantified S5, the formula $\Box\exists x[Fx \wedge \neg\Box Fx]$ is not equivalent to any formula of the first degree.*

Proof. The worlds v_1 and w_1 in the models \mathfrak{A} and \mathfrak{B} satisfy the relation $v_1 \cong_1 w_1$. Consequently, by Lemma 2.2, they satisfy the same sentences of the first degree. However, they differ with respect to the sentence $\Box\exists x[Fx \wedge \neg\Box Fx]$, showing that this formula cannot be expressed as an equivalent first degree formula. \square

4 Finite models

It is a noticeable feature of the construction of the previous section that both the set of worlds and individuals in the models \mathfrak{U} and \mathfrak{B} are infinite. The infinity of the set of worlds and individuals is used in a crucial way in verifying the third condition in Lemma 3.1. Is it possible to simplify the proof by employing models in which the domain of individuals is finite? The purpose of the current section is to show that this approach cannot work.

Let $\mathfrak{U} = \langle V, \mathcal{A}, \Phi \rangle$ be a model for quantified **S5** in which the domain of individuals $\mathcal{A} = \{a_1, \dots, a_m\}$ is finite. Let us use the abbreviation \vec{a} for the sequence a_1, \dots, a_m ; similarly for \vec{x} and \vec{b} . If $v \in V$ is a world in \mathfrak{U} , then we write $\wedge v$ for the state description determined by v ; that is to say, the conjunction $\pm Fa_1 \wedge \pm Fa_2 \wedge \dots \wedge \pm Fa_m$, where $\pm Fa_i$ is Fa_i if $v \Vdash Fa_i$, otherwise $\neg Fa_i$.

Theorem 4.1 *Let $\mathfrak{U} = \langle V, \mathcal{A}, \Phi \rangle$ and $\mathfrak{B} = \langle W, \mathcal{B}, \Psi \rangle$ be models for quantified **S5**, where $\mathcal{A} = \{a_1, \dots, a_m\}$ is finite. If $v_1 \in V$ and $w_1 \in W$ agree on all first degree sentences and $v_1 \Vdash \Box \exists x [Fx \wedge \neg \Box Fx]$, then $w_1 \Vdash \Box \exists x [Fx \wedge \neg \Box Fx]$.*

Proof. Let $\Theta(\mathfrak{U})$ be the following first degree sentence:

$$\exists \vec{x} \left[\bigwedge_{v \in V} \diamond \wedge v[\vec{x}] \wedge \Box \bigvee_{v \in V} \wedge v[\vec{x}] \right].$$

Since $v_1 \Vdash \Theta(\mathfrak{U})$, it follows that $w_1 \Vdash \Theta(\mathfrak{U})$, hence there are $b_1, \dots, b_m \in \mathcal{B}$ so that

$$w_1 \Vdash \left[\bigwedge_{v \in V} \diamond \wedge v[\vec{b}] \wedge \Box \bigvee_{v \in V} \wedge v[\vec{b}] \right]$$

If w is a world in W , then $w \Vdash \bigvee_{v \in V} \wedge v[\vec{b}]$, so that for some $v \in V$, $w \Vdash \wedge v[\vec{b}]$. By assumption, there is an $a_i \in \mathcal{A}$ so that $v \Vdash Fa_i \wedge \neg \Box Fa_i$, so that $w \Vdash Fb_i$, and in addition, there is a $v' \in V$ so that $v' \Vdash \neg Fa_i$. Since $w_1 \Vdash \bigwedge_{v \in V} \diamond \wedge v[\vec{b}]$, there is a $w' \in W$ so that $w' \Vdash \wedge v'[\vec{b}]$, showing that $w' \Vdash \neg Fb_i$. Hence, $w \Vdash Fb_i \wedge \neg \Box Fb_i$, showing that $w_1 \Vdash \Box \exists x [Fx \wedge \neg \Box Fx]$. \square

In the introductory section, we remarked that in the purely propositional version of **S5**, every formula is equivalent to a first degree formula. The proof we have just given is based on this observation, together with the fact that over a finite domain, existential quantifiers can be expanded as finite disjunctions.

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