Substructural Negations

Takuro Onishi*

Abstract

We present substructural negations, a family of negations (or negative modalities) classified in terms of structural rules of an extended kind of sequent calculus, display calculus. In considering the whole picture, we emphasize the duality of negation. Two types of negative modality, impossibility and unnecessity, are discussed and "self-dual" negations like Classical, De Morgan, or Ockham negation are redefined as the fusions of two negative modalities. We also consider how to identify, using intuitionistic and dual intuitionistic negations, two accessibility relations associated with impossibility and unnecessity.

Introduction

We present *substructural negations*, a family of negations classified in terms of structural rules of sequent calculus. The classification has originated from the conception of *negation as (negative) modality*, defined in terms of accessibility relation on frames. Correspondences between various principles concerning negation and constraints on frames have been investigated in the literature [2, 11, 15, 14, 16, 32, 33] [1]. One of our contributions in this paper is to give sequent calculus formulation of those correspondences. To obtain expressive power sufficient for this purpose, we use an extension of traditional sequent calculus, *display calculus*.

Another feature of the present work is an emphasis on duality of negative modal operators. Besides the *impossibility* operator which is more familiar (A is impossible here if it fails everywhere accessible from here), some authors [17, 38] study *unnecessity* operator (A is unnecessary here if it fails somewhere accessible from here). We discuss how to collapse or identify them into one to obtain self-dual negations such as classical, De Morgan, or Okham negation. Redefining them as "fusions" of two negations sheds new light on them.

*takuro.onishi@gmail.com

[1] The recent entry [23] in *Stanford Encyclopedia of Philosophy* on negation in general contains a discussion on this topic.

We also inquire how to identify accessibility relations for impossibility and for unnecessity, which are given independently in the first place. Axioms and structural rules that correspond to the frames in which two relations coincide will be presented. It turns out that we need the expressive power of Bi-Intuitionistic logic that has two, intuitionistic and dual intuitionistic, negations.

After briefly looking at Bi-Intuitionistic logic as our base logic in section 1, we introduce, in terms of frame semantics, four negations, two of impossibility type, two of unnecessity type in section 2. Section 3 is for proof theory, and section 4 for soundness and completeness. In section 5, we discuss a family of negative modalities of various strengths, which was (formerly) pictured in the form of a "kite". The kite for impossibility is well known (5.1). The one for unnecessity is also known, but here we define a translation to obtain a dual kite from the other (5.2). In 5.3 we show how the two kites can be united. Lastly section 6 dicusses identification of accessibility relations.

1 Bi-Intuitionistic Logic

Bi-Intuitionistic logic ($\mathtt{BiInt}$), also known as HB (Heyting-Brouwer) logic, is a conservative extension of Intuitionistic logic with *co-implication* $\leftarrow$ also called dual implication or subtraction [29, 30].

Consider a language $\mathcal{L}_B$ with four connectives, $\wedge, \vee, \rightarrow, \leftarrow$ over countably infinite atoms. The true proposition $\top$ is definable by implication as $q \rightarrow q$ for some fixed atom q. The false proposition $\bot$ is also defined as $q \leftarrow q$ using co-implication. In turn, intuitionistic negation $\neg A$ is defind to be $A \rightarrow \bot$ and dual intuitionistic negation $\sim A$ to be $\top \leftarrow A$.

A Kripke model for $\mathtt{BiInt}$ simply is a Kripke model $\langle W, \leqslant, \models \rangle$ for Intuitionistic logic, where W is a nonempty set of states, $\leqslant$ a reflexive and transitive relation (pre-order) on W, and $\models$ a relation between states and atoms which is *hereditary* along $\leqslant$. The pre-order $\leqslant$ represents development of knowldege or increasing of information. So we refer to it as *information order*. The only difference from Intuitionistic logic is the additional clause for co-implication. We write $x \models A$ for "A is true at a state x". Then the clause for co-implication is given as follows:

$$x \models A \leftarrow B \iff \exists y \leqslant x : y \models A \,\&\, y \not\models B.$$

Note that the heredity property extends to arbitrary formulas including co-implication. I.e. if $x \models A$ and $x \leqslant y$, then $y \models A$ for any formula A and any states x, y in the frame.

Note that $q \leftarrow q$ is false (not true) at every point of any model, hence behaves properly as the false propostion $\bot$. The truth conditions for negation and dual negation are derived as follows:

$$x \models \neg A \, (= A \to \bot) \iff \forall y \geqslant x : y \not\models A;$$
$$x \models {\sim} A \, (= \top \leftarrow A) \iff \exists y \leqslant x : y \not\models A.$$

Validity is defined as usual. A formula is *valid* if it is true at every state of every model. An inference is *valid* if it preserves truth at every state of every model. Presence of co-implication (and the defined dual negation) makes the logic quite symmetrical without collapsing it into classical logic. For example, while $A \wedge \neg A$ is contradictory, $A \vee {\sim} A$ is valid in $\mathtt{BiInt}$. And whereas the inference from A to $\neg\neg A$ is valid but the converse not, one from ${\sim}{\sim}A$ to A is valid and the converse not.

2 Four negative modalities

We add to $\mathcal{L}_B$ four unary operators $\rhd, \lhd, \blacktriangleright, \blacktriangleleft$ that express negative modality. Call the extended language $\mathcal{L}$. The first, $\rhd$, expresses *impossibility*, defined in terms of a binary accsessibility relation $\frown$, which is distinguished from the information order:

$$x \models \rhd A \iff \forall y : x \frown y \Rightarrow y \not\models A.$$

Compatibility is a standard interpretation of the accessibility $\frown$. If A is rejected or excluded as impossible at x ($x \models \rhd A$), and nevertheless A is accepted at y ($y \models A$), then x and y must be *incompatible*. So $x \frown y$ here should be understood as compatibility between x and y. We also consider another impossibility-type operator $\lhd$ that *looks back* through $\frown$:

$$x \models \lhd A \iff \forall y : y \frown x \Rightarrow y \not\models A.$$

The pair $\langle \rhd, \lhd \rangle$ are called *split* or *Galois (connected) negations* [22, 15].

We require the following *frame condition* to assure that truth of $\rhd A$ and $\lhd A$ are hereditary along the information order $\leqslant$:

$$\leqslant \circ \frown \subseteq \frown \supseteq \frown \circ \geqslant \quad (\text{hence} \leqslant \circ \frown = \frown = \frown \circ \geqslant).$$

where "$\circ$" denotes the composition of relations and "$\geqslant$" denotes $\leqslant^{-1}$. $\leqslant \circ \frown \subseteq \frown$ is for heredity of $\rhd$ and $\frown \supseteq \frown \circ \geqslant$ for $\lhd$ [2].

Below are what we call the *basic principles* (rules and axioms) characterising the impossibility operators $\rhd$ [3]. We use here sequent notation.

$$\frac{A \vdash B}{\rhd B \vdash \rhd A} \qquad \top \vdash \rhd\bot \qquad \rhd A \wedge \rhd B \vdash \rhd(A \vee B) \qquad \frac{A \vdash \rhd B}{\overline{B \vdash \lhd A}}$$

[2]The condition which is not only sufficient but necessary for heredity of $\rhd$ is $\leqslant \circ \frown \subseteq \frown \circ \geqslant$. Our condition is for *(strictly) condensed* frames in Došen's term. The difference makes no difference about the set of valid formulas [11].

[3]"Characterising" means soundness and completeness. The deductive system defined by the basic principles is sound and complete with respect to the class of frames with the frame conditions above. Later we prove soundness and completeness of display calculus instead of the basic principles.

$\lhd$ also has the rules and axioms of the same form. The double line indicates that the rule is bidirectional. The leftmost rule may be seen as justifying the name of "negative modality", since the contraposition rule is among the properties possessed by very many, if not all, negation operators [4]. We also note that other two form of de Morgan laws hold here. Only the intuitionistically unacceptable one, $\rhd(A \wedge B) \vdash \rhd A \vee \rhd B$, fails in this basic setteing.

There can be different motivations to study negation as impossibility. The idea traces back to Birkhoff and von Neumann [3] who used *incompatibility* (*orthogonality*, or *perp*) between the states of a Hilbert space to define negation in their quantum logic. Intuitionistic negation $\neg$ is also an impossibility operator since, in Kripke semantics, $\neg A$ is defined to be true at a state if and only if it is *impossible* for A to hold however our body of knowledge extends. Here the modal accessibility for negation is just the information order. Kosta Došen, separating accessibility from information order, studies negative modalities in the context of his investigation into intuitionistic modal logic with M.Božić [4, 5, 8, 9, 10, 11, 13]. His study on negative modalities was intended to form, together with the study of positive modal operators, the theory of unary operators in general. In semantics for relevant logic [36], *De Morgan negation* of A is defined to be true at a state x if and only if A is not true at its "star state" x^*, where $*$ is not just a binary relation, but a function of period two ($x^{**} = x$). Relevant logicians such as Dunn and Restall argue that the $*$-function can be made sense of by understanding it as a special case of compatibility relation [16, 32].

Next let us introduce *unnecessity* operators, $\blacktriangleright$ and $\blacktriangleleft$, which are dual to impossibility operators, and similar to dual intuitioinistic negation [17, 44]. They are defined in terms of yet another accessibility $\smile$:

$$x \models \blacktriangleright A \iff \exists y : x \smile y \;\&\; y \not\models A;$$
$$x \models \blacktriangleleft A \iff \exists y : y \smile x \;\&\; y \not\models A.$$

As in the case of impossibility, the frame condition for unnecessity operators should be assumed in order to ensure heredity:

$$\geqslant \circ \smile \subseteq \smile \supseteq \smile \circ \leqslant \text{ (hence, } \geqslant \circ \smile = \smile = \smile \circ \leqslant).$$

The accessibility $\smile$ may be understood as *exhaustiveness*. If $\blacktriangleright A$ *fails* at x, then any state y related to x must fill up the lack of negative information by supporting A. That is, if $x \smile y$, then $x \models \blacktriangleright A$ or $y \models A$, and similarly $x \models A$ or $y \models \blacktriangleleft A$. Thus, x and y are jointly exhaustive.

The frame condition makes sense under this interpretation. Let $z \geqslant x \smile y$. Then any piece of negative information that fails at z also fails at x since

[4]See the SEP entry [23] for references on negations that do *not* satisfy the contraposition rule.

$z \geqslant x$, hence the lack would be filled up by some information in y since $x \smile y$. Thus y and z are jointly exhaustive. Similarly if $x \smile y \leqslant z$, then since z contains more information than y, obviously x and z are exhaustive.

Below are the basic principles (rules and axioms) for $\blacktriangleright$ and its relation to $\blacktriangleleft$. You can see that it behaves as the dual of $\triangleright$.

$$\frac{A \vdash B}{\blacktriangleright B \vdash \blacktriangleright A} \quad \blacktriangleright\top \vdash \bot \quad \blacktriangleright(A \wedge B) \vdash \blacktriangleright A \vee \blacktriangleright B \quad \frac{\blacktriangleright A \vdash B}{\blacktriangleleft B \vdash A}$$

The only de Morgan law that fails is $\blacktriangleright A \wedge \blacktriangleright B \vdash \blacktriangleright(A \vee B)$.

Let us summarize the definition of frames here.

Definition 2.1 ($\mathtt{BiN}$-frame, model)**.** A $\mathtt{BiN}$-*frame* is a structure $\langle W, \leqslant, \frown, \smile \rangle$, where W is a non-empty set of states, $\leqslant$ is a reflexive and transitive information order on W, and $\frown$ and $\smile$ are binary accessibility relations on W satisfying the following *frame conditions*:

$$\leqslant \circ \frown = \frown = \frown \circ \geqslant \text{ and } \geqslant \circ \smile = \smile = \smile \circ \leqslant.$$

A $\mathtt{BiN}$-*model* is a structure $\langle W, \leqslant, \frown, \smile, \models \rangle$, where $\langle W, \leqslant, \frown, \smile \rangle$ is a $\mathtt{BiN}$-frame and $\models \subseteq W \times \mathtt{Atom}$ satisfies the following *heredity property*: for any $x, y \in W$ and $p \in \mathtt{Atom}$, if $x \models p$ and $x \leqslant y$, then $y \models p$. The relation $\models$ extends to arbitrary formulas inductively preserving the heredity property. Especially the clauses for negative modalities are given as follows:

$$\begin{array}{ll} x \models \triangleright A \Leftrightarrow \forall y : x \frown y \Rightarrow y \not\models A; & x \models \triangleleft A \Leftrightarrow \forall y : y \frown x \Rightarrow y \not\models A; \\ x \models \blacktriangleright A \Leftrightarrow \exists y : x \smile y \;\&\; y \not\models A; & x \models \blacktriangleleft A \Leftrightarrow \exists y : y \smile x \;\&\; y \not\models A. \end{array}$$

We call $\mathtt{BiN}$-frames and $\mathtt{BiN}$-models simply *frames* and *models*. Validity of formulas and inferences are defined as usual. We call $\mathtt{BiN}$ the logic defined by this semantics.

3 Display calculus $\delta\mathtt{BiN}$

In this section we present a display calculus $\delta\mathtt{BiN}$ which is sound and complete with respect to the frame semantics of $\mathtt{BiN}$. As suggested by failure of Cut elimination in the system for S5 [25], the framework of standard sequent calculus is not necessarily apt for modal logic. A similar phenomenon is observed with Bi-Intuitionistic logic [26]. Several extensions to handle with the difficulty have been proposed such as labelled sequent calculus, nested sequent calculus, and deep inference system [6, 19, 20, 21, 26, 27, 28]. We apply the method of display calculus to negative modality on Bi-intuitionistic logic.

The general idea and framework of display calculus (or display *logic*) is originally due to N.Belnap [1]. A traditional sequent $\Gamma \vdash \Delta$ consists of two sequences, multisets, or sets Γ, Δ of formulas. Display calculus generalizes it to arbitrary *structures* built from formulas (and atomic structures) by various kinds of *structural connectives* of meta-level.

For $\mathtt{BiInt}$, two slightly different display formulations are given by Goré and Wansing [18, 42, 43]. The set of the rules below is taken from Wansing's version, while the symbols ; and $>$ are Goré's. Wansing also has been developing the modal display calculus, modifying Belnap's treatment of unary structural connectives that models (positive) modal operators [39, 40, 41]. Unary structural connectives (or punctuation marks) for negation and rules that govern them can be found in Restall's book [33, p.124], though the proof system is not of display style. The display calculus $\delta\mathtt{BiN}$ presented below is just a combination of these items. Novelty of the paper consists in structural rules and their correspondence to constraints on frames, which we will see in section 5.

Definition 3.1 (Structures)**.** The set $\mathcal{S}$ of *structures* is defined by the following grammar:

$$X ::= A \in \mathcal{L} \mid \mathbf{I} \mid \sharp X \mid \flat X \mid X \, ; Y \mid X > Y.$$

A *sequent* is an expression of the form $X \vdash Y$ consisting of structures X, Y.

Definition 3.2 ($\delta\mathtt{BiN}$)**.** The display calculus $\delta\mathtt{BiN}$ consists of the following rules:

Axiom and Cut

$$(\mathrm{id})\ p \vdash p \qquad \frac{X \vdash Y \, ; A \qquad A \, ; X \vdash Y}{X \vdash Y}\,(\mathrm{Cut})$$

Display rules

$$\begin{array}{c} X \, ; Y \vdash Z \\ \hline\hline X \vdash Y > Z \\ \hline\hline Y \, ; X \vdash Z \end{array}\ \begin{array}{l}(;>_1)\\ (;>_2)\end{array} \qquad \begin{array}{c} X \vdash Y \, ; Z \\ \hline\hline X > Y \vdash Z \\ \hline\hline X \vdash Z \, ; Y \end{array}\ \begin{array}{l}(>;_1)\\ (>;_2)\end{array} \qquad \begin{array}{c} \sharp X \vdash Y \\ \hline\hline \flat Y \vdash X \end{array}\,(\sharp\flat_L) \qquad \begin{array}{c} X \vdash \sharp Y \\ \hline\hline Y \vdash \flat X \end{array}\,(\sharp\flat_R)$$

Structural rules

$$\frac{X \, ; \mathbf{I} \vdash Y}{X \vdash Y}\,(\mathbf{I}_L) \qquad \frac{X \vdash Y \, ; \mathbf{I}}{X \vdash Y}\,(\mathbf{I}_R) \qquad \frac{(X \, ; Y) \, ; Z \vdash W}{X \, ; (Y \, ; Z) \vdash W}\,(B_L) \qquad \frac{X \vdash Y \, ; (Z \, ; W)}{X \vdash (Y \, ; Z) \, ; W}\,(B_R)$$

$$\frac{X \vdash Y}{X \, ; Z \vdash Y}\,(K_L) \qquad \frac{X \vdash Y}{X \vdash Y \, ; Z}\,(K_R) \qquad \frac{X \, ; X \vdash Y}{X \vdash Y}\,(W_L) \qquad \frac{X \vdash Y \, ; Y}{X \vdash Y}\,(W_R)$$

Logical rules

$$\frac{A \, ; B \vdash Y}{A \wedge B \vdash Y}\,(\wedge_L) \quad \frac{X \vdash A \qquad Y \vdash B}{X \, ; Y \vdash A \wedge B}\,(\wedge_R) \qquad \frac{A \vdash X \qquad B \vdash Y}{A \vee B \vdash X \, ; Y}\,(\vee_L) \quad \frac{X \vdash A \, ; B}{X \vdash A \vee B}\,(\vee_R)$$

$$\frac{X \vdash A \qquad B \vdash Y}{A \to B \vdash X > Y}\,(\to_L) \quad \frac{X \vdash A > B}{X \vdash A \to B}\,(\to_R) \qquad \frac{A > B \vdash Y}{A \leftarrow B \vdash Y}\,(\leftarrow_L) \quad \frac{X \vdash A \qquad B \vdash Y}{X > Y \vdash A \leftarrow B}\,(\leftarrow_R)$$

$$\frac{X \vdash A}{\rhd A \vdash \sharp X}\,(\rhd_L) \quad \frac{X \vdash \sharp A}{X \vdash \rhd A}\,(\rhd_R) \quad \frac{X \vdash A}{\lhd A \vdash \flat X}\,(\lhd_L) \quad \frac{X \vdash \flat A}{X \vdash \lhd A}\,(\lhd_R)$$

$$\frac{\sharp A \vdash Y}{\blacktriangleright A \vdash Y}\,(\blacktriangleright_L) \quad \frac{A \vdash Y}{\sharp Y \vdash \blacktriangleright A}\,(\blacktriangleright_R) \quad \frac{\flat A \vdash Y}{\blacktriangleleft A \vdash Y}\,(\blacktriangleleft_L) \quad \frac{A \vdash Y}{\flat Y \vdash \blacktriangleleft A}\,(\blacktriangleleft_R)$$

Example 3.3. The basic principles for $\rhd$ are derivable in $\delta \mathsf{BiN}$:

$$\dfrac{\dfrac{A \vdash B}{\rhd B \vdash \sharp A}\,(\rhd_L)}{\rhd B \vdash \rhd A}\,(\rhd_R) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\bot \vdash \mathbf{I}}{\bot \vdash \flat \mathbf{I}}\,(K_R)}{\mathbf{I} \vdash \sharp \bot}\,(\sharp\flat_R)}{\mathbf{I} \vdash \rhd \bot}\,(\rhd_R)}{\top \vdash \rhd \bot}\,(K_L)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\rhd A \vdash \sharp A}}{\rhd A \wedge \rhd B \vdash \sharp A}}{A \vdash \flat(\rhd A \wedge \rhd B)} \qquad \dfrac{\dfrac{\dfrac{B \vdash B}{\rhd B \vdash \sharp B}\,(\rhd_L)}{\rhd A \wedge \rhd B \vdash \sharp B}\,(\wedge_L)}{B \vdash \flat(\rhd A \wedge \rhd B)}\,(\sharp\flat_R)}{A \vee B \vdash \flat(\rhd A \wedge \rhd B)}\,(\vee_L)}{\rhd A \wedge \rhd B \vdash \sharp(A \vee B)}\,(\sharp\flat_R)}{\rhd A \wedge \rhd B \vdash \rhd(A \vee B)}\,(\rhd_R)$$

Structural rules tells us that the nullary $\mathbf{I}$ represents an empty structure and the semicolon behaves like a comma in traditional sequents. You can see from the logical rules that conjunction and disjunction are the object-level counterpart of semicolon on the left and right hand side. Each propositional connective *reflects* the corresponding structural connective [12, 37]. The binary $>$ is reflected at object-level by implication and co-implication on the left and right hand side respectively. And the four negative operators have $\sharp$ and $\flat$ as their home.

Double lines in the display rules indicate that the rules are bi-directional and hence that the upper and lower sequents are equivalent. Given the correspondence between structural and propositional connectives, the rules concerning ; and $>$ express the residuation and its dual at meta-level. $(\sharp\flat_L)$ and $(\sharp\flat_R)$ are the meta-level expression of the fundamental connection between split negations (that is, equivalence between $A \vdash \rhd B$ and $B \vdash \lhd A$, and $\blacktriangleright A \vdash B$ and $\blacktriangleleft B \vdash A$).

An important function of display rules is to *display* an arbitrary substructure of a sequent as a whole antecedent or succedent of a sequent equivalent to the original one. Namely, they establish the defining feature of display calculus, *display property*. Whether a structure is displayed on the left or right hand side is determined by its position or polarity in the original sequent. So let us first define a notion of antecedent/succedent parts.

Definition 3.4 (Antecedent/succedent parts)**.** In $X \vdash Y$, we say that X is an *antecedent part* (AP) and Y is a *succedent part* (SP). And we define:

- If $W ; Z$ is an AP (SP) in $X \vdash Y$, then so are W and Z;
- Whether $W > Z$ is an AP or SP in $X \vdash Y$, W is an AP and Z is an SP;
- If $\sharp W$ is an AP (SP) in $X \vdash Y$, then W is an SP (AP);
- If $\flat W$ is an AP (SP) in $X \vdash Y$, then W is an SP (AP);

Now the display property (or display theorem) is formulated as follows:

Definition 3.5. We say that two sequents are *display equivalent* if they are derivable from each other using only display rules.

Theorem 3.6 (Display property)**.** Let Z be a substructure in $X \vdash Y$. Then,

- If Z is an AP in $X \vdash Y$, then there exists a sequent $Z \vdash W$ which is display equivalent to $X \vdash Y$.
- If Z is an SP in $X \vdash Y$, then there exists a sequent $W \vdash Z$ which is display equivalent to $X \vdash Y$.

Proof. We refer the reader to an elegant proof by Restall [31]. □

Note that in virtue of the display property, logical rules are presented in the "displayed" form in which the principal formulas have no side formulas (or contexts).

Belnap's aim was to make explicit general conditions for Cut elmination through the display propety. Thanks to his proof of the general Cut elimination theorem, it suffices to check several conditions to establish Cut elmination theorem for our system $\delta\mathtt{BiN}$.

Theorem 3.7 (Cut elimination)**.** If $X \vdash Y$ is derivable in $\delta\mathtt{BiN}$, it is derivable without $\mathtt{Cut}$.

Proof. See [1] for the general Cut elimination theorem. Reducibility of Cut in the case where the Cut formula $\rhd A$ is principal in both premises is shown as:

$$
\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ X \vdash \sharp A\end{array}}{X \vdash \rhd A}(\rhd_R) \qquad \dfrac{\begin{array}{c}\Pi_2\\ Y \vdash A\end{array}}{\rhd A \vdash \sharp Y}(\rhd_L)}{X \vdash \sharp Y}(\mathrm{Cut})
\quad \rightsquigarrow \quad
\dfrac{\dfrac{\begin{array}{c}\Pi_2\\ Y \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\Pi_1\\ X \vdash \sharp A\end{array}}{A \vdash \flat X}(\sharp\flat_R)}{Y \vdash \flat X}(\mathrm{Cut})}{X \vdash \sharp Y}(\sharp\flat_R)
$$

□

4 Soundness and completeness

In this section we look at soundness and completeness of $\delta\mathtt{BiN}$ with respect to its frame semantics. The proof is rather standard[5]. So we just sketch the proof.

To state the theorems we define a translation from sequents to formulas.

[5] A completeness proof of the Hilbert system of Bi-Intuitionistic logic with respect to Kripke models is given by Rauszer [30]. Decision procedure, proof search, and counter-model construction for Bi-Intuitionistic (tense) logic based on various extended sequent calculi have been investigated in [6, 19, 20, 21, 26, 27, 28]. Completeness concerning negative modalities is proved by Došen [11]. For a comprehensive exposition of frame semantics for substructural logics and (positive and negative) modalities, see [33]

Definition 4.1. Define a translation τ from sequents to formulas to be:

$$\tau(X \vdash Y) = \tau_1(X) \to \tau_2(Y)$$

$$\begin{array}{rclrcl}
\tau_1(A) &=& A & \tau_2(A) &=& A \\
\tau(\mathbf{I}) &=& \top (= q \to q) & \tau_2(\mathbf{I}) &=& \bot (= q \leftarrow q) \\
\tau_1(X\,;\,Y) &=& \tau_1(X) \wedge \tau_2(Y) & \tau_2(X > Y) &=& \tau_1(X) \to \tau_2(Y) \\
\tau_1(X > Y) &=& \tau_1(X) \leftarrow \tau_2(Y) & \tau_2(X\,;\,Y) &=& \tau_2(X) \vee \tau_2(Y) \\
\tau_1(\sharp X) &=& \blacktriangleright \tau_2(X) & \tau_2(\sharp X) &=& \triangleright \tau_1(X) \\
\tau_1(\flat X) &=& \blacktriangleleft \tau_2(X) & \tau_2(\flat X) &=& \triangleleft \tau_1(X).
\end{array}$$

We say a sequent $X \vdash Y$ is *valid* if the formula $\tau(X \vdash Y)$ is valid.

Theorem 4.2 (Soundness)**.** If $X \vdash Y$ is derivable in $\delta\mathrm{BiN}$, then $X \vdash Y$ is valid.

Proof. Induction on the construction of the derivation of $X \vdash Y$. □

In what follows we use a confused notation, that is, by $X \vdash Y$ ($X \nvdash Y$), we mean that the sequent is (not) derivable in $\delta\mathrm{BiN}$. Completeness proof appeals to well-known notions of *maximal consistent pair* and *canonical model*.

Definition 4.3 (Maximal consistent pair)**.** For any (possibly infinite) sets x, y of formulas, we define

$$\begin{aligned} x \vdash y \iff & \exists B_1, \dots, B_n \in x\ \exists C_1, \dots, C_m \in y : \\ & B_1\,;\cdots;\,B_n \vdash C_1\,;\cdots;\,C_m. \end{aligned}$$

$x \nvdash y$ means that it is not the case that $x \vdash y$. A pair (x, y) is *maximally consistent* if $x \cup y = \mathcal{L}$ (the set of all formulas) and $x \nvdash y$.

Sequents such as $x \vdash A$ or $A; B \vdash y$ which consists of a finite structure and a set of formulas should be analogously understood.

Lemma 4.4 (Pair extension)**.** If $x' \nvdash y'$, then there exists a maximal consistent pair (x, y) such that $x' \subseteq x$ and $y' \subseteq y$.

Proof. Standard. □

Proposition 4.5. For any maximal consistent pair (x, y), the following hold:

(1) $A \in x \iff x \vdash A$ (2) $A \in y \iff A \vdash y$

(3) $A \wedge B \in x \iff A \in x \;\&\; B \in x$ (4) $A \wedge B \in y \iff A \in y$ or $B \in y$

(5) $A \vee B \in x \iff A \in x$ or $B \in x$ (6) $A \vee B \in y \iff A \in y \;\&\; B \in y$

(7) $A \to B \in y \Longrightarrow B \in y$ (8) $A \leftarrow B \in x \Longrightarrow A \in x$

(9) $\rhd A \in y \Longrightarrow$ there is a maximal consistent pair (x', y') such that $x_{\rhd} \cap x' = \emptyset \;\&\; A \in x'$

(10) $\lhd A \in y \Longrightarrow$ there is a maximal consistent pair (x', y') such that $x \cap x'_{\lhd} = \emptyset \;\&\; A \in x'$

(11) $\blacktriangleright A \in x \Longrightarrow$ there is a maximal consistent pair (x', y') such that $x_{\blacktriangleright} \cup x' = \mathcal{L} \;\&\; A \in y'$

(12) $\blacktriangleleft A \in x \Longrightarrow$ there is a maximal consistent pair (x', y') such that $x \cup x'_{\blacktriangleleft} = \mathcal{L} \;\&\; A \in y'$

where $x_\circ = \{A \mid \circ A \in x\}$ for $\circ \in \{\rhd, \lhd, \blacktriangleright, \blacktriangleleft\}$.

Proof. We look at only (11). Assume that $\blacktriangleright A \in x$, i.e. $x \vdash \blacktriangleright A$. First we show that $(x_{\blacktriangleright})^c \nvdash A$, where $(x_{\blacktriangleright})^c = \mathcal{L} \setminus x_{\blacktriangleright}$. Suppose the contrary. Then there are $B_1, \ldots, B_n \notin x_{\blacktriangleright}$ such that $B_1 \wedge \cdots \wedge B_n \vdash A$. By the basic principles for $\blacktriangleright$ and Cut, it follows that $\blacktriangleright A \vdash \blacktriangleright B_1 \vee \cdots \vee \blacktriangleright B_n$. By Cut with $x \vdash \blacktriangleright A$, we have $x \vdash \blacktriangleright B_1 \vee \cdots \vee \blacktriangleright B_n$, hence $\blacktriangleright B_1 \vee \cdots \vee \blacktriangleright B_n \in x$. But by (5), $\blacktriangleright B_i \in x$ for some i, which means that $B_i \in x_{\blacktriangleright}$. A contradiction. Applying Lemma 4.4 to $(x_{\blacktriangleright})^c \nvdash A$, we obtain a maximal consistent pair (x', y') such that $(x_{\blacktriangleright})^c \subseteq x'$ and $A \in y'$. $(x_{\blacktriangleright})^c \subseteq x'$ is equivalent to $x_{\blacktriangleright} \cup x' = \mathcal{L}$. □

This proposition suggests the following definition of canonical model.

Definition 4.6. We define the *canonical model* $\langle W^*, \leqslant^*, \frown^*, \smile^*, \models^* \rangle$ as:

$$\begin{aligned}
W^* &:= \{x \subseteq \mathcal{L} \mid (x, x^c) \text{ is a maximal consistent pair}\},\\
x \leqslant^* x' &\iff x \subseteq x'\\
x \frown^* x' &\iff x_{\rhd} \cap x' = x \cap x'_{\lhd} = \emptyset\\
x \smile^* x' &\iff x_{\blacktriangleright} \cup x' = x \cup x'_{\blacktriangleleft} = \mathcal{L}\\
x \models^* p &\iff p \in x.
\end{aligned}$$

It is immediate that the valuation $\models^*$ is hereditary and that the frame conditions are satisfied. Hence the canonical model is a BiN model.

Now the following lemma almost completes the proof of completeness.

Lemma 4.7. For any $x \in W^*$ in the canonical model and any formula A,

$$x \models A \iff A \in x.$$

Proof. Induction on the complexity of A. Use Lemma 4.5. □

Theorem 4.8 (Completeness)**.** If a sequent is valid, then it is derivable in $\delta\mathsf{BiN}$.

Proof. Suppose that $X \nvdash Y$. Then obviously $\tau_1(X) \nvdash \tau_2(Y)$. By Lemma 4.4, we have a maximal consistent pair (x, y) such that $\tau_1(X) \in x$ and $\tau_2(Y) \in y$. By Lemma 4.7, $x \models \tau_1(X)$ and $x \not\models \tau_2(Y)$ in the canonical model. That is, the sequent $X \vdash Y$ is not valid. □

We note that each accessibility is determined by one of its two conditions because they are equivalent.

Lemma 4.9. In the canonical model,

$$x_{\triangleright} \cap y = \emptyset \iff x \cap y_{\triangleleft} = \emptyset$$
$$x_{\blacktriangleright} \cup x' = \mathcal{L} \iff x \cup x'_{\blacktriangleleft} = \mathcal{L}.$$

Proof. Assume that $x_{\triangleright} \cap y = \emptyset$, and yet $A \in x \cap y_{\triangleleft}$ for some A. Then $\triangleleft A \in y$, so $\triangleleft A \notin x_{\triangleright}$, and hence $\triangleright \triangleleft A \notin x$. But $A \in x$ implies $\triangleright\triangleleft A \in x$ since $A \vdash \triangleright\triangleleft A$ is provable. A contradiction. Therefore $x \cap y_{\triangleleft} = \emptyset$. The converse and the case for $\blacktriangleright, \blacktriangleleft$ are similar. □

This fact tells that once you introduce a negative modality you obtain its split twin for free.

5 The united kite of negation

The four negations discussed so far are very weak. Stronger negations are obtained by imposing various constraints or axioms. The family of such negations has been pictured in the form of a "kite", one for impossibility type, one for unnecessity type. In subsection 5.1, we review the correspondences between frame constraints and axioms investigated in the literature, show that (almost) all of them can be represented by structural rules of our display calculus, and raise problems concerning double negation elimination and correspondence to star semantics. In 5.2, we define a translation of sequents, rules, and frame constraints, which gives the kite of unnecessity from that of impossibility at one time. Comparison of the two kites tells us that identifying impossibility and unnecessity would solve the problem raised. We present axioms, constraints, and structural rules for that.

5.1 The kite of impossibility

Let us begin with the kite of impossibility. Below is a list of corresponding pairs of axioms and frame constraints. All of them already appeared in Došen's work [11] except for FDM (Final de Morgan), which we can find in [33]. See also [2, 13, 17, 32, 44] for detailed proofs and discussions.

Theorem 5.1. In the table below, each sequent (axiom) is valid in a frame if and only if the frame satisfies the constraint in the same row. Moreover they are all *canonical*. That is, for each extension of BiN with some axioms below, its canonical model satisfies the corresponding constraints.

DNI	$A \vdash \rhd\rhd A$	$\forall xy : x \frown y \Rightarrow y \frown x$
Con	$A \to B \vdash \rhd B \to \rhd A$	$\forall xy : x \frown y \Rightarrow \exists z : x \leqslant z \,\&\, y \leqslant z \,\&\, x \frown z$
LNC	$A \wedge \rhd A \vdash B$	$\forall x : x \frown x$
$\rhd\top\bot$	$\rhd\top \vdash \bot$	$\forall x \exists y : x \frown y$
LEM	$B \vdash A \vee \rhd A$	$\forall xy : x \frown y \Rightarrow x \geqslant y$
FDM	$\rhd(A \wedge B) \vdash \rhd A \vee \rhd B$	$\forall xyz : x \frown y \,\&\, x \frown z \Rightarrow$ $\exists w : y \leqslant w \,\&\, z \leqslant w \,\&\, x \frown w$
DNE	$\rhd\rhd A \vdash A$	$\forall x \exists y : x \frown y \,\&\, \forall z : y \frown z \Rightarrow x \geqslant z$

These pairs can be represented by structural rules of $\delta\mathsf{BiN}$. Although we found them by hand, they could be automatically obtained from axioms through the *general recipe* presented in [7].

Theorem 5.2. Each of the following structural rules correponds to the constraint with the same name. That is, a frame satisfies the constraints if and only if the rule preserves the validity in the frame. And Cut elimination holds for any extension of $\delta\mathsf{BiN}$ with any set of these structural rules.

$$\dfrac{X \vdash \sharp Y}{Y \vdash \sharp X}\,(\mathrm{DNI}) \qquad \dfrac{X \vdash \sharp(Y \,;\, Z)}{X \vdash Y > \sharp Z}\,(\mathrm{Con}) \qquad \dfrac{X \vdash \sharp Y}{X \vdash Y > \mathbf{I}}\,(\mathrm{LNC})$$

$$\dfrac{X \vdash \sharp\mathbf{I}}{X \vdash \mathbf{I}}\,(\rhd\top\bot) \qquad \dfrac{X \,;\, Y \vdash Z}{X \vdash \sharp Y \,;\, Z}\,(\mathrm{LEM}) \qquad \dfrac{X \vdash \sharp(Y \,;\, Z)}{X \vdash \sharp Y \,;\, \sharp Z}\,(\mathrm{FDM})$$

Proof. We present derivations of axioms using corresponding structural rules (double lines indicate (consecutive) applications of display rules).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A \to B \vdash A > B}\,(\to_{\mathrm{L}})}{A \to B \,;\, A \vdash B}}{\rhd B \vdash \sharp(A \to B \,;\, A)}\,(\rhd_{\mathrm{L}})}{\rhd B \vdash A \to B > \sharp A}\,(\mathrm{Con})}{A \to B \,;\, \rhd B \vdash \sharp A}}{A \to B \,;\, \rhd B \vdash \rhd A}\,(\rhd_{\mathrm{R}})}{\dfrac{A \to B \vdash \rhd B > \rhd A}{A \to B \vdash \rhd B \to \rhd A}\,(\to_{\mathrm{R}})}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\rhd A \vdash \sharp A}\,(\rhd_{\mathrm{L}})}{\rhd A \vdash A > \mathbf{I}}\,(\mathrm{LNC})}{A \,;\, \rhd A \vdash \mathbf{I}}}{A \wedge \rhd A \vdash \mathbf{I}}\,(\wedge_{\mathrm{L}})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\mathbf{I} \,;\, A \vdash A}\,(\mathrm{K}_{\mathrm{L}})}{\mathbf{I} \vdash \sharp A \,;\, A}\,(\mathrm{LEM})}{\mathbf{I} > A \vdash \sharp A}}{\mathbf{I} > A \vdash \rhd A}\,(\rhd_{\mathrm{R}})}{\mathbf{I} \vdash A \,;\, \rhd A}}{\mathbf{I} \vdash A \vee \rhd A}\,(\vee_{\mathrm{R}})$$

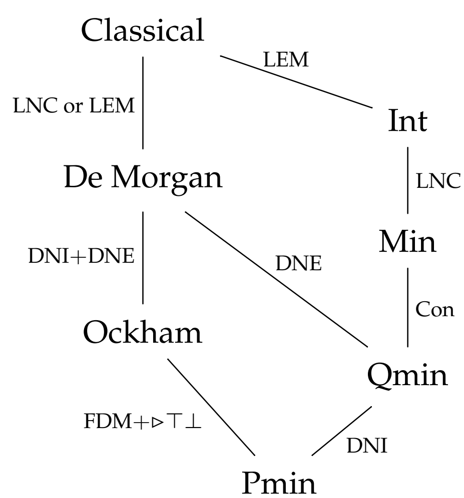
$$\dfrac{\dfrac{\dfrac{A \vdash A}{\rhd A \vdash \sharp A}\,(\rhd_L)}{A \vdash \sharp \rhd A}\,(\mathrm{DNI})}{A \vdash \rhd \rhd A}\,(\rhd_R)
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{A; B \vdash A \wedge B}}{\rhd(A \wedge B) \vdash \sharp(A; B)}\,(\rhd_L)}{\rhd(A \wedge B) \vdash \sharp A; \sharp B}\,(\mathrm{FDM})}{\rhd(A \wedge B) > \sharp A \vdash \sharp B}}{\rhd(A \wedge B) > \sharp A \vdash \rhd B}\,(\rhd_R)}{\rhd(A \wedge B) > \rhd B \vdash \sharp A}}{\rhd(A \wedge B) > \rhd B \vdash \rhd A}\,(\rhd_R)}{\dfrac{\rhd(A \wedge B) \vdash \rhd A; \rhd B}{\rhd(A \wedge B) \vdash \rhd A \vee \rhd B}\,(\vee_R)}$$

$$\dfrac{\dfrac{\dfrac{\mathbf{I} \vdash \top}{\rhd \top \vdash \sharp \mathbf{I}}\,(\rhd_L)}{\rhd \top \vdash \mathbf{I}}\,(\rhd\top\bot)}{\rhd \top \vdash \bot}\,(K_R)$$

Recall that $\top = q \to q$ so $\mathbf{I} \vdash \top$ is derivable in $\delta\mathrm{BiN}$.

Soundness of each rule with respect to the corresponding constraint is easily checked. For Cut elimination, it suffices to check by eye that the conditions for general Cut elmination hold. □

Now familiar negations are obtained by adding some of the above axioms (or correspondingly, constraints or structural rules). The hierarchy was originally depicted by Dunn [15, 14, 16] in the form of a kite, but now no longer looks like so through modifications [38, 44] [6]:



The weakest negation in our framework, $\rhd$ in BiN with no addtional axiom or constraints, is *preminimal* negation (Pmin), which is what we called split negation with its twin $\lhd$ [7]. It is clear that adding DNI (Double Negation Introduction) or symmetry of the accessibility $\frown$, the split negations get equated, i.e. we have $\rhd A \dashv\vdash \lhd A$. The collapsed negation is *quasi-minimal*

[6] An even more comprehensive diagram can be found in [35]. It contains those negations that cannot be captured by the compatibility or exhaustiveness semantics.

[7] Here we follow [17, 44] for the nomenclature of negations.

negation. The axiom Con (minimal Contraposition) turns quasi-minimal negation into *minimal* negation. It is equivalent to the negation in minimal logic defined as $A \to r$ for some fixed, arbitrary atom r.

If one assumes further the axiom LNC (Law of Non-Contradiction), or equivalently requires that $\frown$ be reflexive, $\rhd$ gets absorbed to *intuitionistic* negation. Indeed, the axiom $\rhd\top\bot$ or the seriality constraints suffices. It is easily checked that Con + DNI + $\rhd\top\bot$ is equivalent to Con + LNC. While we have $\rhd A \dashv\vdash \neg A$ under these constraints, $\frown$ is not necessarily equated with $\leqslant$. Instead we then have $\frown\, = \,\leqslant \circ \geqslant$. And notice that the clause for $\neg A$ can be rephrased as:

$$x \models \neg A \iff \forall y : x \leqslant \circ \geqslant y \Rightarrow y \not\models A.$$

This explains the equivalence $\rhd A \dashv\vdash \neg A$.

Finally *classical* negation[8] is obtained by adding LEM (Law of Excluded Middle) or DNE (Double Negation Elimination). The extension is not conservative in a sense because it makes the whole logic classical (Peirce's law $((A \to B) \to A) \to A$ becomes provable). And classical logic is also obtained by MC (Material Conditional) which corresponds to symmetry of the information order $\leqslant$.

$$\text{MC} \quad A \to (B \vee C) \vdash (A \to B) \vee C \qquad \frac{X \vdash Y > (W\,;Z)}{X \vdash (Y > W)\,; Z}\,(\text{MC})$$

Back to quasi-minimal negation and strengthen it with DNE. We obtain *de Morgan* negation. The constraint for DNE implies the *star postulate* (found by Restall [34], the name taken from [24]), in effect that every state has a maximal state among those compatible with x, namely:

$$\forall x \exists y : x \frown y \;\&\; \forall w : x \frown w \Rightarrow y \geqslant w, \tag{*}$$

and the y is the star state x^* of x (we have it that $x \models \rhd A$ iff $x^* \not\models A$). Moreover, the $*$-function is of period two since we have $x \leqslant x^{**}$ by symmetry of $\frown$ (DNI) and $x^{**} \leqslant x$ by DNE. De Morgan negation is reduced to classical negation by adding LNC or LEM.

There is another way to get at de Morgan negation. *Ockham* negation is preminimal negation plus FDM and $\rhd\top\bot$. According to Zhou [44], it is the weakest negation that has star semantcs, or equivalently that is complete with respect to the class of frames satisfying the condition (*) above (He called them *star-crossed* frames). De Morgan negation is equivalently rewritten as Ockham negation (*) plus DNI and DNE ($x = x^{**}$).

We raise two problems here. First, as noticed, no structural rule for DNE has been found. We don't have a proof of that there is none either. But

[8] Or *Ortho* negation if the base logic of conjunction and disjunction is non-distributive.

the general recipe in [7] is not applicable to it (the axiom $\triangleright\triangleright A \vdash A$ is not *soluble* in their term). There is a lacuna to be filled in the whole picture. Secondly, although the extension of $\delta\mathsf{BiN}$ with the axioms FDM $+ \triangleright\top\bot$ (the logic of Ockham negation) is *complete* with respect to star-crossed frames, it does *not correspond* to the condition (*). There is a frame that satisfies the conditions for FDM $+ \triangleright\top\bot$ (hence validates them) but not (*) [9]. Here is a lack of correspondence.

These are problems concerning Ockham and de Morgan (and hence classical) negation. These problems will be solved by introducing another way to obtain these "self-dual" (in the sense shortly made clear) negations, that is, by redefining them as fusions of two negations. To see this, we shoud look at the kite of unnecessity.

5.2 The kite of unnecessity

Corresponding triples of axioms, frame constraints, and structural rules for unnecessity can be obtained through a "dualising" translation from the list for impossibility above. Define the translation $(-)^\bullet$ of formulas as: $p^\bullet = p$ for atom p and

$$\begin{array}{cc}
(A \wedge B)^\bullet = A^\bullet \vee B^\bullet & (A \vee B)^\bullet = A^\bullet \wedge B^\bullet \\
(A \to B)^\bullet = B^\bullet \leftarrow A^\bullet & (A \leftarrow B)^\bullet = B^\bullet \to A^\bullet \\
(\triangleright A)^\bullet = \blacktriangleright A^\bullet & (\triangleleft A)^\bullet = \blacktriangleleft A^\bullet \\
(\blacktriangleright A)^\bullet = \triangleright A^\bullet & (\blacktriangleleft A)^\bullet = \triangleleft A^\bullet
\end{array}$$

The translation extends to sequents built from structures, and inferences:

$$\begin{array}{cc}
\mathbf{I}^\bullet = \mathbf{I} & A^\bullet = A^\bullet \\
(X\,;\,Y)^\bullet = X^\bullet\,;\,Y^\bullet & (X > Y)^\bullet = Y^\bullet > X^\bullet \\
(\sharp X)^\bullet = \sharp X^\bullet & (\flat X)^\bullet = \flat X^\bullet \\
(X \vdash Y)^\bullet = Y^\bullet \vdash X^\bullet & \left(\dfrac{X \vdash Y}{W \vdash Z}\right)^\bullet = \dfrac{Y^\bullet \vdash X^\bullet}{Z^\bullet \vdash W^\bullet}
\end{array}$$

It should be clear how to extend it to expressions involving schematic letters. For example, axioms and rules are translated as:

$$\big(A \wedge \triangleright A \vdash B\big)^\bullet = B \vdash A \vee \blacktriangleright A \quad \big(\triangleright(A \wedge B) \vdash \triangleright A \vee \triangleright B\big)^\bullet = \blacktriangleright A \wedge \blacktriangleright B \vdash \blacktriangleright(A \vee B)$$

$$\left(\frac{X \vdash \sharp Y}{X \vdash Y > \mathbf{I}}\right)^\bullet = \frac{\sharp X \vdash Y}{\mathbf{I} > X \vdash Y} \quad \left(\frac{X \vdash \sharp(Y\,;\,Z)}{X \vdash Y > \sharp Z}\right)^\bullet = \frac{\sharp(X\,;\,Y) \vdash Z}{\sharp X > Y \vdash Z}$$

[9] Consider the frame of the natural numbers with its standard ordering, and the universal compatibility relation, that is, $n \frown m$ for any natural numers n and m. Then the frame conditions and additional conditions for FDM and $\triangleright\top\bot$ are trivially satisfied. But there is no maximal state (number) that is compatible with, say, 0 since 0 is compatible with all natural numbers.

Further we extend it to (first-order expressions of) frame constraints φ. Let $\varphi^\bullet$ be the constraint obtained from φ by replacing all $\frown$ with $\smile$ and all $\leqslant$ with $\geqslant$, and vice versa. For example,

$$\big(\forall xy : x \frown y \Rightarrow x \geqslant y\big)^\bullet = \forall xy : x \smile y \Rightarrow x \leqslant y$$

The following propositions ensure that correspondence among axioms, constraints, and rules is preserved through the translation.

Proposition 5.3. A sequent S is derivable in an extention of $\delta\mathtt{BiInt}$ with a structural rule R if and only if $S^\bullet$ is derivable in $\delta\mathtt{BiInt} + R^\bullet$.

Proposition 5.4. A sequent S is valid in any frame that satisfies a constraint φ if and only if $S^\bullet$ is valid in any frame with the constraint $\varphi^\bullet$.

Proposition 5.5. A constraint φ is satisfied by any frame that validates a sequent S if and only if $\varphi^\bullet$ is satisfied by any frame that validates $S^\bullet$.

The latter two propositions follow from the lemma below:

Lemma 5.6. For any model $M = \langle W, \leqslant, \frown, \smile, \models\rangle$, let $M^\bullet := \langle W, \geqslant, \smile, \frown, \models^\bullet\rangle$, where $x \models^\bullet p$ if and only if $x \not\models p$. It is easy to see that the heredity property and the frame conditions are satisfied in $M^\bullet$. Then for any $x \in W$ and any formula A,

$$x \models A \iff x \not\models^\bullet A^\bullet.$$

Proof. Induction on the complexity of A. We look at two cases.

$$\begin{aligned}
x \models A \to B &\iff \forall y : x \leqslant y \,\&\, y \models A \Rightarrow y \models B \\
&\iff \forall y : y \geqslant x \,\&\, y \models A \Rightarrow y \models B \\
&\iff \forall y : y \geqslant x \,\&\, y \not\models^\bullet A^\bullet \Rightarrow y \not\models^\bullet B^\bullet && \text{(IH)} \\
&\iff x \not\models^\bullet B^\bullet \leftarrow A^\bullet \; [= (A \to B)^\bullet]. \\
x \models \triangleright A &\iff \forall y : x \frown y \Rightarrow y \not\models A \\
&\iff \forall y : x \frown y \Rightarrow y \models^\bullet A^\bullet && \text{(IH)} \\
&\iff x \not\models^\bullet \blacktriangleright A^\bullet \; [= (\triangleright A)^\bullet].
\end{aligned}$$

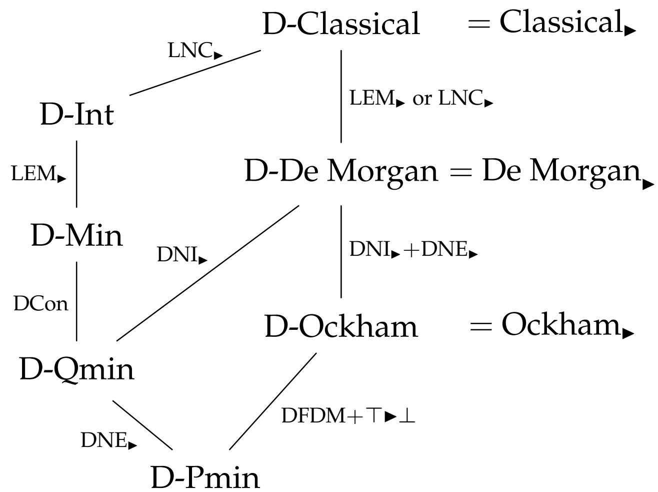
□

Here are the list and kite for $\blacktriangleright$.

$(\mathrm{DNI})^\bullet = \mathrm{DNE}_\blacktriangleright$	$\blacktriangleright\blacktriangleright A \vdash A$	$\forall xy : x \smile y \Rightarrow y \smile x$
$(\mathrm{Con})^\bullet = \mathrm{DCon}$	$\blacktriangleright A \leftarrow \blacktriangleright B \vdash B \leftarrow A$	$\forall xy : x \smile y \Rightarrow \exists z : x \geqslant z \,\&\, y \geqslant z \,\&\, x \smile z$
$(\mathrm{LNC})^\bullet = \mathrm{LEM}_\blacktriangleright$	$B \vdash A \vee \blacktriangleright A$	$\forall x : x \smile x$
$(\triangleright\top\bot)^\bullet = \top\blacktriangleright\bot$	$\top \vdash \blacktriangleright\bot$	$\forall x \exists y : x \smile y$
$(\mathrm{LEM})^\bullet = \mathrm{LNC}_\blacktriangleright$	$A \wedge \blacktriangleright A \vdash B$	$\forall xy : x \smile y \Rightarrow x \leqslant y$
$(\mathrm{FDM})^\bullet = \mathrm{DFDM}$	$\blacktriangleright A \wedge \blacktriangleright B \vdash \blacktriangleright(A \vee B)$	$\forall xyz : x \smile y \,\&\, x \smile z \Rightarrow$ $\exists w : y \geqslant w \,\&\, z \geqslant w \,\&\, x \smile w$
$(\mathrm{DNE})^\bullet = \mathrm{DNI}_\blacktriangleright$	$A \vdash \blacktriangleright\blacktriangleright A$	$\forall x \exists y : x \smile y \,\&\, \forall z : y \smile z \Rightarrow x \leqslant z$

$$\frac{\sharp X \vdash Y}{\sharp Y \vdash X}\,(\mathrm{DNI}_{\blacktriangleright}) \qquad \frac{\sharp(X \,;\, Y) \vdash Z}{\sharp X > Y \vdash Z}\,(\mathrm{DCon}) \qquad \frac{\sharp X \vdash Y}{\mathbf{I} > X \vdash Y}\,(\mathrm{LEM}_{\blacktriangleright})$$

$$\frac{\sharp \mathbf{I} \vdash Y}{\mathbf{I} \vdash Y}\,(\top{\blacktriangleright}\bot) \qquad \frac{X \vdash Y \,;\, Z}{X \,;\, \sharp Y \vdash Z}\,(\mathrm{LNC}_{\blacktriangleright}) \qquad \frac{\sharp(X \,;\, Y) \vdash Z}{\sharp X \,;\, \sharp Y \vdash Z}\,(\mathrm{DFDM})$$



The subscript $(-)_{\blacktriangleright}$ indicates that the negation, axiom or rule is expressed by $\blacktriangleright$. $(-)_{\triangleright}$ will also be used to avoid confusion. The initial "D" is for "dual".

Each node of the above kite is the translation of the corresponding impossibility. For example, the translation of the basic principles of $\triangleright$ (that specifies preminimal negation) specifies dual preminimal negation (D-Pmin) as listed below (items on the same row are associated by the translation). Indeed they are the basic principles of unnecessity. Now you will notice that dual Ockham, de Morgan, and classical negations are just Ockham, de Morgan, and classical negations expressed by $\blacktriangleright$. Each of them satisfies the same principles as the translated one (so we write like "Ockham$_{\blacktriangleright}$"). Thus, these three negations translates into themselves through the dualising $(-)^{\bullet}$, hence they can be called *self-dual* negations.

D-Pmin	$\frac{A \vdash B}{\blacktriangleright B \vdash \blacktriangleright A}$ $\blacktriangleright\top \vdash \bot$ $\blacktriangleright(A \wedge B) \vdash \blacktriangleright A \vee \blacktriangleright B$	$\frac{A \vdash B}{\triangleright B \vdash \triangleright A}$ $\top \vdash \triangleright\bot$ $\triangleright A \wedge \triangleright B \vdash \triangleright(A \vee B)$	Pmin
D-Ockham = Ockham$_{\blacktriangleright}$	$\blacktriangleright A \wedge \blacktriangleright B \vdash \blacktriangleright(A \vee B)$ $\top \vdash \blacktriangleright\bot$	$\triangleright(A \wedge B) \vdash \triangleright A \vee \triangleright B$ $\triangleright\top \vdash \bot$	Ockham$_{\triangleright}$
D-De Morgan = De Morgan$_{\blacktriangleright}$	$A \vdash \blacktriangleright\blacktriangleright A$ $\blacktriangleright\blacktriangleright A \vdash A$	$\triangleright\triangleright A \vdash A$ $A \vdash \triangleright\triangleright A$	De Morgan$_{\triangleright}$
D-Classical$_{\blacktriangleright}$ = Classical$_{\blacktriangleright}$	$B \vdash A \vee \blacktriangleright A$ or $A \wedge \blacktriangleright A \vdash B$	$A \wedge \triangleright A \vdash B$ or $B \vdash A \vee \triangleright A$	Classical$_{\triangleright}$

5.3 Identification of impossibility and unnecessity

As can be seen from the table above, the basic principles of $\blacktriangleright$ contain the additional axioms (FDM and $\triangleright\top\bot$) for Ockham$_{\triangleright}$, expressed by $\blacktriangleright$. Conversely, the additional axioms for Ockham$_{\blacktriangleright}$ have the same form as two of the basic principles for $\triangleright$. This suggests that if we *identify* $\triangleright$ and $\blacktriangleright$, that is, if we add to preminimal (or dual preminimal) negation the axioms $\triangleright A \vdash \blacktriangleright A$ and $\blacktriangleright A \vdash \triangleright A$, then the resultant negation is Ockham negation. Identifying the dual negations is another way to obtain self-dual negations. These axioms can be represented by frame constraints and structural rules.

Proposition 5.7. Below is the corresponding triples for identification of $\triangleright$ and $\blacktriangleright$. They are also canonical.

$$\begin{array}{llll}
\blacktriangleright\triangleright & \blacktriangleright A \vdash \triangleright A & \forall xzw : x \frown z \,\&\, x \smile w \Rightarrow z \leqslant w & \dfrac{X \vdash Y}{\sharp Y \vdash \sharp X}\,(\blacktriangleright\triangleright) \\[2ex]
\triangleright\blacktriangleright & \triangleright A \vdash \blacktriangleright A & \forall x \exists y : x \frown y \,\&\, x \smile y & \dfrac{\flat X \vdash \flat Y}{Y \vdash X}\,(\triangleright\blacktriangleright)
\end{array}$$

Proof. We look at only correspondence and canonicality for $\blacktriangleright\triangleright$. It is routine to check that the constraints validate the axiom and rule. The axiom is derived using the rule as follows:

$$\dfrac{\dfrac{\dfrac{A \vdash A}{\sharp A \vdash \sharp A}\,(\blacktriangleright\triangleright)}{\blacktriangleright A \vdash \sharp A}\,(\blacktriangleright_L)}{\blacktriangleright A \vdash \triangleright A}\,(\triangleright_R)$$

Now suppose that a frame does not satsify the constraint. Then we have x, z, and w such that $x \frown z$, $x \smile w$, but $z \nleqslant w$. Define a valuation of p as:

$$u \models p \iff z \leqslant u.$$

It is clear that this valuation is hereditary. Then since $z \nleqslant w$, we have $w \not\models p$, and hence $x \models \blacktriangleright p$. On the other hand, $z \models p$ since $z \leqslant z$. Therefore $x \not\models \triangleright p$ by $x \frown z$. Thus $\blacktriangleright A \vdash \triangleright A$ is not valid in any frame without the constraint. This extablishes correspondence among the axiom, constraint, and structural rule.

Finally to show canonicality, suppose that $x \frown^* z$ and $x \smile^* w$ in the canonical model. We show $z \subseteq w$. If $A \in z$, then since $x_{\triangleright} \cap z = \emptyset$, we have $A \notin x_{\triangleright}$, that is, $\triangleright A \notin x$. By $\blacktriangleright A \vdash \triangleright A$, it follows that $\blacktriangleright A \notin x$. But since $x \smile^* w$, i.e. $x_{\blacktriangleright} \cup w = \mathcal{L}$, we can conclude that $A \in w$, and hence $z \subseteq w$. $\square$

Let us denote by $\triangleright = \blacktriangleright$ the pair of these axioms or constraints. The point of using $\triangleright = \blacktriangleright$ can be seen clearly from:

Proposition 5.8. The constraint $\triangleright = \blacktriangleright$ is equivalent to the following condition:

$$\forall x \exists y : x \frown y \,\&\, x \smile y \,\&\, \forall z : x \frown z \Rightarrow z \leqslant y \,\&\, \forall w : x \smile w \Rightarrow y \leqslant w. \quad (**)$$

Proof. Assume $\triangleright = \blacktriangleright$. Then for any x, we have y such that $x \frown y$ and $x \smile y$ by $\triangleright\blacktriangleright$. And $x \frown z$, together with $x \smile y$, implies $z \leqslant y$ by $\blacktriangleright\triangleright$. Similarly for the last conjunct. Thus, $\triangleright = \blacktriangleright$ implies (**). Conversely, (**) obviously implies $\triangleright\blacktriangleright$. And if $x \frown z$ and $x \smile w$, then by (**), $z \leqslant y$ and $y \leqslant w$, and hence $z \leqslant w$. Thus $\blacktriangleright\triangleright$ holds. $\square$

The condition (**) is a stronger version of (*), which claims the existence of a special state $x^*(= y$ in the proof) for each state x that is not only maximal among the compatible, but also minimal among the jointly exhaustive with x. Of course we then have:

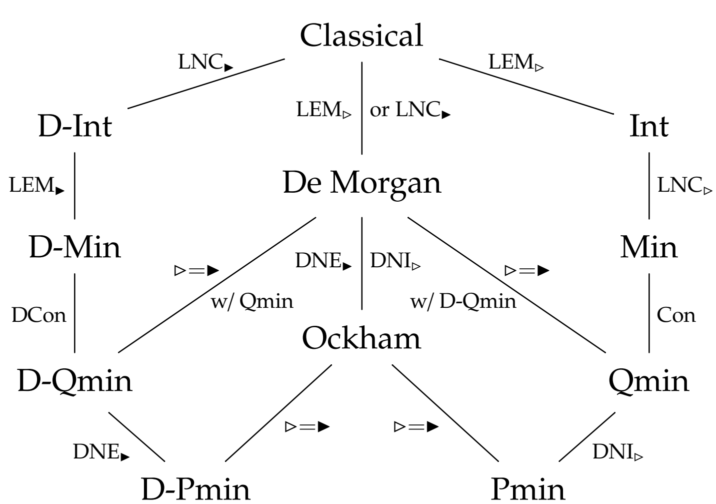
$$x \models \triangleright A \iff x^* \not\models A \iff x \models \blacktriangleright A.$$

Note that now there is correspondence between the axioms and (**), which was lacked in the case of (*). The language of $\triangleright$ (or compatibility) does not have enough expressive power to represent star-crossed frames. Both of $\triangleright$ and $\blacktriangleright$ (or correspondingly, compatibility and exhaustivity) are required. And in particular, we have revealed that assuming the existence of star states is nothing but identifying impossibility with unnecessity.

Moreover this strategy solves the other problem as well: the lack of structural rule for $\mathrm{DNE}_{\triangleright}$ (and one for $\mathrm{DNI}_{\blacktriangleright}$ by the translation), which is necessary for obtaining De Morgan negation. Recall that De Morgan negation is Ockham negation augmented with DNE and DNI. But now Ockham negation is supposed to be the fusion of $\triangleright$ and $\blacktriangleright$. So De Morgan negation then can be obtained by adding $\mathrm{DNI}_{\triangleright}$ and $\mathrm{DNE}_{\blacktriangleright}$ only. And they have structural rules. We do not need $\mathrm{DNE}_{\triangleright}$ and $\mathrm{DNI}_{\blacktriangleright}$:

$$\begin{aligned} \text{De Morgan} &= \text{Ockham} + \mathrm{DNE} + \mathrm{DNI} \\ &= (\triangleright = \blacktriangleright) + \mathrm{DNE} + \mathrm{DNI} \\ &= (\triangleright = \blacktriangleright) + \mathrm{DNE}_{\blacktriangleright} + \mathrm{DNI}_{\triangleright}. \end{aligned}$$

Now we have got the united kite:



Note that Qmin (D-Qmin) should be identified by $\triangleright = \blacktriangleright$ with D-Qmin (Qmin) which has $\mathrm{DNE}_{\blacktriangleright}$ ($\mathrm{DNI}_{\triangleright}$) to get at De Morgan.

Now all the axioms in this figure have its corresponding structural rules. In other words, each negation placed below Classical negation in the kite is obtained from Classical negation by restricting some structural rules. In this sense they are *substructural negations*.

6 Bi-relational frames

We have been working with "tri-relational" frames (and models), namely frames equipped with, besides information order, compatibility and exhaustiveness relations independent from each other. In this section we consider identifying them. It reveals the expressive power of Bi-Intuitionistic logic.

Dunn and Zhou [17] propose the following axioms (in our symbols):

$$\triangleright A \wedge \blacktriangleright B \vdash \blacktriangleright(A \vee B) \qquad \triangleright(A \wedge B) \vdash \triangleright A \vee \blacktriangleright B$$

The logic BiN plus these axioms is complete with respect to the class of *bi-relational* frames in which the unique modal accessibility, say R, takes care of both $\triangleright$ and $\blacktriangleright$, and satisfies only the weaker frame conditions:

$$\leqslant \circ R \subseteq R \circ \geqslant \qquad \geqslant \circ R \subseteq R \circ \leqslant$$

The former is for heredity of $\triangleright$ and the latter for $\blacktriangleright$. The problem is that the pair of axioms does *not correspond* to the constraint $\frown = \smile$ in tri-relational frames[10].

Božić and Došen consider a different idea in the context of intuitionistic (positive) modal logic, $\mathsf{HK}\Box\Diamond$ [5]. The frame for $\mathsf{HK}\Box\Diamond$ is also defined as tri-relational in the first place. To $\Box$ and $\Diamond$, distinct relations, say $R_\Box$ and $R_\Diamond$, are associated, and frame conditions for them are required to ensure heredity. To make it a bi-relational frame logic they added to $\mathsf{HK}\Box\Diamond$ the following axioms that contain intuitionistic negation:

$$\Diamond A \vee \Box\neg A \qquad \neg(\Diamond A \wedge \Box\neg A).$$

But their axioms are not satisfactory either. As in Dunn and Zhou's case, the logic $\mathsf{HK}\Box\Diamond$ plus the above axioms is complete with respect to the class

[10] Consider a frame F of two states x, y, with $x \frown x$, $x \smile x$, $x \frown y$, $y \frown x$, $y \smile x$, $y \frown y$, $y \leqslant x$, $x \leqslant x$, and $y \leqslant y$. In this frame, though $\frown \neq \smile$, the two axioms above are valid.

$\triangleright A \wedge \blacktriangleright B \vdash \blacktriangleright(A \vee B)$: Suppose that $x \models \triangleright A \wedge \blacktriangleright B$. Then $x \not\models A$ by $x \frown x$, and $x \not\models B$ since $x \smile x$ and there is no other complement state of x. So we obtain $x \models \blacktriangleright(A \vee B)$. And if $y \models \triangleright A \wedge \blacktriangleright B$, we have $x \not\models A \vee B$, and hence $y \models \blacktriangleright(A \vee B)$.

$\triangleright(A \wedge B) \vdash \triangleright A \vee \blacktriangleright B$: Suppose $x \models \triangleright(A \wedge B)$. Then $x \not\models A \wedge B$. Suppose in addition that $x \not\models \triangleright A$. Then A holds at x or y. But in either case we have $x \models A$ since $y \leqslant x$. So $x \not\models B$, and hence $x \models \blacktriangleright B$ by $x \smile x$. And similarly if $y \models \triangleright(A \wedge B)$ and $y \not\models \triangleright A$, then $x \models A$ and hence $x \not\models B$. Since $y \smile x$, we obtain $y \models \blacktriangleright B$. Thus $\triangleright(A \wedge B) \vdash \triangleright A \vee \blacktriangleright B$ is also valid in F.

of bi-relational frames, where the unique modal accessibility satisfies only the weaker frame conditions as an accessibility for $\Diamond$. In their terms, the accessibility is (strictly) condensed as one for $\Box$ but not condensed as one for $\Diamond$. And the axioms also do *not correspond* to the constraint $R_{\Box} = R_{\Diamond}$ in tri-relational frames. Indeed, $\Diamond A \vee \Box\neg A$ *does* correspond to the constraint $R_{\Box} \subseteq R_{\Diamond}$, but $\neg(\Diamond A \wedge \Box\neg A)$ does *not* correspond to the converse, $R_{\Diamond} \subseteq R_{\Box}$[11].

The lack of correspondence and duality can be remedied by using dual intuitionistic negation.

Proposition 6.1. The following structural rule, axiom, and frame constraint correspond to each other and canonical in the same sense as above:

$$\frac{\sharp X \vdash Y}{\mathbf{I} > X \vdash \flat(\mathbf{I} > Y)}, \quad B \vdash \rhd\!\sim\! A \vee \blacktriangleright A \quad \text{and} \quad \frown \subseteq \smile.$$

And by the translation $(-)^{\bullet}$, we have another triple:

$$\frac{X \vdash \sharp Y}{\flat(X > \mathbf{I}) \vdash Y > \mathbf{I}}, \quad \blacktriangleright\neg A \wedge \rhd A \vdash B \quad \text{and} \quad \smile \subseteq \frown.$$

Thus, these pairs of rules or axioms correspond to the class of birelational frames (that is, frames in which $\frown = \smile$).

Proof. Below is a derivation of the axiom. An inference appealing to the structural rule is indicated by $(\frown \subseteq \smile)$.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{A \vdash A}{\sharp A \vdash \blacktriangleright A}\,(\blacktriangleright R)}{\mathbf{I} > A \vdash \flat(\mathbf{I} > \blacktriangleright A)}\,(\frown \subseteq \smile)}{\sim A \vdash \flat(\mathbf{I} > \blacktriangleright A)}\,(\sim R)}{\mathbf{I} > \blacktriangleright A \vdash \sharp \sim A}}{\mathbf{I} > \blacktriangleright A \vdash \rhd \sim A}\,(\rhd R)}{\mathbf{I} \vdash \rhd \sim A \,;\, \blacktriangleright A}}{\mathbf{I} \vdash \rhd \sim A \vee \blacktriangleright A}\,(\vee R)$$

Next we show that the axiom is valid in a frame $F = \langle W, \leqslant, \frown, \smile \rangle$ only if $\frown \subseteq \smile$. Suppose that $\frown \not\subseteq \smile$, that is, there are points x, y in W such that $x \frown y$ but $x \not\smile y$. Define a valuation of an atom p as: for any $u \in W$,

$$u \models p \iff x \smile u.$$

This valuation is hereditary in virtue of the frame condition for $\blacktriangleright$. Then by $x \not\smile y$ we have $y \not\models p$. And it follows from $x \frown y$ that $x \not\models \rhd\!\sim\! A$. On the

[11]Consider, for example, a frame F of two states x, y with $xR_{\Box}x$, $xR_{\Diamond}x$, $xR_{\Diamond}y$, $yR_{\Box}x$, $yR_{\Diamond}x$, $yR_{\Diamond}y$, $y \leqslant x$, $x \leqslant x$, and $y \leqslant y$. In this frame, the frame conditions ($\leqslant \circ R_{\Box} \subseteq R_{\Box} \circ \leqslant$ and $\geqslant \circ R_{\Diamond} \subseteq R_{\Diamond} \circ \geqslant$) are satisfied, $R_{\Box} \subseteq R_{\Diamond}$ but not conversely, and the two axioms are valid.

other hand, for all state u, if $x \smile u$, then $u \models p$. Hence $x \not\models \blacktriangleright A$. Thus $x \not\models \triangleright{\sim}A \vee \blacktriangleright A$.

Checking soundness of the structural rule is routine. Lastly let us show the corresponding triple is canonical. Assume $x_{\triangleright} \cap y = \emptyset$ in the canonical model. To show $x_{\blacktriangleright} \cup y = \mathcal{L}$, suppose $A \notin x_{\blacktriangleright}$, that is, $\blacktriangleright A \notin x$. Then since $\triangleright{\sim}A \vee \blacktriangleright A \in x$, we have $\triangleright{\sim}A \in x$, hence ${\sim}A \in x_{\triangleright}$. So ${\sim}A \notin y$, which implies $A \in y$ by the theorem $\mathbf{I} \vdash A \vee {\sim}A$ in Bi-Intuitionistic logic. $\square$

We note that $\triangleright{\sim}$ is just a necessity operator along the accessibility $\frown$ and $\blacktriangleright\neg$ is a possibility operator along $\smile$.

$$\begin{aligned}
x \models \triangleright{\sim}A &\iff \forall y : x \frown y \Rightarrow y \not\models {\sim}A \\
&\iff \forall y : x \frown y \Rightarrow \forall z \leqslant y : z \models A \\
&\iff \forall y : x \frown y \Rightarrow y \models A \\
&\qquad (\because \frown = \frown \circ \geqslant : \text{the frame condition for } \triangleleft) \\
x \models \blacktriangleright\neg A &\iff \exists y : x \smile y \;\&\; y \not\models \neg A \\
&\iff \exists y : x \smile y \;\&\; \exists z \geqslant y : z \models A \\
&\iff \exists y : x \smile y \;\&\; y \models A \\
&\qquad (\because \smile = \smile \circ \leqslant : \text{the frame condition for } \blacktriangleleft)
\end{aligned}$$

So the axiom $B \vdash \triangleright{\sim}A \vee \blacktriangleright A$ says that if A is not necessary along $\frown$ then it is un-necessary along $\smile$, and $\blacktriangleright\neg A \wedge \triangleright A \vdash B$ that it is not the case that A is both possible along $\smile$ and impossible along $\frown$.

In the logic of positive modal operators, $\Box\neg A$ can express impossibility of A along $R_{\Box}$. That is why $\Diamond A \vee \Box\neg A$ corresponds to the constraint $R_{\Box} \subseteq R_{\Diamond}$. On the other hand for the converse $R_{\Diamond} \subseteq R_{\Box}$, we need an expression of unnecessity along $R_{\Diamond}$, but it requires dual intuitionistic negation. In fact, $\neg(\Box A \wedge \Diamond{\sim}A)$, in which $\Diamond{\sim}A$ expresses unnecessity along $R_{\Diamond}$, corresponds to $R_{\Diamond} \subseteq R_{\Box}$.

7 Conclusion

We have seen how our familiar negations are obtained by imposing structural rules or frame constraints upon the weakest negation (weakest in our framework). Conversely, those negations are obtained from the strongest, Classical negation by restricting structural rules. In this sense, they are substructural negations.

This is, we think, more than a matter of wording because we met a phenomenon characteristic to substructural logics, namely, fusion or splitting of logical operators. It was one of important discoveries of linear logic that conjunction and disjunction split into the additive and the multiplicative through restriction of structural rules. Analogously to this, self-dual negation is analyzed as a fusion of two negations dual to each other, or it divides into two by restricting structural rules.

This analysis, as we have seen, makes the whole picture nicer. Double negation laws can be expressed by structural rules. The condition for star-crossed frames has the corresponding axioms and structural rules. It might be said that the analysis is conceptually not so appealing, though. It blurs the distinction between the universal and existential quantifiers (for impossibility and unnecessity respectively). Self-dual negation, then, might be regarded as a *con*fusion. Is it a way of making good sense of them? I believe it is. I expect that a dualist conception exemplified by the analysis presented here would give us a deeper understanding of symmetry underlying negation and logic in general. But to flesh it out requires a more comprehensive formal framework and a more detailed philosophical discussion, which are beyond the reach of this paper [12].

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