Rough consequence and other modal logics

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Abstract

Chakraborty and Banerjee have introduced a rough consequence logic based on the modal logic S5. This paper shows that rough consequence logics, with many of the same properties, can be based on modal logics as weak as K, with a simpler formulation than that of Chakraborty and Banerjee. Also provided are decision procedures for the rough consequence logics and equivalences and independence relations between various systems S and the rough consequence logics, based on them. It also shows that each logic, based on such an S, is theorem equivalent, but not necessarily equivalent, to the modal logic M-S. The paper also shows that rough consequence logic, which was designed to handle rough equality, is somewhat limited for that purpose.

1 Introduction

If α and β are wffs, $\alpha \approx \beta$ (α is roughly equal to β) is defined by:

$$(L\alpha \leftrightarrow L\beta) \wedge (M\alpha \leftrightarrow M\beta)$$

where L and M are the necessity and possibility operators.

In [5] Chakraborty and Banerjee proposed a rough consequence logic, based on S5, with two restricted modus ponens rules, designed to handle rough equality. In this paper we show that one of these rules is superfluous and that many of the interesting properties of rough consequence logic can be derived when modal logics weaker than S5 are used as a base logic.

We will consider rough consequence logics based on the simple modal logic K and stronger logics such as the Deontic logic D, the Feys-von Wright logic T, the Sobocinski logics S4$_n$ (with S4 = S4$_1$) and S5$_n$ (with S5 = S5$_1$) and the logics M^n-S4$_n$, M^n-S5$_n$ and T$^*_n$ of Błaszczuk and Dziobiak [3]. M-S5 is also, by a mapping of Kotas [10], equivalent to Jaśkowski's discussive logic D2, ([8] and [9]).

For each such system S we give a simple decision procedure for $\Gamma \mathrel{|\!\!\!\sim}_{\mathrm{S}} \alpha$ (α is provable from Γ in the rough consequence logic based on S) in terms of a decision procedure (if any) for S, we show how each modal logic is related to its corresponding rough consequence logic and show that $\Gamma \mathrel{|\!\!\!\sim}_{\mathrm{S}} \alpha$ implies but is not generally implied by $\Gamma \vdash_{\mathrm{M-S}} \alpha$. Finally we look at the properties of rough equality in the various modal and rough consequence logics.

2 K and Stronger Modal Logics

Each of the systems we consider includes the axioms of classical propositional logic and has the operators L and M where $M\alpha = \sim L(\sim \alpha)$.

K has the usual modus ponens and substitution rules as well as the Rule of Necessitation:

(N) $\quad \vdash \alpha \quad \Rightarrow \quad \vdash L\alpha.$

and one extra axiom:

K $\quad \vdash L(p \to q) \to (Lp \to Lq).$

Below we will use the Rule of Monotonicity, which holds in K:

(M) $\quad \vdash p \to q \quad \Rightarrow \quad \vdash Mp \to Mq$

and the K-theorem (K7 of Hughes and Cresswell [7]):

K1 $\quad \vdash M(p \to q) \leftrightarrow Lp \to Mq.$

The Deontic Logic D has an extra axiom:

D $\quad \vdash Lp \to Mp.$

The logic T is K with the extra axiom:

T $\quad \vdash Lp \to p$

and includes as theorems D and:

T1 $\quad \vdash p \to Mp$

The Sobocinski logics $S4_n$ have the additional axiom:

$A4_n$ $\quad \vdash M^{n+1}p \to M^n p.$

and the systems $S5_n$, due to Błaszczuk and Dziobiak [3], have, in addition to the postulates of $S4_n$:

$A5_n$ $\quad \vdash M^n L^n p \to L^n p.$

Note that $S4_1 = S4$ and $S5_1 = S5$.

For each of the above systems S and each positive integer n, there is a further system

$$M^n\text{-}S = \{\alpha | \vdash_S M^n \alpha\}.$$

(Note that we will often treat a logical system as the set of its theorems.)

Finally we include the systems T^*_n, which have the postulates of T with the additional rule:

$$\vdash M^{n+1}\alpha \quad \Rightarrow \quad \vdash M^n\alpha.$$

Błaszczuk and Dziobiak [3] showed, following the work of Kotas [10] in the case n=1 and earlier work of Furmanowski [6] and Perzanowski [11], that the systems T_n^* with the properties given below can be axiomatised in this way (where $\subset$ represents "is a proper subset of "):

(i) For every modal logic S such that T$\subseteq$S, M^n-S= M^n-$S4_n$ $\Leftrightarrow$ $T_n^* \subseteq S \subseteq S5_n$;
(ii) $T \subset T_{n+1}^* \subset T_n^* \subset S4_n \subset S5_n$;
(iii) T_n^* is indepenent of $S4_{n+k}$ for positive integers k.

3 Rough Consequence $\mid\!\sim$

A basis logic S is a modal logic such that K$\subseteq$ S $\subseteq$S5. We will use $\vdash_S$ for provability in S and $\mid\!\sim_S$ for provability in the rough consequence logic based on S, as well as for the name of the formal system. If the S is omitted in a definition or theorem, the logic is assumed to be any basis logic, but the same one throughout.

$\mid\!\sim_S$ can be defined (as it was, with S = S5 in Chakraborty and Banerjee [5] and Banerjee and Chakraborty [2]) by:

Definition 1

(i) $\vdash \alpha \quad \Rightarrow \quad \Gamma \mid\!\sim \alpha$
(ii) $\alpha \in \Gamma \quad \Rightarrow \quad \Gamma \mid\!\sim \alpha$

$$R_1 \qquad \frac{\Gamma \mid\!\sim \beta \qquad \vdash M\beta \to M\gamma}{\Gamma \mid\!\sim \gamma}$$

$$(RMP)_2 \qquad \frac{\vdash \beta \qquad \Gamma \mid\!\sim \delta \to \gamma \qquad \vdash L\beta \to L\delta}{\Gamma \mid\!\sim \gamma}$$

$\mid\!\sim\alpha$ represents "α is a rough theorem".

Note that R_1 is the notation of Banerjee [1]. The rule is called $(DR)_1$ in Banerjee and Chakraborty [2] and DR_2 in Chakraborty and Banerjee [5]. We now show that $(RMP)_2$ is derivable from the other postulates, for all our rough consequence logics.

Theorem 1 $(RMP)_2$ follows from R_1 and (i).

Proof. If $\vdash \beta$ and $\vdash L\beta \to L\delta$ then, by (N), $\vdash L\beta$ and so $\vdash L\delta$. K1 then gives $\vdash M(\delta \to \gamma) \to M\gamma$. This with $\Gamma \mid\!\sim \delta \to \gamma$ and R_1 gives $\Gamma \mid\!\sim \gamma$.

This theorem was proved independently, for the S = S5 case, in Banerjee [1]. In [1] she cites an early draft of the present paper that contains this theorem.

A paper, Bunder, Banerjee and Chakraborty [4], examines the relative strengths of rough consequence logics with various conditions other than $\vdash M\beta \to M\gamma$ in R_1.

By R_1 and (M) we have:

Theorem 2

$$R_1' \qquad \frac{\Gamma \mathrel{|\!\sim} \beta \qquad \vdash \beta \to \gamma}{\Gamma \mathrel{|\!\sim} \gamma}$$

By (S4) and R_1 we have:

Theorem 3 If S4 $\subseteq$ S,

$$R_1'' \qquad \frac{\Gamma \mathrel{|\!\sim} M\beta \qquad \vdash M\beta \to M\gamma}{\Gamma \mathrel{|\!\sim} \gamma}$$

and by T1,

Theorem 4 If S4 $\subseteq$ S, $\Gamma \mathrel{|\!\sim} M\beta \iff \Gamma \mathrel{|\!\sim} \beta$.

The following interesting properties of $\mathrel{|\!\sim}$, for the basis logic S5, are from [2] and [5], we show that they also apply for rough consequence logics based on other basis logics.

Theorem 5 $\vdash \alpha \Rightarrow \mathrel{|\!\sim} \alpha$.

Proof By (i).

Theorem 6 (The deduction theorem for $\mathrel{|\!\sim}$) $\Gamma, \alpha \mathrel{|\!\sim} \beta \Rightarrow \Gamma \mathrel{|\!\sim} \alpha \to \beta$.

Proof By induction on the derivation of $\Gamma, \alpha \mathrel{|\!\sim} \beta$. Required are $\vdash M\beta \to M(\alpha \to \beta)$, which comes by (M), and $\vdash M\gamma \to M\beta \Rightarrow \vdash M(\alpha \to \gamma) \to M(\alpha \to \beta)$ which comes using K1.

Theorem 7 $\vdash M\alpha \Rightarrow \Gamma \mathrel{|\!\sim} \alpha$.

Proof If $\vdash M\alpha$, $\vdash M(p \to p) \to M\alpha$. Also, by (i), $\Gamma \mathrel{|\!\sim} p \to p$, so by R_1, $\Gamma \mathrel{|\!\sim} \alpha$.

Theorem 8 If D $\subseteq$ S, $\mathrel{|\!\sim} \alpha \iff \vdash M\alpha$.

Proof $\Leftarrow$ By Theorem 7.
$\Rightarrow$ By induction on the derivation of $\mathrel{|\!\sim} \alpha$.
If $\vdash \alpha$, we have $\vdash L\alpha$ by (N) and, by D, $\vdash M\alpha$.
If $\vdash \alpha$ comes from $\mathrel{|\!\sim} \beta$ and $\vdash M\beta \to M\alpha$, we have $\vdash M\beta$ by the induction hypothesis and so $\vdash M\alpha$.

Note that this implies that, if D $\subseteq$ S, the rough consequence logic based on S is theorem equivalent to the logic M-S.

4 Rules for other connectives

The following are easily provable from R_1 for all basis logics.

Theorem 9

$$\frac{\Gamma \mathrel{|\!\sim} \alpha \qquad \vdash \beta}{\Gamma \mathrel{|\!\sim} \alpha \wedge \beta} \qquad \frac{\vdash \alpha \qquad \Gamma \mathrel{|\!\sim} \beta}{\Gamma \mathrel{|\!\sim} \alpha \wedge \beta}$$

$$\frac{\Gamma \mathrel{|\!\sim} \alpha \wedge \beta}{\Gamma \mathrel{|\!\sim} \alpha} \qquad \frac{\Gamma \mathrel{|\!\sim} \alpha \wedge \beta}{\Gamma \mathrel{|\!\sim} \beta} \qquad \frac{\Gamma \mathrel{|\!\sim} \sim\sim \alpha}{\Gamma \mathrel{|\!\sim} \alpha}$$

$$\frac{\Gamma \mathrel{|\!\sim} \alpha}{\Gamma \mathrel{|\!\sim} \alpha \vee \beta} \qquad \frac{\Gamma \mathrel{|\!\sim} \beta}{\Gamma \mathrel{|\!\sim} \alpha \vee \beta}$$

$$\frac{\Gamma \mathrel{|\!\sim} \alpha \vee \beta \qquad \vdash \alpha \rightarrow \gamma \qquad \vdash \beta \rightarrow \gamma}{\Gamma \mathrel{|\!\sim} \gamma}$$

$$\frac{\vdash \alpha \vee \beta \qquad \Gamma \mathrel{|\!\sim} \alpha \rightarrow \gamma \qquad \vdash \beta \rightarrow \gamma}{\Gamma \mathrel{|\!\sim} \gamma}$$

$$\frac{\vdash \alpha \vee \beta \qquad \vdash \alpha \rightarrow \gamma \qquad \Gamma \mathrel{|\!\sim} \beta \rightarrow \gamma}{\Gamma \mathrel{|\!\sim} \gamma}$$

$$\frac{\Gamma \mathrel{|\!\sim} \alpha \rightarrow \beta \qquad \vdash \alpha \rightarrow \sim \beta}{\Gamma \mathrel{|\!\sim} \sim \alpha} \qquad \frac{\vdash \alpha \rightarrow \beta \qquad \Gamma \mathrel{|\!\sim} \alpha \rightarrow \sim \beta}{\Gamma \vdash \sim \alpha}$$

5 Decision Procedures for $\mathrel{|\!\sim}$ and Relations between $\mathrel{|\!\sim}_S$ and S

The $\vdash M\beta \rightarrow M\gamma$ in R_1, rather than $\vdash \beta \rightarrow \gamma$ (or $\mathrel{|\!\sim} \beta \rightarrow \gamma$), indicates that some $\Gamma \mathrel{|\!\sim} \gamma$ may be provable while $\Gamma \nvdash \gamma$.

On the other hand, the lack of a Γ before the $\vdash$ in R_1 suggests that there may be a provable $\Gamma \vdash \gamma$, for $\Gamma \neq \emptyset$, while $\Gamma \mathrel{|\!\not\sim} \gamma$.

Such results can be derived, for D $\subseteq$ S, from the following simple characterisation of provability in $\mathrel{|\!\sim}$ in terms of that in $\vdash$.

Theorem 10 If D $\subseteq$ S, $\Gamma \mathrel{|\!\sim} \gamma$ if and only if $\vdash M\gamma$ or there is a $\beta \in \Gamma$ such that $\vdash M\beta \rightarrow M\gamma$.

Proof $\Rightarrow$ By induction on the derivation of $\Gamma \mathrel{|\!\sim} \gamma$.
Case 1 $\vdash \gamma$. By (N) and D.
Case 2 $\gamma \in \Gamma$. We have the result with $\beta = \gamma$.
Case 3 $\Gamma \mathrel{|\!\sim} \gamma$ comes by R_1 from $\Gamma \mathrel{|\!\sim} \alpha$ and $\vdash M\alpha \rightarrow M\gamma$.
By the induction hypothesis we have $\vdash M\alpha$ or a $\beta \in \Gamma$ such that $\vdash M\beta \rightarrow M\alpha$. Hence $\vdash M\gamma$ or for a $\beta \in \Gamma$, $\vdash M\beta \rightarrow M\gamma$.
$\Leftarrow$ If $\vdash M\gamma$ the result follows by Theorem 7. If $\vdash M\beta \rightarrow M\gamma$, where $\beta \in \Gamma$, $\Gamma \mathrel{|\!\sim} \beta$, and, by R_1, $\Gamma \mathrel{|\!\sim} \gamma$.

Theorem 10 leads directly to:

Theorem 11 If D $\subseteq$ S, $\Gamma \mathrel{|\!\sim} \gamma \Rightarrow \Gamma \vdash M\gamma$,

and to Theorem 12 which shows that the converse of Theorem 5 fails for T $\subseteq$ S and the converses of Theorems 6, 7 and 11 fail for all basis logics. Theorem 12(ii) also shows the failure of modus ponens in $\mathrel{|\!\sim}$.

Theorem 12 (i) If T $\subseteq$ S, $\mathrel{|\!\sim} M\alpha \to \alpha$, but $\not\mathrel{|\!\sim} Mp \to p$
(ii) $p \to q \mathrel{|\!\sim} p \to q$ and $p \to q,\ p \not\mathrel{|\!\sim} q$
(iii) $p \mathrel{|\!\sim} p$, but $\not\mathrel{|\!\sim} Mp$
(iv) $\alpha, \alpha \to M\beta \vdash M\beta$, but $p, p \to Mq \not\mathrel{|\!\sim} q$

Proof The above unprovability results for $\mathrel{|\!\sim}$ are verified (using Theorem 10) by the unprovability in S5 of $\vdash Mq$, $\vdash Mp \to Mq$, $\vdash M(p \to q) \to Mq$, and $\vdash M(p \to Mq) \to Mq$. These and the unprovability results in S5, and so in any sublogic S, in (i) and (iii) are all easily confirmed by the decision procedure for S5 in Hughes and Cresswell [7].
By T, $\vdash LM\alpha \to M\alpha$ so by K1 and Theorem 7, we have $\mathrel{|\!\sim} M\alpha \to \alpha$, which we need in (i).

Theorem 13, below, outlines the relations between the basis logics and their corresponding rough consequence logics. To prove some of these we need a definition and a lemma.

Definition 2 An occurrence of an L in α is an outer occurrence if it is not within the scope of another L in α.

Lemma 1 If α' is the result of deleting all outer occurrences of L from α and S = K or D, then $\vdash \alpha \ \Rightarrow\ \vdash \alpha'$.

Proof By induction on the proof of $\vdash \alpha$.
If $\vdash \alpha$ is an axiom of K or D, the lemma holds as Axiom K becomes $\vdash (p \to q) \to (p \to q)$ and Axiom D $\vdash p \to \sim\sim p$.
If $\alpha = L\beta$ and $\vdash \alpha$ is obtained by (N) from $\vdash \beta$, we have $\vdash \alpha'$ as $\alpha' = \beta$.
If $\vdash \alpha$ is obtained from $\vdash \beta$ and $\vdash \beta \to \alpha$, then by the induction hypothesis we have $\vdash \beta'$ and $\vdash \beta' \to \alpha'$ and so α'.

We note that this lemma does not apply to the instance $\vdash LMp \to Mp$ of Axiom T.

Theorem 13 (i) If S = K or D, $\mathrel{|\!\sim}_S$ is a proper subsystem of S.
(ii) If T $\subseteq$ S, $\mathrel{|\!\sim}_S$ and S are independent.

Proof (i) Given $p \to q,\ p \not\mathrel{|\!\sim} q$ from Theorem 12(ii), we only need to prove, for S = K or D, that $\Gamma \mathrel{|\!\sim} \alpha \ \Rightarrow\ \Gamma \vdash \alpha$. We do this by induction on the proof of $\Gamma \mathrel{|\!\sim} \alpha$. If $\vdash \alpha$ or $\alpha \in \Gamma$, this is obvious. If $\Gamma \mathrel{|\!\sim} \alpha$ comes by R_1 from $\Gamma \mathrel{|\!\sim} \beta$ and $\vdash\ M\beta \to M\alpha$, we have by the induction hypothesis and Lemma 1, $\Gamma \vdash \beta$ and $\vdash \beta \to \alpha$ and so $\Gamma \vdash \alpha$.
(ii) By Theorem 12(i) and $p \to q,\ p \not\mathrel{|\!\sim} q$.

Theorem 14 For S = K or D, $\mathrel{|\!\sim}_S$ and S are theorem equivalent.

Proof By Theorem 5 and the proof of Theorem 13(i).

Lemma 1 also allows a counterpart to Theorem 10 for K:

Theorem 15 $\Gamma \mathrel{|\!\sim}_K \alpha$ if and only if $\vdash_K \alpha$ or there is a $\beta \in \Gamma$ such that $\vdash \beta \to \alpha$.

Proof Similar to that of Theorem 10, but using Lemma 1.

Theorems 10 and 15 reduce the decision procedure for any $\mid\!\sim_S$ to that for S. The Weakening Theorem for $\mid\!\sim$ holds by an easy induction, Cut Elimination can also be proved for $\mid\!\sim$.

Theorem 16 (Weakening Theorem) $\Gamma_1 \mid\!\sim \alpha \quad \Rightarrow \quad \Gamma_1, \Gamma_2 \mid\!\sim \beta$.

Theorem 17 (Cut Elimination Theorem) $\Gamma_1 \mid\!\sim \alpha, \quad \Gamma_2,\ \alpha \mid\!\sim \beta \quad \Rightarrow \quad \Gamma_1, \Gamma_2 \mid\!\sim \beta$.

Proof If S=K and $\Gamma,\ \alpha \mid\!\sim \beta$ we have, using Theorem 15, $\vdash \alpha \to \beta$ or, for some γ in Γ_2, $\vdash \gamma \to\ \beta$. By (M) we have either $\vdash M\alpha \to M\beta$ or, for that γ, $\vdash M\gamma \to\ M\beta$.
If D $\subseteq$ S and $\Gamma, \alpha \mid\!\sim \beta$, by Theorem 10, we also have either $\vdash M\alpha \to M\beta$ or, for such a γ, $\vdash M\gamma \to\ M\beta$.
If $\Gamma_1 \mid\!\sim \alpha$, by weakening, $\Gamma_1, \Gamma_2 \mid\!\sim \alpha$, so by R_1, if $\vdash M\alpha \to\ M\beta$, we have $\Gamma_1, \Gamma_2 \mid\!\sim \beta$.
If $\vdash M\gamma \to\ M\beta$, by $\Gamma_1, \Gamma_2 \mid\!\sim \gamma$ and R_1 we also have $\Gamma_1, \Gamma_2 \mid\!\sim \beta$.

6 M-S and $\mid\!\sim_S$

We have by Theorem 8:

Theorem 18 If D $\subseteq$ S, M-S and $\mid\!\sim_S$ are theorem-equivalent.

If we extend the definition of M-S to natural deduction systems by:

Definition 3 $\quad \Gamma \vdash_{M-S} \alpha$ iff $\Gamma^M \vdash M\alpha$ where $\Gamma^M = \{M\beta | \beta \in \Gamma\}$,

we have:

Theorem 19 If D $\subseteq$ S, M-S is stronger than or equivalent to $\mid\!\sim_S$.

Proof If $\Gamma \mid\!\sim_S \alpha$, by Theorem 10, there are two cases:
Case 1 $\vdash_S M\alpha$, so then $\vdash_{M-S} \alpha$ and so $\Gamma \vdash_{M-S} \alpha$.
Case 2 For some $\beta \in \Gamma$, $\vdash_S M\beta \to M\alpha$. Also $M\beta \in \Gamma^M$ i.e. $\Gamma^M \vdash_S M\beta$ and so $\Gamma^M \vdash_S M\alpha$, i.e. $\Gamma \vdash_{M-S} \alpha$.
Thus $\mid\!\sim_S$ is a subsystem of M-S.

Theorem 20 M-S5_n is strictly stronger than $\mid\!\sim_{S5_n}$.

Proof By $A5_n$,

$$M^n L^n p,\ L^n p \to Mq \vdash_{S5_n} Mq,$$

so, using K1,

$$M^{n-1} L^n p,\ L^{n-1} p \to q \vdash_{M-S5_n} q.$$

By Theorem 10, this result does not hold for $\mid\!\sim_{S5_n}$. Hence by Theorem 19, M-S5_n is strictly stronger than $\mid\!\sim_{S5_n}$.

The question as to whether, for weaker systems S, M-S is strictly stronger than $\mid\!\sim_S$ remains open.

7 $\mid\!\sim$ and $\approx$

In this section we consider the properties of rough equality ($\approx$) in the various modal and rough consequence logics.

Theorem 21 (i) $\vdash \alpha \approx \alpha$
(ii) $\vdash \alpha \approx \beta \to \beta \approx \alpha$
(iii) $\vdash \alpha \approx \beta \to \beta \approx \gamma \to \alpha \approx \gamma$
(iv) $\mid\!\sim \alpha \approx \beta \quad \Rightarrow \quad \mid\!\sim \beta \approx \alpha$
(v) $\mid\!\sim \alpha \approx \beta, \vdash \ \beta \approx \gamma \quad \Rightarrow \quad \mid\!\sim \alpha \approx \gamma$
(vi) $\vdash \ \alpha \approx \beta \quad \Rightarrow \quad \alpha \mid\!\sim \beta$.

Proof (i) to (iii) By propositional logic.
(iv) If D $\subseteq$ S and $\mid\!\sim \alpha \approx \beta$ we have $\vdash M((L\alpha \leftrightarrow L\beta) \wedge (M\alpha \leftrightarrow M\beta))$ and by propositional logic, $\vdash M((L\beta \leftrightarrow L\alpha) \wedge (M\beta \leftrightarrow M\alpha))$, which is $\mid\!\sim \beta \approx \alpha$.
If S=K, the result follows by (ii) and Theorem 14.
(v) If D $\subseteq$ S and $\mid\!\sim \alpha \approx \beta$ we have $\vdash M((L\alpha \leftrightarrow L\beta) \wedge (M\alpha \leftrightarrow M\beta))$, $\vdash \beta \approx \gamma$ gives $\vdash (L\beta \leftrightarrow L\gamma) \wedge (M\beta \leftrightarrow M\gamma)$, so propositional logic gives $\vdash M(\alpha \approx \gamma)$ i.e. $\mid\!\sim \alpha \approx \gamma$.
If S=K, $\mid\!\sim \alpha \approx \beta$ and $\mid\!\sim \beta \approx \gamma$ we have $\vdash \alpha \approx \beta$ and $\vdash \beta \approx \gamma$ and so $\vdash \alpha \approx \gamma$ by (iii). Hence $\mid\!\sim \alpha \approx \gamma$.
(vi) If $\mid\!\sim \alpha \approx \beta$, we have $\vdash M\alpha \to \ M\beta$ and by $\alpha \mid\!\sim \alpha$ and R_1, $\alpha \mid\!\sim \beta$.

We therefore have that $\approx$ is an equivalence relation in $\vdash_S$, and in $\mid\!\sim_S$ as (i) to (iii) hold there as well. However as we don't have Modus Ponens, properties such as

$$\mid\!\sim \alpha \approx \beta, \ \mid\!\sim \beta \approx \gamma \ \Rightarrow \ \mid\!\sim \alpha \approx \beta$$

can fail unless S=K.

Most properties, so far, have held for all basis sytems. For substitution of rough equality in $\mid\!\sim_S$, however we need S = S_4 or S_5.

Theorem 22 (i) If S=S_5, $\quad \mid\!\sim \alpha \approx \beta, \ \ \Gamma \mid\!\sim \alpha \quad \Rightarrow \quad \Gamma \mid\!\sim \beta$
(ii) If $S_4 \subseteq$ S, $\quad \Gamma \mid\!\sim \alpha \approx \beta, \ \ \mid\!\sim \alpha \quad \Rightarrow \quad \Gamma \mid\!\sim \beta$
(iii) If S=S_5, $\quad \mid\!\sim \alpha \approx \beta, \ \ \Gamma, \alpha \mid\!\sim \gamma \quad \Rightarrow \quad \Gamma, \ \beta \mid\!\sim \gamma$.

Proof (i) If S=S_5 and $\mid\!\sim \alpha \approx \beta$, we have by Theorem 11, $\vdash M(\alpha \approx \beta)$ and so, using $\vdash M(p \wedge q) \to Mp \wedge Mq$, $\quad \vdash M(M\alpha \to M\beta)$ and by K1, $\vdash LM\alpha \to M^2\beta$. By $A4_1$ and $A5_1$ we then have $\vdash M\alpha \to M\beta$. This and $\Gamma \mid\!\sim \alpha$, by R_1 give $\Gamma \mid\!\sim \beta$.
(ii) If $S_4 \subseteq$ S and $\Gamma \mid\!\sim \alpha \approx \beta$, there are two cases by Theorem 10.
(a) If $\vdash M(\alpha \approx \beta)$, we have, as above, $\vdash M(M\alpha \to M\beta)$, so if $\mid\!\sim \alpha$, i.e. $\vdash M\alpha$, we have by (M), $\vdash M^2\beta$ and by $A4_1$, $\vdash M\beta$ i.e. $\Gamma \mid\!\sim \beta$.
(b) If for some γ in Γ, $\vdash M\gamma \to M(\alpha \approx \beta)$ then as in (a), by $A4_1$, we have $\vdash M\gamma \to \ M\beta$ and as $\Gamma \mid\!\sim \gamma$, by R_1, $\Gamma \mid\!\sim \beta$.
(iii) If $\Gamma, \alpha \mid\!\sim \gamma$, there are, by Theorem 10, three cases.
(a), (b) If $\vdash M\gamma$ or $\gamma \in \Gamma$ then, $\Gamma, \ \beta \mid\!\sim \gamma$.
(c) If $\alpha = \gamma$ and $\mid\!\sim \alpha \approx \beta$, then $\mid\!\sim \gamma \approx \beta$, so as $\Gamma, \ \beta \mid\!\sim \beta$, by (i), $\Gamma, \ \beta \mid\!\sim \gamma$.

Note that we cannot generalise these results in the most obvious way, as by Theorems 10 and 15,

$$\alpha, \alpha \approx \beta \not\mid\!\sim \beta.$$

It seems that we can have such results for weaker basis logics only if we replace a $\mid\!\sim$ by a $\vdash$ (as in Theorem 21). For example we have:

Theorem 23 If D $\subseteq$ S, (i) $\mid\!\sim \alpha \approx \beta \;\Rightarrow\; \mid\!\sim \alpha \to \beta$
(ii) $\mid\!\sim \alpha, \vdash \alpha \approx \beta \;\;\Rightarrow\;\; \mid\!\sim \beta$.

Proof (i) If D $\subseteq$ S and $\vdash \alpha \approx \beta$, by D we have $\vdash L\alpha \to M\beta$ and by K1, $\vdash M(\alpha \to \beta)$ and so, by Theorem 7, $\mid\!\sim \alpha \to \beta$.
(ii) If $\mid\!\sim \alpha$ and $\vdash \alpha \approx \beta$, we have, by Theorem 8, $\vdash M\alpha$ and $\vdash M\alpha \to M\beta$ and so $\vdash M\beta$ i.e. $\mid\!\sim \beta$.

Some interesting rough equivalences and non-rough equivalences are given in the final theorem.

Theorem 24 (i) If T $\subseteq$ S, $\mid\!\sim L\alpha \leftrightarrow \alpha$
(ii) If T$\subseteq$ S, $\mid\!\sim M\alpha \leftrightarrow \alpha$
(iii) $\not\mid\!\sim_{S5} Lp \approx p$
(iv) $\not\mid\!\sim_{S5} Mp \approx p$.

Proof (i) By T1, $\vdash L\alpha \to ML\alpha$, so by K1, $\vdash M(\alpha \to L\alpha)$ and by Theorem 7, $\mid\!\sim \alpha \to L\alpha$. Using T, we have, by Theorem 9 (i) $\mid\!\sim L\alpha \leftrightarrow \alpha$.
(ii) T1, Theorem 12(i) and Theorem 9(i) give the result.
(iii) and (iv) By Theorem 10 and the fact that $\nvdash_{S5} Mp \to M(L\alpha)$.

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