Towards Paraconsistent Inquiry

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Abstract

In this paper, we discuss Hintikka's theory of interrogative approach to inquiry with a focus on bracketing. First, we dispute the use of bracketing in interrogative models of inquiry arguing that bracketing provides an indispensable component of an inquiry. Then, we suggest a formal system based on strategy logic and logic of paradox to describe the epistemic aspects of an inquiry, and obtain a naturally paraconsistent system. The formalism depends on Logic of Paradox and Strategy Logic. We then apply our framework to some cases to illustrate its use.

Key words Hintikkan Interrogative model of inquiry, paraconsistency, strategy logic.

1 Introduction

Hintikka's interrogative inquiry is a well-known example of a dynamic epistemic game procedure which can result in an increase in knowledge. In a nutshell, in an interrogative inquiry, the inquirer is given a theory and a question. He then tries to answer the question based on the theory by posing questions to nature or an oracle. The initial formulation of this procedure is rather broad and informal. The main aim of this paper is to address various issues in interrogative models of inquiry, and suggest a formal analysis. First, we motivate our formalism with some philosophical observations at length and then present a formal framework which can provide an expressive syntax for the issues we raise in this article. Then we apply the formalism to various examples regarding interrogative inquiry.

For Hintikka, "[the] interrogative model is simplicity itself" [18]. That is certainly true, but also a call for extensions. Hintikka argued that the basic model of interrogative inquiry had some room for *ignorance, but not uncertainty*, which led him to introduce probabilistic elements to the theory [18]. Later on, first-order and tableaux based extensions of interrogative models were also introduced [21]. Even if there have been various extensions of interrogative models, to the best of our knowledge, none of them seem to address non-classical logical concerns [14, 16]. In this work, we argue that the theory needs a non-classical logical touch as it already contains some such elements implicitly rendering the theory *not* deductively explosive.

Deductive explosion describes the situation where any formula can be deduced from an inconsistent set of formulas. In deductively explosive logics, we have $\{\varphi, \neg\varphi\} \vdash \psi$ for any formula φ and ψ where $\vdash$ is a logical consequence relation. In this respect, both classical and intuitionistic logics are known to be deductively explosive. Paraconsistent logic, on the other hand, is the umbrella term for the logical systems where the logical consequence relation is not explosive [10, 35, 36]. Paraconsistency is a proof theoretical concept, and its semantical counterpart, *dialetheism*, suggests that there are true contradictions.

Paraconsistency presents itself in various areas in epistemology and ontology [36, 27, 34]. More relevantly, paraconsistent discussive systems were analyzed by Jaśkowski [23]. Carnielli, in a relatively recent paper, briefly discussed similar issues without suggesting a formal framework, and argued that "the problem of coping with contradictory information belongs to interrogative games" [8]. This motivates our work.

2 Hintikka's Theory of Interrogative Inquiry

Hintikka introduced his interrogative theory of scientific inquiry in 1984, and proposed a game theoretical reading for it [17, 19]. Some extensions of the system (probabilistic, first-order) were also suggested later [18, 21, 22]. Additionally, from a view point of philosophy of science, interrogative inquiry relates to the discussions on *context of discovery* [13]. Such an approach has also been analyzed formally, relating the interrogative inquiry to Hintikkan game theoretical semantics [30, 31]. Epistemically, the inquiry game can be viewed as a game between the *Inquirer* and the *Nature* (or *Oracle*) who will be denoted as I and N respectively.

In the interrogative model of inquiry (IMI, henceforth), the inquirer is given a starting theory T and a question usually of the form "Q or not-Q?" to be answered, even though the given question does not necessarily have to be in the form of a polar question. Hintikka assumes T as a "truly *general* theory" (his emphasis), and does not allow it to have constants, and consequently imposes that the question is formulated in the same language as T [17]. These two conditions are quite natural. Limiting the inquiry to (complex or ground) propositional objects makes the questioning aspect sensible. Otherwise, simply put, what can be questioned about the constants? Secondly, if the questions were to be syntactically formulated in a different formal framework than the theory we are working with, this mismatch would make the inquiry almost impossible. In short, algebraic group theory cannot be expected to produce an answer for questions about the nutritional needs of pregnant women.

In an IMI game, the *Inquirer* has two options. He is allowed to ask questions to the oracle or draw conclusions by using T and the answers he has already received (and supposedly by using the derivation rules of classical logic). But, what can the player I ask? Hintikka introduces several restrictions on the admissibility of questions. First, he imposes that "the presuppositions of the question to be asked" should already have been proved by I [17]. Second, Hintikka introduces his *Atomistic Postulate* as another restriction, even though it is not generally considered as an essential feature of IMI. This postulate stipulates that the player I may ask N "only questions about the truth or falsity of particular atomic sentences" concerning the model. The reason for this is quite simple: for the complex sentences with Boolean connectives, the proof rules make it possible for I to deduce further results. We believe these restrictions are quite natural, translating the framework of scientific and epistemic inquiry into the language of formal systems. One of those methods in this translation that Hintikka suggests is *bracketing*.

3 Bracketing and Paraconsistency

Hintikka introduces bracketing as a tool to omit irrelevant or uncertain answers during an interrogative inquiry.

> An important aspect of this general applicability of the interrogative model is its ability to handle uncertain answers - that is, answers that may be false. The model can be extended to this case simply by allowing the inquirer to tentatively disregard ("bracket") answers that are dubious. (...) Equally obviously, further inquiry might lead the inquirer to reinstate ("unbracket") a previously bracketed answer. This means thinking of interrogative inquiry as a self-corrective process.
>
> [20, p. 3]

This is clearly a very sensible idea. Not everything uttered in a dialogue may be relevant or truthful. We may need to be able to select the answers that are relevant to our inquiry. Also, undoing must be allowed in the form of unbracketing a previously omitted piece of knowledge. This is where strategizing becomes central, and Hintikka describes the need for an answer-selection procedure as follows.

> In a typical application of interrogative inquiry - for instance in the cross-examination of a witness in a court of law - the inquirer cannot simply accept all answers at their face value. They can be false. Hence we must have rules allowing the rejection or, as I will call it, the "bracketing of an answer", and rules governing such bracketing.
>
> But this seems totally unrealistic. How can we possibly hope to formulate realistic rules for the rejection or acceptance of any answers - any data - that an inquirer might ever receive?
>
> [20, p. 223]

Hintikka suggests a rather relaxed and "natural" framework for bracketing; yet, in this paper, we maintain that bracketing is an overkill and suffers from various problems. We categorize them as epistemic, game theoretical, and heuristic problems.

Epistemically, there seems to be a major problem in bracketing, as Hintikka pointed out as well. In an inquiry, how can the inquirer know which answers to ignore? How can he know what to reject or accept? This epistemic problem empties the notion of bracketing. In other words, if inquiry is a procedure during which we want to acquire and learn information, this implies that we *did not* have that information before. Moreover, by the rules of the IMI game, we also did not have the presuppositions that may lead us to that piece of information.

By iteration, this line of argument can be carried on *ad infinitum.* We simply do not know what to ignore or what to accept, unless our understanding of inquiry is limited to the situations where we only want to *hear* the answer from our opponent. In short, an interrogative inquiry should not be confused with what I call a *legal* inquiry. In a legal inquiry, the task is only to make the opponent *confess* while the inquirer has sufficient background information, priors or presuppositions about the information to be confessed. This is where Hintikka's legal inquiry differs from an epistemic inquiry. Because, in an epistemic inquiry, we are supposed to be searching and looking for some information that we did not have before. We cannot discard some responses in favor of or against some questions or propositions - simply because we do not know the answer.

The epistemic problem appears to be connected to the issue of derivation in an inquiry. Rules of the IMI game allow us to use the previous answers we obtained during our inquiry. But this does not necessarily mean that we need to incorporate *all* the answers we have received into the inquiry. Some answers may be helpful, some may not. This procedure calls for a choice mechanism. In an investigative deduction, how can we know which propositions and answers to use?[1].

A game theoretical response can be given to eliminate this problem. It can be suggested that, in an IMI game, the inquirer simply chooses the assumptions and responses that help him win the game. If the inquirer can win the game with a particular set of assumptions, then he can adopt those assumptions for a win. If he cannot, then he simply selects another set of assumptions and answers, and keeps playing. However, this objection undermines the *agency and rationality* of the players. In a game theoretical setting, each player follows a *strategy*, and employs a method to choose their moves, and, by definition, the strategy is predetermined based on some understanding of rationality and players' priors and epistemics. Players decide how they will play before they start playing the game. If we allow them to exercise their choice of moves based on their *a posteriori* success, that means that they did not have an *a priori* strategy before the game. Therefore, such an objection clashes with the basic definition of a strategy - a function that tells the player which move(s) to make at each state based on what moves the other players have made.

Additionally, bracketing poses another game theoretical problem as it seems to ignore the element of rationality in the game. In an inquiry game, all parties have an intrinsic prior commitment to play the game to engage in the dialogue, and solve or win the interrogative inquiry which can be viewed as a game. Questions and answers should be assumed somewhat relevant to each other, otherwise the dialogue would turn into two parallel simultaneous monologues. Suggesting the use of bracketing for such a trivial purpose is unnecessary as it ignores the rational commitment of the players in the inquiry. Putting it game theoretically, irrelevant answers may be signals or a part of a strategy, thus can be an integral part of the game - especially in imperfect information games.

Finally, bracketing suffers from various problems from a heuristic point of view as well. First, let us remember the Lakatosian concept of "proofs that do

[1]Clearly, this is an age-old question which goes back to Aquinas and Aristotle, and latter's use of the term 'analytic' in argumentation and deductions makes it historically relevant to our discussion here [29]. Hintikka himself mentions this issue very briefly on passing [20]. A further connection to what proposition can be used how many times in a deduction or an inquiry relates this debate to linear logic [15].

not prove" which is directly relevant and helpful to our investigation. Lakatosian methodology of proofs and refutations, as exemplified in *Proofs and Refutations*, discusses the significant roles of thought experiments and unsound deductions in mathematical reasoning, among many other things [26, 5, 4, 9, 24]. "Proofs that do not prove" are the proofs that are wrong in some ways, yet help us develop better proofs or improve the actual false proof. Lakatos discusses this idea in detail, and explains its role in concept formation with many historical examples. For Lakatosian epistemology, in an evolutionary and practice based sense, mathematical concepts develop, improve and then they are falsified, proven and disproven along their conceptual development. Mathematical activity continues, concepts are redeveloped, proofs are reexamined.

In short, "proof attempts" and "proofs that do not prove" help us improve the proofs. However, if we bracket the "proofs that do not prove", we risk the growth of (mathematical) knowledge, and lose the opportunity to *learn from our mistakes.* Clearly, not every mistake is didactic, and in order to distinguish them we need to resort to the semantics and the model. Additionally, what we suggest here does not mean that we should include every single bit of information in the deductive process. In other words, the presence of contradictory statements, in practice, do not *explode* the formal system or render it trivial. It may even help us develop the theory further, even if we expect the emergence of additional inconsistencies or even if we expect to obtain a consistent final theory at the end.

However, it can be claimed that our proposal is rather pessimistic, and assumes a non-cooperative nature of inquiry. A non-cooperative inquiry, in this context, is a form of inquiry where the parties do not necessarily help each other in the process. However, cooperative inquiries suffer from the aforementioned epistemic, game theoretical and heuristic issues as well. Imagine a child asking questions to her parents who are trying to answer her question. For instance, when a child receives an incorrect answer from her parents (as a result of an honest mistake in the case of a cooperative inquiry, or as a result of a lie in the case of a non-cooperative inquiry, for example), the child may not identify it as a wrong answer and accepts it regardless. In this case, we may not expect the child to apply bracketing even if the parents try to deceive her, or even if the parents try to help her.

In conclusion, if bracketing is not intended for epistemic, game theoretical and heuristic reasons, then it is employed to avoid contradictions. In an inquiry, it is possible to receive answers that contradict each other - such answers may be a part of a strategy (lying, cheating, in non-cooperative cases), they may be honest mistakes (in cooperative cases), or communication problems (noise in the communication in either cooperative or non-cooperative cases). Therefore, if we are committed to classical logic as the underlying formalism of IMI, then contradictions will trivialize the system according to the classical logical presuppositions. In order to avoid trivialization, we will argue that IMI necessitates a paraconsistent framework.

Now, we can discuss whether IMI is explosive, or more importantly, if it should be explosive. Let us now elaborate why we answer both questions negatively.

Let us consider an inquiry where p and $\neg p$ are received as responses. First, the very existence of both p and $\neg p$ is an information or a game theoretical signal. The history of science and mathematics offers numerous cases where a scientific inquiry progressed under the very existence of inconsistencies [36]. It can also be argued that the nature of the contradictions and the contradictory propositions provide some information that can be used in the course of the inquiry, and this necessitates the *study* of inconsistencies as opposed to simply ignoring them. Second, the very existence of contradictory statements provides a broader understanding of rationality - a rational player in paraconsistent IMI is not a dreamer, but a realistic inquirer who does not let his system collapse under the presence of a mere contradiction or conflicting information.

Additionally, the reason as to why Jaśkowski's discussive logic is not explosive applies to IMI as well [23]. In an inquiry, assume that player I receives two answers p and $\neg p$ at different times during the inquiry. Nevertheless, it is completely possible that there exists a proposition q which is nowhere true in the model. Thus, q may not be deducible under the presence of a contradiction. Therefore, for some p and q, we observe $p, \neg p \not\vdash q$ concluding that inquiries are not explosive. This shows that the interrogative inquiry is perhaps one of the canonical applications of paraconsistent epistemology.

On the other hand, there are some partial, classical logic oriented responses to our remarks. In [21, §8,], the authors discuss the strategic aspects of the question-answer process of the interrogative inquiry, and give various rules for bracketing which do not seem to add anything new to the discussion. Formal epistemological concerns raised by Hintikka above can be remedied by considering a variety of non-classical logics. For instance, linear logics of Girard offers a "resource" based analysis of proofs [15]. Such a proof-theoretical approach may give more meaning to questions and answers by considering whether they have been used earlier in the dialogue. This line of research may associate interrogative models to intuitionistic logics and substructural logics. Similarly, dialogical logic has some relevance to paraconsistency bridging over to interrogative inquiries [38, 39]. Additionally, the dynamic aspects of IMI have been studied by Hamami who views IMI and dynamic epistemic logics complimentary [16]. Such an approach centers its attention to the *questioning* aspect of IMI without underlining the role of classical logic in the process. Similarly, IMI can be seen as an applicable tool in some other issues in logic [14].

Based on these observations, we can now discuss a formal framework for paraconsistent interrogative inquiry.

4 Formal Matters: ParaIMI

In this section, we suggest a formal framework to express the philosophical concerns we raised about IMI. We achieve it by combining Priest's logic of paradox (LP), Hintikka's Interrogative Model of Inquiry (IMI) and a dynamic game logic.

LP introduces an additional truth value P, called *paradoxical* (P), for the sentences which are both true and false. The truth value "true" (T) is for the sentences which are true but not false. Similarly, the truth value "false" (F) is for those which are false but not true. Based on these definitions, the truth tables for LP are given as follows [32].

Implication and bi-implication are defined in the standard fashion. As Priest remarked, the following do *not* always hold in LP as expected [32].

- $\varphi, \neg\varphi \models \psi$

	$\neg$
T	F
P	P
F	T

$\wedge$	T	P	F
T	T	P	F
P	P	P	F
F	F	F	F

$\vee$	T	P	F
T	T	T	T
P	T	P	P
F	T	P	F

- $\varphi, \neg\varphi \vee \psi \models \psi$
- $\varphi, \varphi \to \psi \models \psi$
- $\varphi \to \psi, \neg\psi \models \neg\varphi$
- $\varphi \to \psi \wedge \neg\psi \models \neg\varphi$

LP is simple and easy-to-work with, but does not sufficiently address the game theoretical and epistemic issues we discussed. The system we need should be (i) inconsistency friendly, (ii) strong enough to describe game theoretical strategies in an inquiry, (iii) capable of handling multi-agent cases, and (iv) expressive enough to describe actions/moves in games. For this aim, we use a segment of *strategy logic* where strategies are taken as logical primitives [40, 41, 2]. In strategy logic, one can reason about the strategies and describe preconditions for moves beyond the traditional Boolean connectives. We call our system ParaIMI - paraconsistent interrogative inquiry.

Let us now present the formal matters. First, let Σ be the set of moves common to set of players n, and denote the set of propositional variables by $\mathbf{P}$. We follow the common methodology and interpret the game on a tree where each node denotes a game state, and the labeled edges indicate the moves. Additionally, we also have a tool that designates the turns of the players. We define ParaIMI models as follows.

Definition 4.1. A ParaIMI game tree T is defined as a tuple $T = (W, \Rightarrow, w_0)$ where W is a non-empty set with $w_0 \in W$, and the partial function $\Rightarrow: W \times \Sigma \to W$ specifies the labeled edges of the tree where the labels represent the moves. The extensive form ParaIMI game tree then is the pair (T, t) where T is as before and $t : W \to n$ specifies whose turn it is at each state. A ParaIMI model M then is defined as the tuple $M = (T, t, V)$ where T and t are as before, and V is a valuation function assigning subsets of W to propositional variables.

Let us explicate the model further. First, the model allows multiple players simply to allocate more than one inquirer and oracle. In an inquiry, it is perfectly possible to have more than one inquirer with matching or competing strategies directing questions to the same set of oracles. The turn function t is introduced as inquirers and oracles may not take turns regularly. Finally, the set of moves Σ is common to all agents to reflect the reasonable restrictions that Hintikka imposed on IMI - a shared syntax for the players.

A *strategy* is defined as a subtree of T that specifies what moves to play for a certain player when it is his turn. Therefore, for a player $i \in n$, strategy s_i for i is a function $s_i : W_i \to \wp(\Sigma)$ where W_i is the set of $w \in W$ where $t(w) = i$ that is W_i is the set of states where it is i's turn to play. Strategies give rise to strategy trees per player, and the formulas are evaluated at strategy trees. There are several reasons for this choice. First, an inquiry is a strategic process, not an automated, fully deductive reasoning. Therefore, evaluating the truth of logical formulas with respect to a strategy tree and oracles' answers is a sensible idea. Second, strategy trees localize the model. This is an important point from a game theoretical perspective. The agents have local information and local strategies, as they may not know their opponent's strategy. Finally, this formalism allocates the oracles. The information that the oracle provides can be taken as a precondition when it is evaluated at a strategy tree. Then, the agent can decide which move to make given what the oracle has said. Let us now formally introduce strategy trees [41].

Definition 4.2. For a player i, and a strategy $s(i)$, the strategy tree $T_{s(i)} = (W_{s(i)}, \Rightarrow_{s(i)}, w_0, t_{s(i)})$ is the least subtree of T satisfying (1) $w_0 \in W_{s(i)}$, (2) for any w in $W_{s(i)}$, if $t(w) = i$; then there exists a w' in $W_{s(i)}$ and a move a such that $\Rightarrow_{s(i)} (w, a) = w'$ (or $w \overset{a}{\Rightarrow}_{s(i)} w'$). If $t(w) \neq i$; then for all w' with $w \overset{a}{\Rightarrow} w'$ for some a, we have $w \overset{a}{\Rightarrow}_{s(i)} w'$.

The above definition specifies that the root (the start of the game or the current state) is included in the strategy tree, and when it is the player's turn, his moves at those states are included in the strategy tree. If it is not his turn, then all possible accessible nodes are included in the tree just because the player cannot know how the opponents may play. We will write T_s instead of T_{s_i} to simplify the notation when it is obvious.

It is essential to underline that the very definition of a strategy tree illustrates how we can avoid bracketing. The strategies count in all possible moves of the opponent, they do not exclude (or bracket out) some of them. If the inquiry is between an inquirer and nature, then the oracle's answer is assumed to be among nature's possible moves.

The syntax of ParaIMI is given as follows.

$$p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \rightsquigarrow_i a$$

where $p \in \mathbf{P}$ and $a \in \Sigma$. We will drop the subscripts when possible for easy reading. In order to reflect Hintikka's concerns, we did not specify the logical constants. Yet, from a formal perspective, they can easily be appended.

The formula $\varphi \rightsquigarrow_i a$ with the dynamic operator $\rightsquigarrow_i$ means that when the precondition φ is satisfied, then the agent i makes an a move. In other words, it is a controller to see whether the move that i made is compatible with the strategy or not. This makes it clear why we evaluate formulas at the strategy trees.

Now, what does $\varphi \rightsquigarrow_i a$ signify in the context of inquiry? In a scientific inquiry for instance, it can denote the steps to take after making an observation: when scientist i makes an observation (φ), then based on this observation, he proceeds to some action a (some further experimentation, calculation, revision, etc). The key point is that the whole process ($\varphi \rightsquigarrow_i a$) may or may not agree with his strategy, which is his research program aiming at proving a scientific hypothesis in this case. And then the framework of ParaIMI allows us to describe this situation. We believe this is one of the strengths of ParaIMI and justifies the use of strategy logic to explore it.

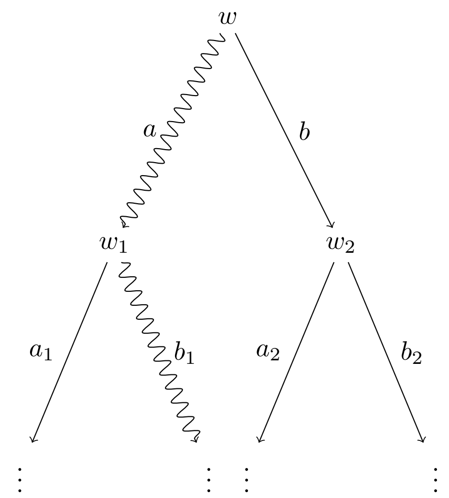
The semantics of the Booleans are standard, hence skipped. We give the semantics of the dynamic operator $\rightsquigarrow_i$ as follows.

$$T_{s(i)}, w \models \varphi \rightsquigarrow_i a \text{ iff } M, w \models \varphi \text{ implies } a = out_s(w)$$

where $out_s(w)$ is an outgoing labeled edge from w with respect to the strategy s. The semantical definition simply indicates that when a precondition is met, the strategy tells us which action to take. If φ has the paradoxical truth value P, for instance, this formula states what move to make when a paradoxical sentence (such as Russell's Sentence) is the precondition. This eliminates the need to bracket some propositions out to maintain consistency.

Now, we can briefly compare ParaIMI with the one given in [21]. One of the most interesting theorems in the aforementioned work, the *Yes - No Theorem*, states that any conclusion that can be deduced by using various instrumental questions can be deduced by yes - no questions. The Yes - No Theorem fails in ParaIMI in its stated form. The reason is that for some formulas, $\varphi \vee \neg\varphi$ is not a truly yes-no question since the formula φ may be paradoxical, and the paradoxical truth value P is a fixed-point in LP under the negation operator. This suggests that it is not possible to obtain P from the other truth values with a negation and this renders polar questions in IMI not sufficient.

As we mentioned earlier, the oracle's answers must be among the possible moves of the player nature. These answers produce a strategy tree that determines the play of the inquirer I. In this case, the oracle *guides* the inquirer I by giving him the exact specifications as depicted below.



This diagram represents the strategy tree of the inquirer I where the curly branches identify the actual actions/moves taken based on the oracle's answers. For instance, at state w_1, the oracle is asked a question, and the answer I receives directs him to b_1. Thus, we observe the following.

Lemma 4.3. *In ParaIMI, the subtree that is constructed after the responses of the oracle is a subtree of the strategy tree of I.*

The proof of the above lemma follows from the definitions. In order to see how the argument works, assume otherwise. If the responses of the oracle are not part of the strategy, then the player cannot make any moves based on the responses. Also, by definition, every possible move of the player nature is included in the inquirer's strategy in which the oracle's responses lie. This result ensures that the oracle's answers are not only correct, but also relevant.

Now, we call a strategy tree *minimal* if it does not contain a smaller strategy subtree with the same root. A minimal strategy tree need not be unique.

Lemma 4.4. *An inquiry that is conducted based on the oracle's responses at each state is the unique minimal strategy tree.*

Proof. Let us call the sub-tree constructed based on the oracle's responses as T_o. First, let us show it is minimal. For a contradiction, assume there is a tree $T'_o \subset T_o$ constructed by the oracle's responses. However, as $T'_o \subset T_o$, T_o contains some moves (branches) based on the oracle's responses which are not included in T'_o. But, T'_o was also constructed by the responses of the oracles, and must have included those branches and moves (responses). Contradiction shows that T_o is minimal.

Now, let us show it is unique. Assume T is another minimal tree constructed based on the oracle's responses with the same length. We will show T and T_o are identical.

First, we define the length $l(T)$ of a tree T as the shortest path between the root and the terminal nodes. The proof is by induction on the length of the trees l.

For, $l = 0$, by definition, the root belongs to both T_o and T. So, the claim holds.

Assume that both T_o and T are identical up to $l = n$ in order to show that the claim holds $l = n+1$. At length n, the strategy tree is constructed by asking which move to make to the oracle. The answer then included in to both T_o and T to obtain a tree of length $n + 1$. In this case, both T_o and T remain identical at length $n + 1$.

By induction, the claim holds for any n. So, T_o and T are identical. Thus, T_o is unique. □

Lemma 4.5. *Every strategy has a unique minimal strategy that coincides with the oracle's strategy.*

Proof. Given a strategy σ for I, Lemma 4.3 shows that the oracle's responses form a subtree T_o of T_σ where T_σ is the strategy tree based on σ. By Lemma 4.4, T_o is unique and minimal, and combining these two result, σ has a unique minimal subtree T_o. □

Notice that each strategy, with or without the help of the oracle, can be reduced to a minimal strategy. However, for the uniqueness, we need an input from the oracle.

We only assumed that the oracle is a truth-teller. However, it is possible to impose further conditions on the oracle. An oracle is said to have *state-based rationality* if at each state in the ParaIMI model, the oracle gives the most rational response (not only the correct one), that is, the answer that brings the highest pay-off. Such a state based rationality can be represented by a proposition [11, 42]. The strategy that is constructed with the responses of the oracle that has state-based rationality will be called the *rational strategy*. In the following, we assume one oracle for simplicity as the ideas can easily be generalized to the situations with multiple oracles.

Lemma 4.6. *If we assume state-based rationality for the oracle, than the inquiry conducted with the help of the oracle has the rational strategy.*

Proof. The proof follows from the definitions. Let the oracle have state-based rationality. For a contradiction, assume that the inquiry conducted with the help of the oracle does not admit the rational strategy. By definition, this suggests that, at least at one state, the oracle does not offer the most rational response. But this violates the initial assumption of state-based rationality for the oracle. Thus, the inquirer's strategy must be rational. □

Now we observe that the rational strategy need not be a minimal strategy.

Lemma 4.7. *Every rational strategy has a minimal rational strategy.*

Proof. Let σ be a rational strategy. By Lemma 4.5, σ has a minimal strategy σ'. Then, we need to show that σ' is rational.

Assume not. Then, there is a response by the oracle in σ' that is not rational. However, this response also belongs to σ which was assumed to be rational. The contradiction shows that $\sigma' \subseteq \sigma$ is minimal. □

Based on the lemmas we have given above, the following theorem follows.

Theorem 4.8. *The best strategy for the inquiry game in ParaIMI can be obtained by introducing the oracle's answers and the assumption of state-based rationality as a set of premise to the inquiry.*

Proof. Notice first that this is the generalized version of the Strategy Theorem in [21]. We will only sketch the proof idea here.

By the previous lemmas, it follows that the minimal rational strategy can be constructed by the oracle's answers and the assumption of state-based rationality. Therefore, treating the rationality as a proposition that holds at the states that are part of the rational strategy, we can simply turn them into assumptions in the inquiry. Additionally, if this strategy is not the best one, then there is a better one, which needs to be included in the rational strategy. This violates the initial assumption (towards contradiction that) the strategy was not the best one. □

It is important to observe the strengths of ParaIMI which stem from LP. For an arbitrary strategy tree T_s and a state w on T_s, we observe that $T_s, w \models \varphi \vee \neg\varphi \rightsquigarrow a$ does not always hold as there is a truth value beyond true and false. This means that polar questions are not sufficient for ParaIMI. In the classical IMI, we need to raise one question, say, $\varphi \vee \neg\varphi$. In ParaIMI, the corresponding questions, for an action a are as follows:

- $\varphi \vee \neg\varphi \rightsquigarrow a$
- $\varphi \vee P \rightsquigarrow a$
- $\neg\varphi \vee P \rightsquigarrow a$

This level of complexity may raise some doubts about the computational cost of the process. The following argument resolves this issue.

Theorem 4.9. *The computational complexity of asking questions to the oracle and constructing a rational strategy tree in ParaIMI is the same as performing the same operation in the classical IMI.*

Proof. In ParaIMI, the paradoxical truth value complicates the inquiry only by a constant. Therefore, both IMI and ParaIMI belong to the same complexity class. □

The above theorem can also be viewed as another argument against bracketing on the basis of computational cost.

Another interesting issue in dynamic games and classical logic is duality. Classically, if a player has a winning strategy to verify the formula φ, then he cannot have a winning strategy for its negation $\neg\varphi$. This result does not hold in ParaIMI, due to the paradoxical truth value. If the inquirer has a winning strategy, that is a strategy that certainly brings him a win for a paradoxical formula; then he also has a strategy for the negation of that paradoxical formula, which is still paradoxical. Therefore, the duality between negations and possessing winning strategies is broken in ParaIMI.

Theorem 4.10. *In IMI, if the inquirer I can rationally strategize about an inquiry about φ and prove it with the help of an oracle, then I cannot prove $\neg\varphi$. This is not the case in ParaIMI.*

Proof. The argument is straight-forward for (classical) IMI.

In order to see why the argument fails in ParaIMI, assume φ has a truth value P. Then, by definition, $\neg\varphi$ has truth value P as well. Thus, it is possible for a rational strategy to prove φ and $\neg\varphi$ at the same time in ParaIMI. □

Strategy trees can be instrumental in understanding the epistemic aspects of interrogative inquiries. What we have discussed above indicates that given a rational strategy σ, it is possible to obtain a minimal strategy that coincides with the oracle's. In this process, the inquirer I poses questions to the oracle. Given a rational strategy σ, we call the minimum number of questions that I needs to ask to the oracle in order to construct the minimal rational strategy σ' as *jumps*, and denote it with $j(T_\sigma)$ for a given strategy tree T_σ.

Proposition 4.11. *For a strategy tree T, $j(T) \leq l(T)$.*

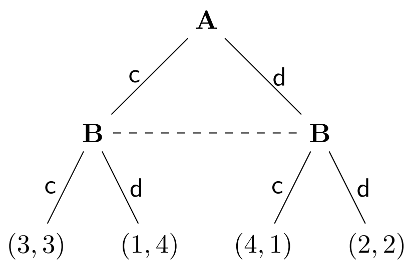
Proof. The proof directly follows from the definitions. Because asking more questions to the oracle than the length of the tree T_σ for a rational strategy σ would lead to constructing a minimal rational strategy tree $T_{\sigma'}$ for a minimal rational strategy σ' where $l(T_{\sigma'}) > T_\sigma$. This contradicts the minimality of σ'. □

We call an inquiry game *determined* if all the players know their pay-offs.

Proposition 4.12. *Interrogative inquiries conducted with an oracle with state-based rationality are determined.*

Proof. Since LP is decidable, and for a rational strategy σ, $j(T_\sigma)$ is bounded, each T_σ has to be finite. If the pay-offs are not known, this entails that at some point in the game, the oracle did not give out the rational response that might have brought the highest pay-off. This contradicts with the state-based rationality of the oracle. Thus, the inquirer knows his pay-off once the oracle has state-based rationality. □

It is important to see that jumps help the inquirer to handle the epistemic indistinguishability in the game. Let us consider the simple game of Prisoners' Dilemma in its extensive normal form where the utility pair (u_A, u_B) denotes the utility of the players **A** and **B** respectively. In this context, **A** and **B** are considered as inquirers. The moves c and d denote "cooperate" and "defect" respectively. In the extensive form game tree, the dots between **B**'s moves indicate that they are epistemically indistinguishable for **A**. They are the exact points for **A** for an oracle inquiry. Based on the below utilities for both agents, the traditional game theoretical view suggests that the rational move is to defect (make a d move).[2] Nevertheless, as long as the players do not know their opponent's move, they will never know their exact pay-offs.



An oracle can solve this problem. If **B** asks the oracle how **A** has played, then he can know his exact pay-off. Then, jumps are the epistemic states where agents can eliminate some epistemic possibilities based on oracle's answers. In this fashion, it is possible to identify jumps with epistemic indistinguishable states. The inquirer therefore is supposed to dissolve the epistemic indistinguishabilities in the game by posing questions to the oracle. Epistemic indistinguishable states form information sets. In the above example, the states connected by a dotted-line form an information set.

Theorem 4.13. *In a finite strategy tree T, the total number of the information sets is bound by $j(T)$.*

Proof. Assume otherwise and suppose the total number of the information sets is $n > j(T)$ for a given strategy tree T. Therefore, there will be an epistemic state w in an information set which is indistinguishable for player I (such as the states for **B** in Prisoners' Dilemma), and I will not be a able to inquire to the oracle to determine the minimal rational strategy in order to maximize his pay-off. Thus, I will not have a minimal rational strategy. But, according to Lemma 4.4, we know that the minimal rational strategy exists for I. Thus, we obtain a contradiction and $n > j(T)$ cannot be the case. □

The above theorem simply specifies that the inquires for the oracle include the epistemic ones.

*

[2] It is an essential and fruitful debate whether the traditional view of rationality presupposes or necessitates classical logic, and whether paraconsistent logic entails a broader reading of game theoretical rationality. However, for the purposes of the current paper, we will leave this discussion aside.

Our treatment of ParaIMI does not address the erotetic aspects of IMI, instead focuses on its strategic, dynamic and interactive aspects. There have been some formal attempts to achieve this task which goes back to Belnap and Steel [7, 1, 43, 21, 44]. Additionally, some formal approaches to classical IMI were also presented relatively recently [14, 16, 28]. To the best of our knowledge, some of the ideas we have put forward in this article have been discussed in a different framework by Batens [6]. Batens motivates his formalism with the need to "construct explanations from inconsistent knowledge", and shows how a "coherent theory of the process of explanation is obtained from inconsistent knowledge" (*ibid*). Both the current paper and [6] argue that inconsistencies need *not* be removed *during* the process of explanation and "one cannot simply discard" an inconsistent theory just because it is inconsistent. Batens, in his work, uses an inconsistency-adaptive logic to develop his interrogative model, and his achievement can briefly be summarized as replacing the classical logical framework of [21] with inconsistency-adaptive logics, and consequently implement the necessary changes both in the model and in the deductive rules accordingly.

5 Some Uses of ParaIMI

ParaIMI can provide a wider perspective in illustrating various aspects of inquiry. In this section, we focus on some applications to show how ParaIMI can be useful in expressing inquisitive and inconsistency-friendly aspects of interrogative inquiry.

The first example brings together bracketing and inquiry in an interesting way, and illustrates it quite clearly as to why bracketing must be avoided in a scientific inquiry.

Example 5.1. Let us consider Lakatos's seminal work *Proofs and Refutations* [26, 24, 9, 3, 4]. *Proofs and Refutations* is written in an interrogative dialogue form where the inquirers examine the conjecture and battle with examples and counter-examples. In the dialogue, the inquirers try to figure out whether Euler's conjecture holds. The conjecture, as it is specified initially in *Proofs and Refutations*, claims that for all polyhedra, we have $V - E + F = 2$ where V, E, F are the numbers of vertices, edges and faces of the polyhedron in question respectively. Let us denote the conjecture by χ.

At some stage of the inquiry, the dialogue includes both sentences χ and $\neg\chi$ with their supporting examples and refuting counter-examples. Different sets of inquirers deduce the aforementioned sentences based on some distinct set of examples and counterexamples. For instance, regular cube verifies the conjecture and yields χ. Similarly, hollow cube (which is a cube from which a smaller cube is removed) is a counter-example and refutes the conjecture yielding $\neg\chi$. For Lakatos, this is one of the important stages of the growth of knowledge as the conjecture is modified and improved at each stage. Lakatosian methodology has a strategy which describes what to do when contradictions emerge without trivializing the system.

Let us assume that based on a regular cube (for which the conjecture holds) and hollow-cube (for which the conjecture does not hold), we have the statements χ and $\neg\chi$. In this situation, Lakatosian strategy has a method that suggests to redefine the terms (which is "polyhedron" in this case). Let us denote this move with $\mathsf{ReDefine}$, and the associated strategy tree with T_r. Therefore, after the cube and hollow-cube are observed, at some state w of T_r, we have $T_r, w \models \chi \wedge \neg\chi$ based on the previous answers (cube and hollow-cube) we have received. Since our Lakatosian methodology tells us what to do in this contradictory situation, we also observe that $T_r, w \models (\chi \wedge \neg\chi) \rightsquigarrow \mathsf{ReDefine}$, or simply, by a slight abuse of notation, we observe that $T_r, w \models P \rightsquigarrow \mathsf{ReDefine}$ holds: when a contradictory statement of this sort is reached, redefine the terms. Notice that it is possible that at some state in the inquiry game, the statement $P \rightsquigarrow \mathsf{ReDefine}$ may fail if redefining the terms is not available methodologically, or needs to be avoided to prevent an *ad hoc* strategy, which Lakatos also addressed.

Lakatos describes various other actions for the cases when inconsistencies emerge. These actions include re-examining the proof, incorporating lemmas to the statement, declaring some counter-examples as exceptions (or *monsters* as Lakatos calls them), and define these strategies within various strategic frameworks. Therefore, ParaIMI for Lakatosian methodology has satisfiable statements such as $P \rightsquigarrow \mathsf{ReExamineProof}$, $P \rightsquigarrow \mathsf{IncorporateLemma}$, and $P \rightsquigarrow \mathsf{DeclareExceptions}$ where P is the paradoxical truth value.

As observed, the Lakatosian method does not bracket out the contradictions in an inquiry, and this is one of the central tenets of the method of proofs and refutations even if the Lakatosian method is not paraconsistency-friendly. $\square$

We can generalize the above idea to belief revision.

Example 5.2. Paraconsistent logic can directly be applied to belief revision theory [12, 34, 27]. From an IMI view point, the interrogative inquiry can be considered between the inquirer and the nature where the inquirer discovers an inconsistency and asks questions as to how to revise the formal model of the inquiry. The questions he asks, in this case, are in the form of various rules of belief revision theory such as contraction and revision rules.

For this example, let us consider the following case. Assume that you thought you had your wallet with you with some cash in it. Then, you go to a restaurant to eat, and after the meal, you want to pay the bill. You try to reach your wallet only to realize that you left it home. This new information, added to your belief set, creates a contradiction. According to your current state w, it is both true and false that you believe that you have your wallet with you. Your "thought" suggests that you had your wallet with you, and at the same time, the evidence suggests that you simply did not have it. Then, as a rational person, you know what to do. You either explain the situation to the waiter, or call a friend for help. Therefore, in this belief model, non-classically, we have $T_s, w \models P \rightsquigarrow \mathsf{CallAFriend}$ for some strategy tree T_s. Belief revision theory, being a classical logical theory originally, aims at resolving the contradiction by revising the theory, and offers descriptive ways on *how* to revise the theory [12].

Notice that the contribution that ParaIMI provides here is to introduce actions and strategies in a paraconsistent model. This amounts to the fact that some different ways of model revision can be represented with respect to various strategic approaches. Also, it can express inconsistencies, and describe what the agent should do when the inconsistencies are present. This is a rather improved approach to the phenomena than what is presented in [21].

Priest offered some insights on the subject as well.

> People often have inconsistent commitments. (...) Yet it is absurd to

> suppose that a person who has inconsistent commitments is thereby committed to everything. If, by oversight, I believe both that I will give a talk on campus at noon, but also that I will be in town at noon, this hardly commits me to believing that the Battle of Hastings was in 1939. (...) Worse, it is not at all clear that an ideally rational agent must have consistent beliefs. Sometimes there is overwhelming evidence for inconsistent beliefs.
>
> [34]

However, our take on the subject is rather different than Priest's. We allow (and accept) that some agents may strive for being an ideally rational agent. Some of such agents can be computer abstractions, agents of artificial intelligence or queries in some databases with possible inconsistencies. Nevertheless, even if that is the case, there is some state in the model at which inconsistencies occur and they do not collapse the model. Following the initial thought-experiment with the wallet, when you have an inconsistent belief whether you have your wallet with you or not, you first acknowledge the inconsistent belief, and then *choose* to revise it. Thus, before the revision and until the verification of the revision, the model is inconsistent, yet allows a (classical logic induced) rational deduction - the deduction that says that the model should be revised. However, some other logical frameworks can be given, as Priest did in [34], in such a way that after the emergence of the inconsistent belief, the agent may choose some other action.

In our example, we have $T_s, w \models P \rightsquigarrow \mathsf{CallAFriend}$, in some other, we can endorse $T_u, w \models P$ without any action attached where s and u are strategies with their associated strategy trees. Therefore, without resorting to any assumption of rationality, ParaIMI can express various strategic actions under the presence of an inconsistent situation. □

A relatively recent work discusses the interrogative and erotetic aspects of IMI from a formal perspective. We now observe how ParaIIMI can enrich the discussions of such formalisms.

Example 5.3. Let us now discuss a case from Genot [14]. In his work, Genot offers a formal framework for IMI with tableaux method based on [21]. A central theme of this proposal is to incorporate the theory of belief revision and game theory into IMI, and bracketing plays an important role in Genot's formalization. Following Genot's representation, by $!\varphi$ let us denote the action of answering with the statement φ. Namely, once asked $\varphi \vee \neg\varphi$, an agent can respond by an action $!\varphi$. Similarly, by $?\varphi$, we denote the action of asking whether it is the case that φ.

$$
\begin{array}{ccccc}
?\mathsf{T}(\varphi_1 \vee \varphi_2) & \qquad & ?\mathsf{T}\neg(\varphi_1 \wedge \varphi_2) & \qquad & \varphi \in A \\
| & & | & & | \\
!\mathsf{T}\varphi_i & & !\mathsf{T}\neg\varphi_i & & ?\mathsf{T}(\varphi \vee \neg\varphi) \\
(\varphi_i \in A) & & (\neg\varphi_i \in A) & & | \\
 & & & & !\mathsf{T}\varphi
\end{array}
$$

Figure 1: Genot's IMI game trees for some connectives [14].

Here, we reproduce some of the semantic trees that Genot gives where A denotes the set of available answers. We now show how they can be represented in ParaIMI. The first two trees in Figure 1 are classical. In the setting of ParaIMI, we represent them with $\varphi_1 \vee \varphi_2 \rightsquigarrow !\varphi_i$ where $\varphi_i \in A$, and $\neg(\varphi_1 \wedge \varphi_2) \rightsquigarrow !\neg\varphi_i$ where $\neg\varphi_i \in A$.

The third tree in Figure 1 uses the resolution rule which is not valid in ParaIMI as we remarked earlier in Section 4. However, the prefix T assures that the truth value is T (as opposed to P or F) which validates the scheme in ParaIMI. In this case, it is a case of resolution restricted only to one truth value of ParaIMI. In ParaIMI, we can introduce an additional prefix P which stands for the truth value P. We suggest the following additional rules:

$$
\begin{array}{c}
?\mathsf{P}(\varphi_1 \vee \varphi_2) \\
\swarrow \quad \searrow \\
\begin{array}{cc} !\mathsf{T}\varphi_i & !\mathsf{P}\varphi_i \\ (\varphi_i \in A) & \end{array}
\end{array}
\qquad
\begin{array}{c}
?\mathsf{P}\neg(\varphi_1 \wedge \varphi_2) \\
| \\
!\mathsf{P}\neg\varphi_i
\end{array}
\qquad
\begin{array}{c}
\varphi \in A \\
| \\
?\mathsf{P}(\varphi \vee \neg\varphi) \\
| \\
!\mathsf{P}\varphi
\end{array}
$$

The last case is interesting. Here, if the question itself for an answer φ is paradoxical, then φ itself turns out to be paradoxical. Clearly, this heavily depends on the form of the question $(\varphi \vee \neg\varphi)$ and the truth table for disjunction. These rules can easily be extended to other cases and connectives, and we leave it to the reader. □

6 Conclusion

In this paper, we first discussed bracketing in detail, and argued against its use in IMI. Motivated by our observations, we proposed a paraconsistent framework for IMI, and detailed its use in various examples. Our contributions included various game logical results that related ParaIMI to some other work in the field, and underlined the benefits of using an inconsistency-friendly formalism.

There are several issues which we did not focus in this work. For instance, we did not discuss the computational aspects of IMI or ParaIMI in detail. An interesting observation is that interrogative inquiry and *inquiry* in some declarative programming languages (such as prolog) have some interesting similarities. The way such languages handle negation (closed world assumption) and negations in (paraconsistent or not) interrogative inquiries present exciting new research directions.

The careful reader may have noticed that we paid some but insufficient attention to various Lakatosian concepts. The connection between paraconsistent inquiry and Lakatosian methodology is stronger than it might seem. The path between the two goes through Hegelian dialectic, and requires a special treatment which falls outside the scope of the current paper [4, 33, 37, 25].

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