

The Completeness of Carnap's Predicate Logic

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Abstract

The paper first proves the completeness of the (non-modal) first-order predicate logic presented in Carnap's 1946 article 'Modalities and quantification'. By contrast the modal logic defined by the semantics Carnap produces is unaxiomatisable. One can though adapt Carnap's semantics so that a standard completeness proof for a Carnapian version of predicate S5 turns out to be available.

The purpose of this note is to prove the completeness of the non-modal predicate logic found in Carnap 1946¹, and of a Carnapian version of predicate S5, with respect to a semantics a little different from Carnap's. I will say at the outset that the results and proof methods used are not surprising or significantly innovative; and the reason for the note is simply that, as far as I am aware, no one has proved specifically the completeness of *Carnap's* 1946 predicate logic. In setting this out I shall avoid Carnap's use of German letters, which make his work difficult for current readers. Since Carnap's labels for axioms and so on are specific to his paper I have used names which are more indicative of their content. Further, I shall base my presentation of Carnap's logic on \perp and \supset rather than on Carnap's \sim , \vee and \cdot . For quantifiers I use $\forall x$ in place of (x) , and $\exists x$ in place of $(\exists x)$.² Carnap's language of FL (first order logic) contains a denumerable infinity of individual variables and constants, which can be referred to simply as 'variables' or 'constants'. Carnap uses the term 'matrix' for a (well-formed) formula (wff), and the term 'sentence' for a closed wff, i.e., a wff with no free variables. When it comes to interpreting FL Carnap (p. 49) assumes denumerable domains. For this reason he is able to represent a domain by a denumerable set of individual constants — a distinct constant for each individual. The

¹Carnap does not claim to prove completeness, and on p. 53 he says that it is not clear whether his axiom system is complete, since Gödel's completeness theorem (Gödel, 1930) "cannot be directly applied" to it. Carnap's article predates the completeness proof for first-order predicate logic found in Henkin 1949. The present note provides a Henkin proof for an axiomatic system equivalent to Carnap's FC.

²The other truth functions are all definable in terms of \supset and \perp ; in particular $\alpha \vee \beta$ may be defined as $(\alpha \supset \beta) \supset \beta$, $\sim \alpha$ as $\alpha \supset \perp$ and $\alpha \wedge \beta$ as $\sim(\alpha \supset \sim \beta)$. Carnap takes \sim , \vee and \cdot (\wedge) as primitive, but for the points made here these differences do not matter. $\exists x \alpha$ may be defined as $\sim \forall x \sim \alpha$. Carnap does not introduce an explicit definition, but does use \exists on p. 53.

presence of constants enables there to be atomic sentences like φab . Truth and falsity in FL are relative to a *state-description* (Carnap 1946, p. 50, and 1947, p. 9). Formally, a state-description s for a language \mathcal{L} of first-order predicate logic is a set of atomic sentences or their negations, such that for every n -place predicate φ of \mathcal{L} , and every n -tuple of constants a_1, \dots, a_n , either $\varphi a_1 \dots a_n \in s$ or $\sim \varphi a_1 \dots a_n \in s$, but not both. In place of relativising the truth of an open formula to an assignment to the variables Carnap defines truth only for sentences. If s is any state-description, I write $s \models \alpha$ to mean that α is true in s and $s \models \alpha$ to mean that α is false in s . The truth of a sentence α in a state-description s may be defined formally by the following valuation rules:

$[V\varphi]$ $s \models \varphi a_1 \dots a_n$ if $\varphi a_1 \dots a_n \in s$, and $s \models \varphi a_1 \dots a_n$ if $\sim \varphi a_1 \dots a_n \in s$

$[V=]$ $s \models a = b$ iff a and b are the same constant

$[V\perp]$ $s \models \perp$

$[V\supset]$ $s \models \alpha \supset \beta$ iff $s \models \alpha$ or $s \models \beta$

$[V\forall]$ $s \models \forall x\alpha$ iff $s \models \alpha[a/x]$ for every constant a , where $[\alpha/x]$ is α with a replacing every free x .³

$s \models \alpha$ means that α is true in s . Carnap does not speak of truth in a state-description, but speaks of the *range* of a sentence — which is a set of state-descriptions. Where I have said $s \models \alpha$, Carnap will say that s is in the range of α . Carnap's 'rules of ranges' (p. 50) then become $[V\varphi]$ – $[V\forall]$, allowing for the adoption of \perp and \supset as the primitive truth-functional operators. Say that a sentence α is *valid*, or in Carnap's words *L-true*, iff $s \models \alpha$ for every state-description s .

Carnap produces an axiomatic basis for what he calls 'FC'. In Carnap's terminology a 'logic' is a semantic specification of a set of formulae, while a 'calculus' is an axiomatic specification. 'FC' stands for 'functional calculus', and I shall use it to refer to the version presented here of Carnap's axiomatic system. Carnap refers to axioms as 'primitive sentences'. In place of Carnap's (\forall) to indicate a universal closure of a wff, I shall write $\forall\alpha$. More specifically, where α is any wff, $\forall\alpha$ denotes α preceded by a (finite) sequence of universal quantifiers, in any order, which includes one for every variable free in α . (Where α is a sentence then α itself is an instance of $\forall\alpha$.) With that notation Carnap's first axiom schema, D8–1a (p. 52), can be written as:

³This feature of Carnap's semantics can be thought of as a version of the 'substitutional' interpretation of the quantifiers — to the extent that Leblanc 1973, p. 2, refers to the substitutional account of quantification as 'Carnap's account'. Leblanc 1969 (following Beth 1959, pp. 263–267) provides a completeness proof for a Carnap-like system. What makes the present completeness proof different is the feature of identity reflected in $[V=]$, that $a = b$ is always false when a and b are distinct constants. This latter feature has some similarities with the interpretation of Wittgenstein's *Tractatus-Logico-Philosophicus* (Wittgenstein 1922, for instance at 5.53) studied in Rogers and Wehmeier 2012, pp. 540–555, though Wittgenstein appears to suggest that identity statements, like say $a = b$, and even presumably $x = y$, cannot even be stated.

PC Any wff of the form $\forall\alpha$ where α is a PC-tautology, i.e., where α can be obtained from a valid formula of the classical propositional calculus by uniform substitution of wff of \mathcal{L} for its propositional variables.⁴

and the remaining schemata can be represented as:

$$\forall\supset \forall(\forall x(\alpha \supset \beta) \supset (\forall x\alpha \supset \forall x\beta))$$

VQ $\forall(\alpha \supset \forall x\alpha)$, where x is not free in α .

$\forall 1a$ $\forall(\forall x\alpha \supset \alpha[y/x])$, where $\alpha[y/x]$ is just like α except in having free y in place of free x , where y is any variable for which x is free

$\forall 1b$ $\forall(\forall x\alpha \supset \alpha[a/x])$, where $\alpha[a/x]$ is just like α except in having a in place of free x , where a is any constant

I1 $\forall x x = x$

I2 $\forall(x = y \supset (\alpha \supset \beta))$, where α and β are alike except that α has free x in 0 or more places where β has free y .⁵

I3 $a \neq b$ where a and b are different constants.

The only transformation rule is modus ponens:

MP $\vdash \alpha, \vdash \alpha \supset \beta \rightarrow \vdash \beta$

In MP of course, given that $\vdash \alpha$ and $\vdash \alpha \supset \beta$, then α and β and $\alpha \supset \beta$ must all be sentences. Carnap's axiomatisation follows that of Quine 1940 in restricting theorems to closed wff. In what follows proofs will be given in outline only, especially where full proofs may be found of these or similar theorems on pp. 80-115 of Quine's book. The principal non-standard feature is I3, which entails that all state descriptions are infinite, and, as Carnap points out on p. 53, sentences like $\exists x\exists y x \neq y$ are true in universes containing more than one individual.

⁴This is slightly more general than the definition Carnap uses, since it requires that the \forall s in instances of $\forall\alpha$, where α is an instance of a PC tautology, include all prefixes consisting of universal quantifiers, in any order, whose variables are those free in α . In this respect I am following Fitch 1941, commenting on Quine 1940, whose first edition contained an axiom schema stating that the order of two adjacent universal quantifiers makes no logical difference. This issue is a complex one discussed in some detail in Fine 1983. What Carnap says on p. 51 is that ““()(\mathcal{M}_k)” denotes the *closure* of \mathcal{M}_k , i.e. the sentence formed out of \mathcal{M}_k by prefixing universal quantifiers for all variables occurring freely in \mathcal{M}_k , in their inverse alphabetical order.” He then (footnote 14, p. 52) cites Berry 1941 as establishing that this is sufficient to enable the proof that the order of a sequence of universal quantifiers is immaterial.

⁵Carnap, D8-1g, (1946, p. 52), appears to require that β must have free y wherever α has free x , but that seems too stringent, since $\forall x\forall y(x = y \supset (\varphi x x \supset \varphi x y))$ is valid.

We cannot, in Carnap's basis, appeal to the rule of universal generalisation in the form, $\vdash \alpha \rightarrow \vdash \forall x \alpha$, because Carnap's theorems are all sentences. Nor can we appeal to the rule $\vdash \alpha \rightarrow \vdash \forall x(\alpha[x/a])$, (where $\alpha[x/a]$ is α with free x replacing every occurrence of a in α), since, when a and b are distinct constants we have $\vdash a \neq b$, but we do not have $\vdash \forall x x \neq b$.⁶ However we can prove the following:

Theorem 1. *If $\vdash \alpha$, and a is any constant, then, for some $n \geq 0$, there are constants b_1, \dots, b_n such that $\vdash \forall x((x \neq b_1 \wedge \dots \wedge x \neq b_n) \supset \alpha[x/a])$, provided x does not become bound in $\alpha[x/a]$ as a result of the substitution. (We allow the degenerate case in which there are no b s, so that here $n = 0$, and we have simply $\vdash \forall x \alpha[x/a]$.)*

Proof. The proof is by induction on the proof of α , and depends on the fact that if $\vdash \alpha$ then α is closed. First suppose that α is an instance of one of the axiom schemata PC-I2. Then $\forall x(\alpha[x/a])$ remains an instance of that very same schema. This may be seen by inspection. So for these cases the theorem holds where $n = 0$. The remaining case is I3:

$$\mathbf{I3'} \vdash \forall x(x \neq b \supset x \neq b)$$

$$\mathbf{I3''} \vdash \forall x(x \neq a \supset a \neq x)$$

I3' is an instance of PC, and I3'' follows from I1 and I2 using $\forall 1b$.⁷

Further, MP preserves the rule, since suppose we are given $\vdash \alpha$ and $\vdash \alpha \supset \beta$. Then, by the induction hypothesis, we may assume that there are b_1, \dots, b_n , and c_1, \dots, c_k such that

$$(i) \vdash \forall x((x \neq b_1 \wedge \dots \wedge x \neq b_n) \supset \alpha[x/a])$$

and

$$(ii) \vdash \forall x((x \neq c_1 \wedge \dots \wedge x \neq c_k) \supset (\alpha[x/a] \supset \beta[x/a]))$$

From (i) and (ii), by PC and $\forall \supset$ we have

$$(iii) \vdash \forall x((x \neq b_1 \wedge \dots \wedge x \neq b_n \wedge x \neq c_1 \wedge \dots \wedge x \neq c_k) \supset \beta[x/a])$$

and so the theorem holds for β . □

The next three results are proved in an appendix:

⁶Compare Carnap's remarks in T7.2 and T7.3 in Carnap 1946, p. 51f.

⁷This may be proved as follows:

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|-----------------|-----|--|
| I2 $\forall 1b$ | (1) | $\forall x(a = x \supset (a = a \supset x = a))$ |
| (1)PC | (2) | $\forall x(a = a \supset (x \neq a \supset a \neq x))$ |
| (2)I1 | (3) | $\forall x(x \neq a \supset a \neq x)$ |

Theorem 2. *If $\alpha \supset \beta$ then, for some b_1, \dots, b_n , $\vdash \alpha \supset \forall x((x \neq b_1 \wedge \dots \wedge x \neq b_n) \supset \beta[x/a])$, provided a does not occur in α*

Theorem 3. *Where $\forall x\alpha$ is a sentence and b_1, \dots, b_n are any constants then, $\vdash (\alpha[b_1/x] \wedge \dots \wedge \alpha[b_n/x] \wedge \forall x((x \neq b_1 \wedge \dots \wedge x \neq b_n) \supset \alpha)) \supset \forall x\alpha$*

Lemma 4. $\vdash \exists y(\alpha[y/x] \supset \forall x\alpha)$ provided y is not free in $\forall x\alpha$.

In proving soundness we make use of the following lemma:

Lemma 5. *Where s is any state-description and α is any wff, where the variables free in α are x_1, \dots, x_n , and $s \models \alpha[a_1/x_1, \dots, a_n/x_n]$ for all constants a_1, \dots, a_n ; then $s \models \forall \alpha$.*

Theorem 6. *If $\vdash \alpha$ then α is L-true*

Proof. Soundness may be proved in the usual way, by proving that all axioms are valid and that MP preserves validity. Lemma 5 guarantees that the schematic axioms may be stated, without affecting validity, by leaving off the initial \forall , and replacing each distinct free variable by a distinct constant not otherwise occurring, and we shall consider them in this form. Obviously, if α is an instance of a tautology then, for every state-description s , $s \models \alpha$. For \forall^\supset , suppose $s \models \forall x(\alpha \supset \beta) \supset (\forall x\alpha \supset \forall x\beta)$. So $s \models \forall x(\alpha \supset \beta)$, $s \models \forall x\alpha$, but $s \models \forall x\beta$; and so for some a , $s \models \beta[a/x]$. But $s \models \alpha[a/x] \supset \beta[a/x]$, and $s \models \alpha[a/x]$, and so $s \models \beta[a/x]$. For VQ, suppose that $s \models \alpha \supset \forall x\alpha$. So $s \models \alpha$ and $s \models \forall x\alpha$. But x is not free in α , and so $s \models \alpha$. For $\forall 1a$ suppose that $s \models \forall y(\forall x\alpha \supset \alpha[y/x])$. Then for some a , $s \models \forall x\alpha \supset \alpha[a/x]$ and so $s \models \forall x\alpha$ and $s \models \alpha[a/x]$ which is a contradiction. This also covers $\forall 1b$. For I1 a and a are the same constant, and $s \models a = a$. For I2, if $s \models a = b$ then these are the same constant, and α and β are the same sentence, and so $s \models \alpha \supset \beta$. Finally if a and b are different constants then $s \models a \neq b$. MP preserves validity since if $s \models \alpha$ and $s \models \beta$ then $s \models \alpha \supset \beta$. \square

In proving completeness we have to shew that if $s \models \alpha$ for all state-descriptions then $\vdash_{\text{FC}} \alpha$. In fact we would like to get the stronger form of completeness by shewing that if a set Λ of sentences is FC-consistent then there is a state-description s such that $s \models \alpha$ for every $\alpha \in \Lambda$. At this point we have to bear in mind that the set of state-descriptions is relative to a language \mathcal{L} . For take the set $\Omega = \{\sim \forall x\varphi x, \varphi a_1, \dots, \varphi a_n, \dots\}$ where $\varphi a_i \in \Omega$ for every constant a_i in \mathcal{L} . This set is not true in any state-description for \mathcal{L} . We can, however, extend \mathcal{L} to a language \mathcal{L}^+ by adding a denumerable infinity of new constants not in \mathcal{L} , so that Ω is satisfiable in \mathcal{L}^+ , though there will be a new Ω^+ unsatisfiable in \mathcal{L}^+ . This means that where Λ is a consistent set of sentences of a language \mathcal{L} then Λ may not be satisfiable in \mathcal{L} , though there will be an extension \mathcal{L}^+ of \mathcal{L} such that Λ is satisfiable with respect to the state-descriptions, not of \mathcal{L} , but of \mathcal{L}^+ .⁸

⁸This is a known feature of substitutional quantification. See for instance Dunn and Belnap 1968. It

Call a set Λ of sentences of \mathcal{L} , *FC-consistent* (or simply *consistent*) iff there are no $\alpha_1, \dots, \alpha_n \in \Lambda$ such that $\vdash_{FC} \sim(\alpha_1 \wedge \dots \wedge \alpha_n)$. Suppose Λ is a consistent set of sentences of a language \mathcal{L} . A set Γ of sentences is *maximal* in \mathcal{L} iff for every sentence α of \mathcal{L} , either $\alpha \in \Gamma$ or $\sim\alpha \in \Gamma$. Since FC is finitary Λ will still be consistent in \mathcal{L}^+ , where \mathcal{L}^+ contains infinitely many constants not in \mathcal{L} , and therefore not in Λ . The presence of I3 requires a change in what is called on p. 257 of Hughes and Cresswell 1996 the \forall -property. In a language without constants a set Δ has \forall -property iff for every wff in \mathcal{L} of the form $\forall x\alpha$ there is some variable y such that $\alpha[y/x] \supset \forall x\alpha \in \Delta$, and the standard technique for giving a consistent set the \forall -property, as for instance described in theorem 14.1 on p. 258 of Hughes and Cresswell 1996, has as a consequence that y does not occur in α . However, I3 makes the sentence $a \neq b \supset \forall x x \neq b$ inconsistent where a is any constant distinct from b , and therefore inconsistent where a is not in $x \neq b$. For that reason we shall introduce the *Carnapian* \forall -property, or \forall_C -property:

Definition. \forall_C -prop. A set Δ of sentences of a language \mathcal{L} of FC has the \forall_C -property in \mathcal{L} iff for every sentence $\forall x\alpha$ in \mathcal{L} there are constants a and b_1, \dots, b_n in \mathcal{L} such that $\alpha[a/x] \supset \forall x((x \neq b_1 \wedge \dots \wedge x \neq b_n) \supset \alpha) \in \Delta$.

Lemma 7. Suppose Γ is a maximal-consistent set of sentences with the \forall_C -property, and suppose that $\forall x\alpha$ is a sentence such that $\alpha[a/x]$ is in Γ for every constant a of \mathcal{L} . Then $\forall x\alpha \in \Gamma$.

Proof. If Γ has the \forall_C -property then there is some a , such that for some b_1, \dots, b_n , $\alpha[a/x] \supset \forall x((x \neq b_1 \wedge \dots \wedge x \neq b_n) \supset \alpha) \in \Gamma$. So, if $\alpha[c/x] \in \Gamma$ for every constant c , then in particular, $\alpha[b_i/x] \in \Gamma$ for every $1 \leq i \leq n$. So, since Γ is maximal-consistent, $\alpha[b_1/x] \wedge \dots \wedge \alpha[b_n/x] \wedge \forall x((x \neq b_1 \wedge \dots \wedge x \neq b_n) \supset \alpha) \in \Gamma$, and so, by theorem 3, $\forall x\alpha \in \Gamma$. \square

For a Carnapian language we may define the (ordinary) \forall -property of a set Δ of sentences to hold when, for every sentence $\forall x\alpha$, there is some constant a such that $\alpha[a/x] \supset \forall x\alpha \in \Delta$. From lemma 7 we have the following:

Corollary 8. If Γ is a maximal-consistent set with the \forall_C -property then Γ has the \forall -property.

The case of $a \neq b \supset \forall x x \neq b$ shews that in some cases the \forall -property requires a constant already occurring in α . In this case we require $b \neq b \supset \forall x x \neq b$.

Theorem 9. Where Λ is an FC-consistent set of sentences of a language \mathcal{L} then there is a consistent set Δ with the \forall_C -property in a language \mathcal{L}^+ which contains infinitely many constants not in \mathcal{L} , such that $\Lambda \subseteq \Delta$

entails a failure of compactness in the sense that, although every finite subset of a set Λ of sentence of a language \mathcal{L} may be satisfied by a state description for \mathcal{L} , Λ itself may not, though it will be satisfiable in a language with infinitely constants not in \mathcal{L} . The difference between the present completeness proof and more standard ones is that in the present case, though not in the standard case, the extension of the language is effectively accompanied by an extension of the domain.

Proof. Theorem 9 is proved by specifying a sequence $\Delta_0, \Delta_1, \dots, \Delta_n, \dots$ defined as follows. Assume that there is an enumeration of wff of \mathcal{L} each of which contains just one free variable, so that α_n contains free x_n , and so on. Δ_0 is Λ , and, for $1 \leq n$, Δ_n is $\Delta_{n-1} \cup \{\alpha_n[a/x_n] \supset \forall x_n((x_n \neq b_1 \wedge \dots \wedge x_n \neq b_k) \supset \alpha)\}$, where a is the first constant not already occurring in α or in any member of Δ_{n-1} and b_1, \dots, b_k are constants such that $\Delta_{n-1} \cup \{\alpha_n[a/x_n] \supset \forall x_n((x_n \neq b_1 \wedge \dots \wedge x_n \neq b_k) \supset \alpha)\}$ is consistent. For this to succeed it needs to be shewn that such a Δ_n exists. If not, then

$$(i) \text{ For every } b_1, \dots, b_k, \text{ there are } \beta_1, \dots, \beta_m \in \Delta_{n-1} \text{ such that} \\ \vdash_{\text{FC}} (\beta_1 \wedge \dots \wedge \beta_m) \supset \sim(\alpha[a/x_n] \supset \forall x_n((x_n \neq b_1 \wedge \dots \wedge x_n \neq b_k) \supset \alpha)).$$

From (i) we have that there are $\gamma_1, \dots, \gamma_i \in \Delta_{n-1}$ such that

$$(ii) \vdash_{\text{FC}} (\gamma_1 \wedge \dots \wedge \gamma_i) \supset \alpha[a/x_n]$$

and so, by theorem 2, since a does not occur in Δ_{n-1} , there are constants c_1, \dots, c_h such that

$$(iii) \vdash_{\text{FC}} (\gamma_1 \wedge \dots \wedge \gamma_i) \supset \forall x_n((x_n \neq c_1 \wedge \dots \wedge x_n \neq c_h) \supset \alpha).$$

Now (i) holds for every b_1, \dots, b_k and so, in particular for c_1, \dots, c_h , and so there are $\delta_1, \dots, \delta_j \in \Delta_{n-1}$ such that

$$(iv) \vdash_{\text{FC}} (\delta_1 \wedge \dots \wedge \delta_j) \supset \sim \forall x_n((x_n \neq c_1 \wedge \dots \wedge x_n \neq c_h) \supset \alpha).$$

and so

$$(v) \vdash_{\text{FC}} (\gamma_1 \wedge \dots \wedge \gamma_i \wedge \delta_1 \wedge \dots \wedge \delta_j) \supset \sim \forall x_n((x_n \neq c_1 \wedge \dots \wedge x_n \neq c_h) \supset \alpha).$$

But from (iii) we have

$$(vi) \vdash_{\text{FC}} (\gamma_1 \wedge \dots \wedge \gamma_i \wedge \delta_1 \wedge \dots \wedge \delta_j) \supset \forall x_n((x_n \neq c_1 \wedge \dots \wedge x_n \neq c_h) \supset \alpha).$$

which with (v) yields

$$(vii) \vdash_{\text{FC}} \sim(\gamma_1 \wedge \dots \wedge \gamma_i \wedge \delta_1 \wedge \dots \wedge \delta_j)$$

in which case Δ_{n-1} would be inconsistent. Let Δ be the union of all the Δ_n s. Δ is consistent for the usual reasons. Further Δ has the \forall_C -property in \mathcal{L}^+ . \square

Finally let Γ be a maximal-consistent set containing Δ . By corollary 8, theorem 9 establishes that Γ has the \forall -property.

Now let s_Γ , be the state-description which consists of $\varphi a_1 \dots a_n \in s_\Gamma$ if $\varphi a_1 \dots a_n \in \Gamma$, and $\sim \varphi a_1 \dots a_n \in s_\Gamma$ if $\sim \varphi a_1 \dots a_n \in \Gamma$.

Theorem 10. $s_\Gamma \models \alpha$ iff $\alpha \in \Gamma$.

Proof. The result is defined for $\varphi a_1 \dots a_n$, and is obvious for \perp . If $s_\Gamma \models a = b$, then these are the same constant, and from I1 we then have $a = b \in \Gamma$. If $s_\Gamma \models a \neq b$, then these are distinct constants, and from I3 we have $a \neq b \in \Gamma$. The induction for \perp and \supset is standard. Suppose $s_\Gamma \models \forall x\alpha$. Then there is a constant a such that $s_\Gamma \models \alpha[a/x]$. So, by the induction hypothesis $\alpha[a/x] \notin \Gamma$, and therefore, by $\forall 1b$, $\forall x\alpha \notin \Gamma$. Suppose $\forall x\alpha \notin \Gamma$. Since Γ has the \forall -property there is some a such that $\alpha[a/x] \supset \forall x\alpha$. But then $\alpha[a/x] \notin \Gamma$, and so, by the induction hypothesis, $s_\Gamma \models \alpha[a/x]$ and so $s_\Gamma \models \forall x\alpha$. \square

Completeness follows in the usual way.

The main interest of Carnap 1946 is not, however, ordinary non-modal predicate logic. The purpose of Carnap's article is to study modal logic. If \mathcal{L} is a language of ordinary first-order logic we may extend \mathcal{L} to obtain a language \mathcal{L}_M . \mathcal{L}_M is simply the modal predicate language which extends \mathcal{L} by the addition of the (monadic) necessity operator \Box , and the formation rule that if α is a wff then so is $\Box \alpha$. The idea behind Carnap's account of modality is that \Box (which he writes as N^9) is to be interpreted in such a way that $\Box \alpha$ is to be true iff α is *valid* — or, as Carnap puts it on p. 54, iff α is L-true. We shall make this precise as follows. Where s is a state-description $[V\varphi]$ - $[V\forall]$ shew how to extend \models to all LPC wff. To extend \models to all wff of \mathcal{L}_M in accordance with D9-5a (p. 54) we add the condition

$$[V\Box_{\text{MFL}}] s \models \Box \alpha \text{ iff } s' \models \alpha \text{ for every state description } s'.$$

Say that a wff is *C-valid* iff it is true in every state-description, when \models satisfies $[V\varphi]$ - $[V\forall]$ and $[V\Box_{\text{MFL}}]$. One result is that if α is S5-valid then α is C-valid. However (quantified) S5 is not complete for C-validity. This is because, where φa is an atomic wff, $\sim \Box \varphi a$ is C-valid even though it is not a theorem of S5 — and similarly with any atomic wff. Carnap specifies an axiomatic system of modal predicate logic, which he calls MFC. His first set of axioms (p. 54) is simply a basis sufficient for S5:

LPC \Box Where α is one of the LPC axioms PC-I3 then both α and $\Box \alpha$ are axioms of MFC.¹⁰

$$\mathbf{K} \quad \Box \forall (\Box (\alpha \supset \beta) \supset (\Box \alpha \supset \Box \beta))$$

$$\mathbf{T} \quad \forall (\Box \alpha \supset \alpha)$$

$$\mathbf{5} \quad \Box \forall (\Box \alpha \vee \Box \sim \Box \alpha)$$

⁹The symbol \Box for necessity was devised by F.B. Fitch in 1945, and first appeared in print in Barcan 1946. I only met Carnap once, in 1970, shortly before his death. I asked him whether he or anyone else he knew of had ever used L for a necessity operator before Feys 1950, but he could not recall any information about that. (I find it convenient to use \Box for *logical* necessity, which satisfies S5, and L for the kind of necessity formalised by weaker modal systems.)

¹⁰Carnap separates this into a schema (D10-1a) which guarantees that PC is part of the basis, and then in D10-1, e to k, repeats our $\forall \supset$ -I3.

BF_C $\Box\forall(\Box\forall x\alpha \supset \forall x\Box\alpha)$

BF $\Box\forall(\forall x\Box\alpha \supset \Box\forall x\alpha)$ ¹¹

(A straightforward induction on proofs establishes the rule of necessitation, that if α is a theorem, so is $\Box\alpha$.¹²) Carnap then includes some non-standard axioms, which shew that he is attempting to axiomatise C-validity. These are what Carnap calls ‘Assimilation’, ‘Variation and Generalization’ and ‘Substitution for Predicates’. (Carnap 1946, p. 54f.) In our notation these can be expressed as follows:

Ass $\Box\forall x\forall y\forall z_1\dots\forall z_n((x \neq z_1 \wedge \dots \wedge x \neq z_n) \supset (\Box\alpha \supset \Box\alpha[y/x]))$, where α contains no free variables other than x, y, z_1, \dots, z_n , and no constants and no occurrences of =.

VG $\Box\forall x\forall y\forall z_1\dots\forall z_n((x \neq z_1 \wedge \dots \wedge x \neq z_n \wedge y \neq z_1 \wedge \dots \wedge y \neq z_n) \supset (\Box\alpha \supset \Box\alpha[y/x]))$, where α contains no free variables other than x, y, z_1, \dots, z_n , and no constants.

SP $\Box\forall(\Box\alpha \supset \Box\beta)$, where α is a non-modal wff, and β is obtained from α by uniform substitution of a complex expression for a predicate.

None of these axiom schemata is easy to process, but it is not difficult to see what the simplest instances would look like. A very simple instance, which is of both Ass and VG is

$$\mathbf{Ass}_\varphi \quad \Box\forall x\forall y\forall z(x \neq z \supset (\Box\varphi xyz \supset \Box\varphi yyz))$$

To establish the validity of \mathbf{Ass}_φ it is sufficient to shew that if a and c are distinct constants then $\Box\varphi abc \supset \Box\varphi bbc$ is valid. This is trivially so, since there is some s such that $s \models \varphi abc$, and therefore for every s , $s \models \Box\varphi abc$, and so, for every s , $s \models \Box\varphi abc \supset \varphi bbc$. More telling is the case of SP. Let φ be a one-place predicate and consider

$$\mathbf{SP}_\varphi \quad \Box\forall x(\Box\varphi x \supset \Box(\varphi x \wedge \sim\varphi x))$$

In this case α is φx , while β is $\varphi x \wedge \sim\varphi x$, so that, in Carnap’s words, β ‘is formed from α by replacing every atomic matrix containing φ by the current substitution form of β ’.

¹¹This principle is Carnap’s T9-2 on p. 54 of Carnap 1946. The current name, ‘Barcan Formula’, comes from an equivalent formula first introduced in Barcan 1946, p. 2, Axiom 11. BF appears to have been first called the ‘Barcan Formula’ in Prior 1956, p. 60, where it is shewn derivable in S5. Even though Carnap includes \mathbf{BF}_C as an axiom, it is in fact derivable from the rest of the basis using only principles of non-modal logic together with \mathbf{LPC}_\Box and K. By $\forall 1b$ we have $\Box(\forall x\alpha \supset \alpha[a/x])$ and so, by K, $\Box\forall x\alpha \supset \Box\alpha[a/x]$ for every constant a . So choose some a not in $\Box\forall x\alpha$. Then, by theorem 2, we have that there are b_1, \dots, b_n such that $\vdash \Box\forall x\alpha \supset \forall x(x \neq b_1 \wedge \dots \wedge x \neq b_n) \supset \Box\alpha$. But we also have $\Box\forall x\alpha \supset \Box\alpha[b_i/x]$ for $1 \leq i \leq n$, and so, by theorem 3, we have $\Box\forall x\alpha \supset \forall x\Box\alpha$.

¹²The only non-trivial case is the axiom T, where we require $\Box\forall(\Box\alpha \supset \alpha)$, which is Carnap’s theorem T4-1u on p. 42.

Thus we have $\vdash \sim \Box(\varphi a \wedge \sim \varphi a)$, and so $\vdash \sim \Box \varphi a$.¹³

A completeness result for C-validity is not possible. For, although Carnap appears to have it in mind to axiomatise C-validity, it is easy to see that it is not recursively axiomatisable. For, by $[V\Box_{\text{MFL}}]$, where α is any LPC wff, α is not LPC-valid iff $\sim \Box \alpha$ is C-valid, and so, if C-validity were axiomatisable then LPC would be decidable.¹⁴ There is a hint on p. 57 that Carnap may have recognised this. He is certainly aware that the kind of reduction to normal form, with which he achieves the completeness of propositional S5, is unavailable in the predicate case, since it would lead to the decidability of LPC.

It is however worth looking at a modification of Carnap's semantics which brings it into line with the current indexical semantics for modal logic.

Let Σ be a class of state-descriptions, where each $s \in \Sigma$ satisfies $[V\varphi]$ - $[VV]$, and $[V\Box_{\text{MFL}}]$ is replaced by

$$[V\Box_{\text{MFL}\Sigma}] s \models \Box \alpha \text{ iff } s' \models \alpha \text{ for every } s' \in \Sigma$$

Say that a sentence α is valid in Σ iff $s \models \alpha$ for every $s \in \Sigma$, and say that α is Σ^+ -valid iff α is valid in every Σ . With this change it is easy to see that $\sim \Box \varphi a$ is not Σ^+ -valid, since one need only consider, for instance, a class Σ whose only member is a state-description s , where s contains φa . In fact one can prove that a Carnapian version of the standard axiomatic basis for predicate S5 is complete for this semantics. To avoid confusion with Carnap's MFC, which *does* contain the non-standard axioms, I shall refer to the system for which completeness is to be proved as FC+S5. That is to say, completeness can be proved with respect to the semantics based on an arbitrary set of state-descriptions, for the basis which consists of LPC_{\Box} -BF, but *not* containing Ass, VG or SP.

It is easy to establish that all the axioms of FC + S5 are Σ^+ -valid, and that MP preserves Σ^+ -validity. For completeness we need some standard results from modal logic. Where Λ is a set of sentences let $L^-(\Lambda)$ denote the set consisting precisely of every sentence β for which $\Box \beta$ is in Λ . I.e., $L^-(\Lambda) = \{\beta : \Box \beta \in \Lambda\}$.

Lemma 11. *Let Λ be a set of sentences containing $\sim \Box \alpha$. Then $L^-(\Lambda) \cup \{\sim \alpha\}$ is FC+S5-consistent.*¹⁵

The following theorem should be compared with theorem 14.2 on p. 259 of Hughes and Cresswell 1996.

¹³I.e., where β is $\varphi x \wedge \sim \varphi x$, it replaces α 's φx . If α had been more complex and contained φy as well as φx , then the replacement would have given $\varphi y \wedge \sim \varphi y$, and so on, where care needs to be taken to prevent any free variable being bound as a result of the replacement. Substitution of complex expressions for predicates is a complex business. For a hint of some of the complexities of substitution rules for predicates see Church 1956 p. 289f. Carnap spells out the rule for substitution for predicates in D10-1p on p. 55.

¹⁴Although that is so it has not always been regarded as a criticism. Cocchiarella 1975 for one welcomes the various unusual features of Carnap's logic. (See for instance pp. 45 and 54f.)

¹⁵See lemma 6.4 on p. 117 of Hughes and Cresswell, 1996.

Theorem 12. *If Γ is a maximal-consistent set of sentences in some language (say \mathcal{L}^+) of modal predicate logic, and Γ has the \forall -property in \mathcal{L}^+ , and α is a sentence such that $\Box\alpha \notin \Gamma$, then there is a consistent set Δ of sentences of \mathcal{L}^+ with the \forall -property such that $L^-(\Gamma) \cup \{\sim\alpha\} \subseteq \Delta$.*

Proof. We define a sequence of sentences $\gamma_0, \gamma_1, \gamma_2, \dots$ etc. γ_0 is $\sim\alpha$. Given γ_n we define γ_{n+1} as follows. Let $\forall x\delta$ be the $n+1$ th sentence of that form and let a be the first constant such that

$$(i) L^-(\Gamma) \cup \{\gamma_n \wedge (\delta[a/x] \supset \forall x\delta)\} \text{ is consistent.}$$

Let γ_{n+1} be $\gamma_n \wedge (\delta[a/x] \supset \forall x\delta)$. Since γ_0 is $\sim\alpha$, $L^-(\Gamma) \cup \{\gamma_0\}$ is consistent from lemma 11. We shew that provided $L^-(\Gamma) \cup \{\gamma_n\}$ is consistent there will always be an a which satisfies (i). Suppose there were not. Then, for every constant a , there will exist some $\{\beta_1, \dots, \beta_k\} \subseteq L^-(\Gamma)$ such that

$$(ii) \vdash (\beta_1 \wedge \dots \wedge \beta_k) \supset (\gamma_n \supset \sim(\delta[a/x] \supset \forall x\delta))$$

so, by principles of S5¹⁶

$$(iii) \vdash (\Box\beta_1 \wedge \dots \wedge \Box\beta_k) \supset \Box(\gamma_n \supset \sim(\delta[a/x] \supset \forall x\delta))$$

But Γ is maximal consistent and $\Box\beta_1, \dots, \Box\beta_k \in \Gamma$, and so

$$(iv) \Box(\gamma_n \supset \sim(\delta[a/x] \supset \forall x\delta)) \in \Gamma.$$

Now (iv) holds for *every* constant a , and Γ has the \forall -property. What this means is this. Let z be some variable not occurring in δ or in γ_n , and consider $\forall z\Box(\gamma_n \supset \sim(\delta[z/x] \supset \forall x\delta))$. Since Γ has the \forall -property there will be a constant a such that

$$(v) \Box(\gamma_n \supset (\sim(\delta[a/x] \supset \forall x\delta))) \supset \forall z\Box(\gamma_n \supset \sim(\delta[z/x] \supset \forall x\delta)) \in \Gamma$$

and so, by (iv)

$$(vi) \forall z\Box(\gamma_n \supset \sim(\delta[z/x] \supset \forall x\delta)) \in \Gamma$$

and so, by BF,

$$(vii) \Box\forall z(\gamma_n \supset \sim(\delta[z/x] \supset \forall x\delta)) \in \Gamma$$

Since z does not occur in γ_n or δ then by \forall^\supset and VQ we have in Γ

$$(viii) \Box(\gamma_n \supset \forall z \sim(\delta[z/x] \supset \forall x\delta))$$

But by lemma 4

¹⁶See p. 117 of Hughes and Cresswell 1996.

$$(ix) \vdash \exists z(\delta[z/x] \supset \forall x\delta)$$

But then $\Box \sim \gamma_n \in \Gamma$ and so $\sim \gamma_n \in L^-(\Gamma)$ which would make $L^-(\Gamma) \cup \{\gamma_n\}$ inconsistent. \square

Now assume that Λ is an FC+S5-consistent set of sentences, and that Γ is a maximal-consistent set of sentences with the \forall_C -property constructed in accordance with the procedure of theorem 9. By corollary 8, Γ has the \forall -property. Let Γ^+ be the set of all maximal consistent sets w with the \forall -property, such that $L^-(\Gamma) \subseteq w$. The presence of **T** ensures that $L^-(\Gamma) \subseteq \Gamma$, and therefore that $\Gamma \in \Gamma^+$. Let Σ be the set of all state descriptions s_w where, for $w \in \Gamma^+$, s_w is the state-description which consists of $\varphi a_1 \dots a_n \in$ if $\varphi a_1 \dots a_n \in w$, and $\sim \varphi a_1 \dots a_n \in s_w$ if $\varphi a_1 \dots a_n \notin w$. We then have the following analogue of theorem 10:

Theorem 13. *For any $w \in \Gamma^+$, $s_w \models \alpha$ iff $\alpha \in w$*

Proof. The inductive proof is as for theorem 10, except for the induction for \Box . Suppose that $\Box\beta \in w$. Then $\Box\beta \in \Gamma$, since if not, by **5**, $\Box \sim \Box\beta \in \Gamma$, and then $\sim \Box\beta \in w$. But if $\Box\beta \in \Gamma$ then $\beta \in w'$ for every $w' \in \Gamma^+$, and so, by the induction hypothesis, $s_{w'} \models \beta$ for every $s_{w'} \in \Sigma$, and so $s_w \models \Box\beta$. If $\Box\beta \notin w$ then $\sim \Box\beta \in \Gamma$, since if $\Box\beta \in \Gamma$ then $\Box\Box\beta \in \Gamma$ and then $\Box\beta \in w$. But if $\sim \Box\beta \in \Gamma$ then, by theorem 12, there is some $w' \in \Gamma^+$ such that $\beta \notin w'$. So, by the induction hypothesis, $s_{w'} \not\models \beta$, and so $s_w \not\models \Box\beta$. \square

From theorem 13 it follows that the members of Λ are true in s_Γ , and therefore, since $\Gamma \in \Gamma^+$, true in some $s \in \Sigma$, thus establishing the completeness of FC+S5 with respect to Σ^+ -validity.

Finally we shall revisit Ass-SP. As noted, neither they nor any other recursively specified axiom set is complete for C-validity. However, a rather trivial completeness proof is possible based on Σ^+ -validity. MFC is FC + S5, with the addition of all instances of Ass-SP. Let a sentence α be MFC-valid iff α is valid in every class Σ of state-descriptions on which every instance of Ass-SP is valid. It is trivial that MP preserves MFC-validity.¹⁷ So if $\vdash_{\text{MFC}} \alpha$ then α is MFC-valid. If $\neg_{\text{MFC}} \alpha$ then, where Δ consists of all instances of Ass-SP, $\Delta \cup \{\sim \alpha\}$ is FC+S5-consistent. So, according to theorem 13, there is a class Σ of state descriptions such that, for some $s \in \Sigma$, every instance of Ass-SP is true at s but α is not; which is just to say that α is not MFC-valid.

Appendix: Proofs of theorems 2 and 3, and lemma 4:

The proofs of the following three results have no theoretical interest, except to establish that they do indeed follow from Carnap's axioms, as presented in this paper. Where the

¹⁷Further, assuming the consistency of the part of set theory needed to define the class of all state-descriptions for a denumerable language, all of Ass-SP will be valid on that class, and so MFC will be consistent.

justification for a line of the proof is given by merely citing earlier lines of the proof the line is obtained from an instance of **PC** or $\forall \supset$ and applications of MP.

Theorem 2:

Proof. The theorem may be proved as follows. Assume

$$(1) \alpha \supset \beta$$

Then, by theorem 1, there are b_1, \dots, b_n such that

$$(2) \forall x((x \neq b_1 \wedge \dots \wedge x \neq b_n) \supset (\alpha \supset \beta)[x/a])$$

But a does not occur in α and so

$$(3) \forall x((x \neq b_1 \wedge \dots \wedge x \neq b_n) \supset (\alpha \supset \beta[x/a]))$$

So, by PC and $\forall \supset$,

$$(4) \forall x \alpha \supset \forall x((x \neq b_1 \wedge \dots \wedge x \neq b_n) \supset \beta[x/a])$$

But α is a sentence since $\alpha \supset \beta$ is a theorem, and so, by VQ

$$(5) \alpha \supset \forall x \alpha$$

$$(4)(5) \text{ PC} \quad (6) \alpha \supset \forall x((x \neq b_1 \wedge \dots \wedge x \neq b_n) \supset \beta[x/a])$$

□

Theorem 3:

Proof.

- | | | |
|------------------------|------|---|
| I1, I2 | (1) | $\forall x(x = b_i \supset b_i = x)(1 \leq i \leq n)$ |
| I2, $\forall 1b$ | (2) | $\forall x(b_i = x \supset (\alpha[b_i/x] \supset \alpha))$ |
| (1)(2) | (3) | $\forall x(x = b_i \supset (\alpha[b_i/x] \supset \alpha))$ |
| (3) | (4) | $\forall x(\alpha[b_i/x] \supset (x = b_i \supset \alpha))$ |
| (4) | (5) | $\forall x(\alpha[b_i/x]) \supset \forall x(x = b_i \supset \alpha)$ |
| (5) | (6) | $\forall x((\alpha[b_1/x] \wedge \dots \wedge \alpha[b_n/x]) \supset \forall x((x = b_1 \vee \dots \vee x = b_n) \supset \alpha))$ |
| VQ | (7) | $(\alpha[b_1/x] \wedge \dots \wedge \alpha[b_n/x]) \supset \forall x(\alpha[b_1/x] \wedge \dots \wedge \alpha[b_n/x])$ |
| (6)(7) | (8) | $(\alpha[b_1/x] \wedge \dots \wedge \alpha[b_n/x]) \supset \forall x((x = b_1 \vee \dots \vee x = b_n) \supset \alpha)$ |
| PC | (9) | $\forall x(((x = b_1 \vee \dots \vee x = b_n) \supset \alpha) \supset (((x \neq b_1 \wedge \dots \wedge x \neq b_n) \supset \alpha) \supset \alpha))$ |
| (9) $\forall \supset$ | (10) | $\forall x((x = b_1 \vee \dots \vee x = b_n) \supset \alpha) \supset \forall x(((x \neq b_1 \wedge \dots \wedge x \neq b_n) \supset \alpha) \supset \alpha)$ |
| (10) $\forall \supset$ | (11) | $\forall x((x = b_1 \vee \dots \vee x = b_n) \supset \alpha) \supset (\forall x((x \neq b_1 \wedge \dots \wedge x \neq b_n) \supset \alpha) \supset \forall x\alpha)$ |
| (8)(11) | (12) | $(\alpha[b_1/x] \wedge \dots \wedge \alpha[b_n/x] \wedge \forall x((x \neq b_1 \wedge \dots \wedge x \neq b_n) \supset \alpha)) \supset \forall x\alpha$ |

□

Lemma 4:

- | | | |
|-------------------------------|-----|--|
| PC | (1) | $\forall x\alpha \supset (\alpha[a/x] \supset \forall x\alpha)$ |
| $\forall 1b$ PC Def \exists | (2) | $(\alpha[a/x] \supset \forall x\alpha) \supset \exists y(\alpha[y/x] \supset \forall x\alpha)$ |
| Quine 1940, p. 112, *170 | (3) | $\forall y\alpha[y/x] \supset \forall x\alpha$ |
| (1)(2)(3) PC | (4) | $\forall y\alpha[y/x] \supset \exists y(\alpha[y/x] \supset \forall x\alpha)$ |
| PC | (5) | $\forall y(\sim \alpha[y/x] \supset \forall x\alpha) \supset \alpha[y/x]$ |
| (5) $\forall \supset$ | (6) | $\forall y\sim(\alpha[y/x] \supset \forall x\alpha) \supset \forall y\alpha[y/x]$ |
| (6) PC Def \exists | (7) | $\sim \forall y\alpha[y/x] \supset \exists y(\alpha[y/x] \supset \forall x\alpha)$ |
| (4)(7) PC | (8) | $\exists y(\alpha[y/x] \supset \forall x\alpha)$ |

□

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