Fraenkel–Carnap Questions for Equivalence Relations

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Abstract: An equivalence is a binary relational system $\mathfrak{A} = (A, \rho_{\mathfrak{A}})$ where $\rho_{\mathfrak{A}}$ is an equivalence relation on A. A simple expansion of an equivalence is a system of the form $(\mathfrak{A}a_1 \cdots a_n)$ were $\mathfrak{A}$ is an equivalence and $a_1, \ldots, a_n$ are members of A. It is shown that the Fraenkel–Carnap question when restricted to the class of equivalences or to the class of simple expansions of equivalences has a positive answer: that the complete second-order theory of such a system is categorical, if it is finitely axiomatizable.

1 Introduction

In the late 1920's Fraenkel and Carnap independently raised the question of whether or not every semantically complete, finitely axiomatizable theory is categorical. Carnap, but not Fraenkel, restricted attention to theories formulated in the simple theory of types. A positive answer to Carnap's question implies that a finitely axiomatizable theory is semantically complete iff it is categorical. Carnap announced a positive answer. However, his proof was flawed. It appears that the question was forgotten until the beginning of this century. A related result was established by the late 1960's: that for $n \geqslant 2$, a semantically complete, finitely axiomatizable and satisfiable theory formulated in an n^{th}-order language has models of a single cardinality and that this cardinal is *characterizable* in the language in the sense that there is a sentence in the language true on all and only interpretations of the language of that cardinality [8, 9]. It follows that if the non-logical vocabulary is empty, then every finitely axiomatizable, semantically complete theory is categorical. A proof of this consequence, due to Dana Scott, is outlined in [4].

The Fraenkel–Carnap question was rediscovered by Awodey, Carus and Reck [2, 3, 4]. Later papers have considered restricting the Fraenkel–Carnap question to theories formulated in second-order languages [13, 14]. Even in this case, the Fraenkel–Carnap question remains open. Some partial positive answers have been obtained. The general approach has been to restrict attention to subclasses of the class of interpretations of the language and to show that the complete theory of a member of the class is categorical, if that theory is finitely axiomatizable [3, 13, 14]. Some of these answers are consequences of results in first-order model theory. For example, it follows from Theorem 22 [7, page 712] that the second-order theory of any finite interpretation is categorical, whether or not that theory is finitely axiomatizable. When the non-logical vocabulary is finite it can be shown, in addition, that the first-order theory of a finite interpretation is both finitely axiomatizable and categorical [5, page 106]. The purpose of this paper is to add to the list of classes for which the Fraenkel–Carnap question has a positive answer. Consider the class of binary relational systems (A, ρ) were ρ is an equivalence relation on A. Members of this class are called *equivalences*. There are infinite equivalences whose second-order theories are finitely axiomatizable. For example, let A be countably infinite and ρ be the identity relation of A. It is shown that the Fraenkel–Carnap question for the class of infinite equivalences has a positive answer. In broad outline, reasoning is analogous to that for Dedekind algebras [13]: associate a cardinal valued function with each member of the class (in the case of equivalences, the partition function for that equivalence), and show that the functions encode the structure of their associated members in the sense that two members are isomorphic iff their associated functions are identical. Reasoning proceeds by showing that if ϕ is a second-order sentence all of whose models are equivalences and there are models of ϕ of cardinality β that are not isomorphic, then there is a second-order sentence true on some models of ϕ of cardinality β, but false on others. Thus, the theory generated by ϕ is not semantically complete. The fact that the sentence ϕ has models of cardinality β whose partition functions are different allows for the selection of a model of ϕ of cardinality β that is minimal among the models of ϕ of that cardinality in a way that can be expressed by a second-order sentence. The sentence expressing this minimality distinguishes between models of ϕ.

Given L, a second-order language with a finite non-logical vocabulary, and Δ, a class of interpretations for L, Δ *satisfies the Fraenkel–Carnap property* provided the complete second-order theory of any member of Δ is categorical, if it is finitely axiomatizable. The Fraenkel–Carnap property can be reformulated in a way that suggests natural generalizations. One of these is the *quasi Fraenkel–Carnap property*. Let $\mathfrak{A}$ be an interpretation for L. A *pure sentence in* L is a sentence whose non-logical vocabulary is empty. The *pure theory of* $\mathfrak{A}$ *in* L is the set of pure sentences in L that are true on $\mathfrak{A}$. $\mathrm{Th}(\mathfrak{A})$ is the theory of $\mathfrak{A}$ in L. A set of sentences, T, is a *basis for* $\mathrm{Th}(\mathfrak{A})$ provided the logical consequences of T are exactly the members of $\mathrm{Th}(\mathfrak{A})$. $\mathrm{Th}(\mathfrak{A})$ is *finitely axiomatizable* provided some finite subset of $\mathrm{Th}(\mathfrak{A})$ is a basis for $\mathrm{Th}(\mathfrak{A})$. $\mathrm{Th}(\mathfrak{A})$ is *quasi-finitely axiomatizable* provided some finite subset of $\mathrm{Th}(\mathfrak{A})$ together with the pure theory of $\mathfrak{A}$ is a basis for $\mathrm{Th}(\mathfrak{A})$. $\mathfrak{A}$ *is finitely characterizable* provided there is T, a finite subset of $\mathrm{Th}(\mathfrak{A})$, such that all models of T are isomorphic; and $\mathfrak{A}$ *is quasi-finitely characterizable* provided there is T, a finite subset of $\mathrm{Th}(\mathfrak{A})$, such that all models of T of the same cardinality are isomorphic. Δ *satisfies the Fraenkel–Carnap property* provided every member of Δ whose theory is finitely axiomatizable is finitely characterizable.

If $\mathfrak{A}$ is quasi-finitely characterizable, then $\mathrm{Th}(\mathfrak{A})$ is quasi-finitely axiomatizable [14, Lemma 4.1, page 287]. Δ *satisfies the quasi Fraenkel–Carnap property* provided for all $\mathfrak{A} \in \Delta$, if $\mathrm{Th}(\mathfrak{A})$ is quasi-finitely axiomatizable, then $\mathfrak{A}$ is quasi-finitely characterizable. Assume that L is *homogeneous* in the sense that for each member of the non-logical vocabulary of L there is a variable in the vocabulary of L of the same grammatical category. Let $\mathfrak{A}$ be an interpretation of cardinality β. If $\mathrm{Th}(\mathfrak{A})$ is finitely axiomatizable, then all models of $\mathrm{Th}(\mathfrak{A})$ are of cardinality β and β is *characterizable in* L in the sense that there is a pure sentence in L true on all and only interpretations of cardinality β [14, Corollary 2.2, page 286]. If $\mathfrak{A}$ is quasi-finitely characterizable and β is characterizable, then $\mathfrak{A}$ is finitely characterizable. Hence, any class that satisfies the quasi Fraenkel–Carnap property also satisfies the Fraenkel–Carnap property.

L_Q is a second-order language whose non-logical vocabulary consists of a single binary relational constant, Q. Interpretations for L_Q are *binary relational systems*, that is, ordered pairs $\mathfrak{A} = (\mathsf{A}, \rho_\mathfrak{A})$ where A is a non-empty set (the *domain of* $\mathfrak{A}$) and $\rho_\mathfrak{A}$ is a binary relation on A. Sentences in L_Q are interpreted in the binary relational systems in the "standard" way. An *equivalence* is a binary relational system in which $\rho_\mathfrak{A}$ is an equivalence relation on A. It is shown below that the class of infinite equivalences satisfies the Fraenkel–Carnap property.

2 THE CLASS OF INFINITE EQUIVALENCES

$\mathbb{E}$ is the class of infinite equivalences. $\mathbb{E}$ is *a finitary class in* L_Q in the sense that there is a finite set of sentences in L_Q whose models are exactly the members of $\mathbb{E}$. For $\mathfrak{A}$ and $\mathfrak{B}$, interpretations for L_Q, $\mathfrak{A}$ and $\mathfrak{B}$ are *equivalent in* L_Q iff $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B})$. The following is direct from Theorem 5.1 of [14, page 288] and the fact that L_Q is homogeneous.

LEMMA 1 *The following are equivalent*:

1. $\mathbb{E}$ *satisfies the quasi Fraenkel–Carnap property*; *and*

2. *for all infinite cardinals* β *and all sentences* ϕ *in* L_Q*, if all models of* ϕ *are in* $\mathbb{E}$ *and all models of* ϕ *of cardinality* β *are equivalent in* L_Q*, then all models of* ϕ *of cardinality* β *are isomorphic.*

The function f is *an isomorphism from* $\mathfrak{A}$ *to* $\mathfrak{B}$ iff f is a bijection from A to B and for all $a, d \in A$, $(a, d) \in \rho_{\mathfrak{A}}$ iff $(f(a), f(d)) \in \rho_{\mathfrak{B}}$. $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic* iff there is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. An *automorphism on* $\mathfrak{A}$ is an isomorphism from $\mathfrak{A}$ to itself. If C is a non-empty subset of A, $\mathfrak{A}[C]$ is the *subsystem of* $\mathfrak{A}$ *generated by* C. C is the *domain* of $\mathfrak{A}[C]$ and $\rho_{\mathfrak{A}[C]} = \rho_{\mathfrak{A}} \cap C^2$.

Assume that $\mathfrak{A}$ is an equivalence of cardinality β. $A/\rho_{\mathfrak{A}}$ is the partition on A induced by $\rho_{\mathfrak{A}}$. Members of this partition are the equivalence classes of the members of A relative to $\rho_{\mathfrak{A}}([a]_{\mathfrak{A}})$. These classes are the *cells* of the partition. $\mathbb{P}(\mathfrak{A})$, the *partition function for* $\mathfrak{A}$, is a cardinal valued function defined on the cardinals less than or equal to β. The value of this function at the cardinal κ is the number of cells in $A/\rho_{\mathfrak{A}}$ of cardinality κ. In essence, $\mathbb{P}(\mathfrak{A})$ encodes the structure of $\mathfrak{A}$.

LEMMA 2 *Assume that* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are members of* $\mathbb{E}$*. Then, the following are equivalent*:

1. $\mathfrak{A}$ *and* $\mathfrak{B}$ *are isomorphic*; *and*

2. $\mathbb{P}(\mathfrak{A}) = \mathbb{P}(\mathfrak{B})$.

Proof: Suppose that f is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ and that $a \in A$. $[a]_{\mathfrak{A}}$ has the same cardinality as $[f(a)]_{\mathfrak{B}}$. Thus for all cardinals κ less than or equal to the cardinality of $\mathfrak{A}$, $\mathbb{P}(\mathfrak{A})(\kappa) \leqslant \mathbb{P}(\mathfrak{B})(\kappa)$. Since the inverse of f is an isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$, $\mathbb{P}(\mathfrak{B})(\kappa) \leqslant \mathbb{P}(\mathfrak{A})(\kappa)$, and $\mathbb{P}(\mathfrak{A}) = \mathbb{P}(\mathfrak{B})$.

Suppose that $\mathbb{P}(\mathfrak{A}) = \mathbb{P}(\mathfrak{B})$. For each cardinal κ less than or equal to the cardinality of $\mathfrak{A}$, $A/\rho_{\mathfrak{A}}[\kappa]$ is the set of cells in $A/\rho_{\mathfrak{A}}$ of cardinality κ. $B/\rho_{\mathfrak{B}}[\kappa]$ is the set of cells in $B/\rho_{\mathfrak{B}}$ of cardinality κ. By supposition, there is a bijection f_κ from $A/\rho_{\mathfrak{A}}[\kappa]$ to $B/\rho_{\mathfrak{B}}[\kappa]$. Let g be the union of the f_κ. g is a bijection from $A/\rho_{\mathfrak{A}}$ to $B/\rho_{\mathfrak{B}}$ that preserves the cardinality of cells. Let C be a cell in $A/\rho_{\mathfrak{A}}$. $\mathfrak{A}[C]$ is an equivalence. In the same way, $\mathfrak{B}[g(C)]$ is an equivalence. f_C is any bijection from C to $g(C)$. f_C is an isomorphism from $\mathfrak{A}[C]$ to $\mathfrak{B}[g(C)]$. Let f be the union, over the cells in $A/\rho_{\mathfrak{A}}$, of the f_C. So f is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. □

If D is a set, then *card* D is the cardinal number of the set D. $\phi(S)$ is a formula whose one and only free variable is the set variable S. $\phi(S, S')$ is a formula whose only free variables are the set variables S and S' where S and S' are different and the first free occurrence of S in $\phi(S, S')$ occurs before (to the left of) the first free occurrence of S' in $\phi(S, S')$. If $\mathfrak{A}$ is a binary relational system and D and D' are subsets of A, $\mathfrak{A} \models \phi(S)[D]$ indicates that D satisfies $\phi(S)$ in $\mathfrak{A}$; and $\mathfrak{A} \models \phi(S, S')[D, D']$ indicates that (D, D') satisfies $\phi(S, S')$ in $\mathfrak{A}$, when D is the value of S and D' is the value of S'.

In the following, β is a non-zero cardinal, Γ is a collection of cardinals each of which is less than or equal to β, Σ is a set of ordered pairs whose coordinates are cardinals less than or equal to β. The pure formula $\phi(S)$ *describes* Γ *in* β iff for all $\mathfrak{A}$, a binary relational system of cardinality β, and all D, a subset of A, $\mathfrak{A} \models \phi(S)[D]$ iff *card* $D \in \Gamma$. Γ is *describable in* β iff some pure formula $\phi(S)$ describes Γ in β. The pure formula $\phi(S, S')$ *describes* Σ *in* β iff for all $\mathfrak{A}$, a binary relational system of cardinality β, all D, D′ subsets of A, $\mathfrak{A} \models \phi(S, S')[D, D']$ iff $(\mathit{card}\,D, \mathit{card}\,D') \in \Sigma$. Σ is *describable in* β iff there is a pure formula $\phi(S, S')$ that describes Σ in β. The pure formula $\phi(S)$ *describes* κ *in* β iff $\phi(S)$ describes $\{\kappa\}$ in β. κ is *describable in* β iff $\{\kappa\}$ is describable in β.

LEMMA 3 *Assume that* Γ *is a non-empty set of cardinals each* $\leqslant \beta$ *and that* Γ *is describable in* β. *Then, the least member of* Γ *is describable in* β.

Proof: Let $\phi(S)$ be that pure formula that describes Γ in β. $\lesssim(S, S')$ is a pure formula such that for all $\mathfrak{A}$, a binary relational system, all D and D′, subsets of A, $\mathfrak{A} \models \lesssim(S, S')[D, D']$ iff $\mathit{card}\,D \leqslant \mathit{card}\,D'$. $\psi(S)$ is the pure formula

$$(\phi(S) \;\&\; \forall S'(\phi(S') \supset \lesssim(S, S'))). \tag{1}$$

$\psi(S)$ describes the least member of Γ in β. □

LEMMA 4 *Assume that all models of* ϕ *are equivalences and that* Γ *is the collection of those cardinals* $\kappa \leqslant \beta$ *such that there are models of* ϕ *of cardinality* β *whose partition functions differ at* κ. *Then,*

1. Γ *is empty iff all models of* ϕ *of cardinality* β *are isomorphic*; *and*

2. *if* Γ *is non-empty, then* Γ *is describable in* β.

Proof: Assume that ϕ and Γ are as above. The first condition is immediate from Lemma 2.

Suppose that Γ is non-empty. Reasoning proceeds by constructing a pure sentence that describes Γ in β. $\mathrm{Cell}(x, S)$ is the formula

$$(S(x) \;\&\; \forall y(S(y) \equiv Q(x, y))). \tag{2}$$

If $\mathfrak{C}$ is an equivalence, $a \in C$ and D is a subset of C, then $\mathfrak{C} \models \mathrm{Cell}(x, S)[a, D]$ iff $D = [a]_{\mathfrak{C}}$. $\mathrm{Cell}(S)$ is the formula

$$\exists x \mathrm{Cell}(x, S). \tag{3}$$

The set D satisfies $\mathrm{Cell}(S)$ in the equivalence $\mathfrak{C}$ iff D is a cell in $C/\rho_{\mathfrak{C}}$.

If $\mathfrak{C}$ is an equivalence, $D \in C/\rho_{\mathfrak{C}}$ and D′ is a subset of C, then *D′ is a witness set for D in* $\mathfrak{C}$ iff D′ includes one and only one member from each cell in $C/\rho_{\mathfrak{C}}[\mathit{card}\,D]$. The existence of witness sets for each cell is a consequence of the axiom of choice. Reasoning proceeds by constructing a formula that defines the witness relation in $\mathfrak{C}$. There is a pure formula $\mathrm{Eq}(S, S')$ such that for all $\mathfrak{A}$, a binary relational system, and all D, D', subsets of A, $\mathfrak{A} \models Eq(S, S')[D, D']$ iff $\mathit{card}\, D = \mathit{card}\, D'$.

Witness(S', S) is the conjunction of the following:

$$\exists x(S'(x) \,\&\, S(x)), \tag{4}$$

$$\text{Cell}(S), \tag{5}$$

$$\forall S''((\text{Cell}(S'') \,\&\, \text{Eq}(S'', S)) \supset \exists x(S''(x) \,\&\, S'(x))), \tag{6}$$

$$\forall y(S'(y) \supset \exists S''(\text{Cell}(y, S'') \,\&\, \text{Eq}(S'', S)), \tag{7}$$

$$\forall xy((x \neq y \,\&\, S'(x) \,\&\, S'(y)) \supset {\sim}Q(x, y)). \tag{8}$$

If $\mathfrak{C}$ is an equivalence, D and D' are subsets of C, then $\mathfrak{C} \models \text{Witness}(S', S)[D', D]$ iff $D \in C/\rho_{\mathfrak{C}}$ and D' is a witness set for D. While the partition function for $\mathfrak{C}$ is not definable in $\mathfrak{C}$, it is ***representable in*** $\mathfrak{C}$ in the sense that there is a formula $\phi(S, S')$ such that $\mathfrak{C} \models \phi(S, S')[D, D']$ iff $\mathbb{P}(\mathfrak{C})(\mathit{card}\, D) = \mathit{card}\, D'$. Such a formula ***represents*** $\mathbb{P}(\mathfrak{C})$ ***in*** $\mathfrak{C}$. $PF(S, S')$ is the disjunction of the following:

$$\exists S'''S''''(\text{Eq}(S, S''') \,\&\, \text{Eq}(S', S'''') \,\&\, \text{Witness}(S'''', S''')), \tag{9}$$

$$({\sim}\exists S''(\text{Cell}(S'') \,\&\, \text{Eq}(S'', S)) \,\&\, \forall x {\sim} S'(x))). \tag{10}$$

$PF(S, S')$ represents the partition function $\mathbb{P}(\mathfrak{C})$ in $\mathfrak{C}$.

If R is a binary relational variable and ψ is a formula in which the binary relational constant, Q, occurs but R does not occur, then $\psi(R)$ is the result of replacing every occurrence of Q in ψ by R. Let R and R' be binary relational variables not occurring in ϕ or in $PF(S, S')$. $G(S)$ is the pure formula

$$\exists RR'S'S''(\phi(R) \,\&\, \phi(R') \,\&\, PF(S, S')(R) \,\&\, PF(S, S'')(R') \,\&\, {\sim}Eq(S', S'')). \tag{11}$$

$G(S)$ describes Γ in β. □

THEOREM 5 *Assume that all models of* ϕ *are equivalences and that there are models of* ϕ *of cardinality* β *that are not isomorphic. Then, there are models of* ϕ *of cardinality* β *that are not equivalent.*

Proof: Let Γ be as in Lemma 4. By Lemma 2, Γ is non-empty. Let κ be the least member of Γ. By Lemma 4 and Lemma 3, κ is describable in β. Let $\psi(S)$ describe κ in β. Let $\Theta = \{\mathbb{P}(\mathfrak{A})(\kappa) | \mathfrak{A}$ is a model of ϕ of cardinality $\beta\}$. By assumption, Θ contains at least two cardinals. Let α be the least member of Θ and γ be any member of $\Theta - \{\alpha\}$. There are $\mathfrak{A}$ and $\mathfrak{B}$, models of ϕ of cardinality β, such that $\mathbb{P}(\mathfrak{A})(\kappa) = \alpha$ and $\mathbb{P}(\mathfrak{B})(\kappa) = \gamma$. Reasoning proceeds by showing that these models of ϕ are not equivalent. Suppose that $\alpha = 0$. The following sentence is true on $\mathfrak{A}$ but false on $\mathfrak{B}$:

$$\forall S(\text{Cell}(S) \supset {\sim}\psi(S)). \tag{12}$$

Suppose that α is not 0. The following sentence is true on $\mathfrak{A}$ but false on $\mathfrak{B}$:

$$\exists SS'(PF(S,S') \,\&\, \psi(S) \,\&\, \forall RS''((\phi(R) \,\&\, PF(S,S'')(R)) \supset \precsim(S',S''))). \quad (13)$$

□

The following is immediate from the above, Lemma 1 and the observation that L_Q is homogeneous.

COROLLARY 1 *$\mathbb{E}$ satisfies both the quasi Fraenkel–Carnap property and the Fraenkel–Carnap property.*

3 SIMPLE EXPANSIONS OF EQUIVALENCES

For each $n \geqslant 1$, $K_n = \{Q\} \cup \{c_1, \ldots, c_n\}$ where $c_1, \ldots, c_n$ are distinct individual constants. L_n is the second-order language with non-logical vocabulary K_n. Interpretations for L_n are of the form $(\mathfrak{A}a_1 \cdots a_n)$ where $\mathfrak{A}$ is a binary relational system, $a_1, \ldots, a_n$ are members of A and c_i denotes a_i in $(\mathfrak{A}a_1 \cdots a_n)$. $(\mathfrak{A}a_1 \cdots a_n)$ is a *simple expansion of* $\mathfrak{A}$. $\mathbb{E}_n = \{(\mathfrak{A}a_1 \cdots a_n) \,|\, \mathfrak{A} \in \mathbb{E}\}$. It is shown in this section that $\mathbb{E}_n$ satisfies the quasi Fraenkel–Carnap property. Since L_n is homogeneous, $\mathbb{E}_n$ also satisfies the Fraenkel–Carnap property.

If $(\mathfrak{A}a_1 \cdots a_n)$ and $(\mathfrak{B}b_1 \cdots b_n)$ are interpretations for L_n and f is a function, f is an *isomorphism* from $(\mathfrak{A}a_1 \cdots a_n)$ to $(\mathfrak{B}b_1 \cdots b_n)$ provided f is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ and $f(a_i) = b_i$, for all i. $(\mathfrak{A}a_1 \cdots a_n)$ and $(\mathfrak{B}b_1 \cdots b_n)$ are *isomorphic* provided that there is some isomorphism from $(\mathfrak{A}a_1 \cdots a_n)$ to $(\mathfrak{B}b_1 \cdots b_n)$.

Since $\mathbb{E}$ is a finitary class in L_Q, $\mathbb{E}_n$ is finitary in L_n. Hence by Theorem 5.1 [14], to show that $\mathbb{E}_n$ satisfies the quasi Fraenkel–Carnap property it suffices to show that for all ϕ, a sentence in L_n, and all cardinals β, if all models of ϕ are in $\mathbb{E}_n$ and all models of ϕ of cardinality β are equivalent in L_n, then all models of ϕ of cardinality β are isomorphic.

LEMMA 6 *Assume that ϕ is a sentence in L_n all of whose models are in $\mathbb{E}_n$ and that β is an infinite cardinal. Then, if all models of ϕ of cardinality β are equivalent in L_n, then there is ψ, a sentence in L_Q, such that*

1. *$\mathfrak{A}$ is a model of ψ iff there are $a_1, \ldots, a_n$ in A such that $(\mathfrak{A}a_1 \cdots a_n)$ is a model of ϕ;*
2. *all models of ψ are in $\mathbb{E}$;*
3. *all models of ψ of cardinality β are equivalent in L_Q; and*
4. *all models of ψ of cardinality β are isomorphic.*

Proof: Suppose that all models of ϕ of cardinality β are equivalent in L_n. Let $x_1, \ldots, x_n$ be distinct individual variables not occurring in ϕ. φ is the result of replacing every occurrence of c_i in ϕ by an occurrence of x_i, for all i. ψ is $\exists x_1 \cdots x_n \varphi$. Conditions 1, 2 and 3 are immediate. Condition 4 follows from condition 3 and Theorem 5. □

Lemma 6 essentially reduces the problem of showing that $\mathbb{E}_n$ satisfies the quasi Fraenkel–Carnap property to the study of automorphisms on the members of $\mathbb{E}$. To understand this, consider the following. Suppose that $(\mathfrak{A} a_1 \cdots a_n)$ and $(\mathfrak{B} b_1 \cdots b_n)$ are models of ϕ of cardinality β. Then $\mathfrak{A}$ and $\mathfrak{B}$ are models of ψ of cardinality β. There is an isomorphism f from $\mathfrak{A}$ to $\mathfrak{B}$. f is an isomorphism from $(\mathfrak{A} a_1 \cdots a_n)$ to $(\mathfrak{B} f(a_1) \cdots f(a_n))$. Hence, $(\mathfrak{A} a_1 \cdots a_n)$ and $(\mathfrak{B} f(a_1) \cdots f(a_n))$ are equivalent in L_n. Further, if g is any automorphism on $\mathfrak{B}$ that takes $f(a_i)$ to b_i, for all i, then g is an isomorphism from $(\mathfrak{B} b_1 \cdots b_n)$ to $(\mathfrak{B} f(a_1) \cdots f(a_n))$. Thus, the composition of f and g is an isomorphism from $(\mathfrak{A} a_1 \cdots a_n)$ to $(\mathfrak{B} b_1 \cdots b_n)$. The next lemma provides conditions necessary and sufficient for the existence of an automorphism on a member of $\mathbb{E}$.

LEMMA 7 *Assume that $\mathfrak{A}$ is an infinite equivalence and that $a_1, \ldots, a_n$, $d_1, \ldots, d_n$ are in A. Then the following are equivalent:*

1. *there is f, an automorphism on $\mathfrak{A}$, such that $f(a_i) = d_i$ for all i; and*
2. *for all i, j, if $1 \leqslant i, j \leqslant n$, then*
   - *(a)* $a_i \neq a_j$ *iff* $d_i \neq d_j$;
   - *(b)* $(a_i, a_j) \in \rho_{\mathfrak{A}}$ *iff* $(d_i, d_j) \in \rho_{\mathfrak{A}}$; *and*
   - *(c) card* $[a_i]_{\mathfrak{A}} =$ *card* $[d_i]_{\mathfrak{A}}$.

Proof: Condition 2 is immediate from Condition 1. Suppose that condition 2 holds. There are two cases to consider. Supppose that for all i and j, if $i \neq j$, then $[a_i]_{\mathfrak{A}}$ and $[a_j]_{\mathfrak{A}}$ are disjoint. It follows from condition 2(c) that for all i, there is f_i, a bijection from $[a_i]_{\mathfrak{A}}$ to $[d_i]_{\mathfrak{A}}$ such that $f_i(a_i) = d_i$. Let $h = f_1 \cup \cdots \cup f_n$. h is bijection from $[a_1]_{\mathfrak{A}} \cup \cdots \cup [a_n]_{\mathfrak{A}}$ to $[d_1]_{\mathfrak{A}} \cup \cdots \cup [d_n]_{\mathfrak{A}}$ and $h(a_i) = d_i$, for all i. Further, h is an isomorphism from $\mathfrak{A}[[a_1]_{\mathfrak{A}} \cup \cdots \cup [a_n]_{\mathfrak{A}}]$ to $\mathfrak{A}[[d_1]_{\mathfrak{A}} \cup \cdots \cup [d_n]_{\mathfrak{A}}]$. $A = [a_1]_{\mathfrak{A}} \cup \cdots \cup [a_n]_{\mathfrak{A}}$ iff $A = [d_1]_{\mathfrak{A}} \cup \cdots \cup [d_n]_{\mathfrak{A}}$. Thus, if $A = [a_1]_{\mathfrak{A}} \cup \cdots \cup [a_n]_{\mathfrak{A}}$, then h is the desired automorphism on $\mathfrak{A}$. Suppose that $A \neq [a_1]_{\mathfrak{A}} \cup \cdots \cup [a_n]_{\mathfrak{A}}$. Then $A \neq [d_1]_{\mathfrak{A}} \cup \cdots \cup [d_n]_{\mathfrak{A}}$. Let $B = A$-$([a_1]_{\mathfrak{A}} \cup \cdots \cup [a_n]_{\mathfrak{A}})$ and $C = A$-$([d_1]_{\mathfrak{A}} \cup \cdots \cup [d_n]_{\mathfrak{A}})$. Both $\mathfrak{A}[B]$ and $\mathfrak{A}[C]$ are equivalences of the same cardinality. If g is an isomorphism from $\mathfrak{A}[B]$ to $\mathfrak{A}[C]$, then $h \cup g$ is an automorphism on $\mathfrak{A}$. Reasoning proceeds by applying Lemma 2 to show that $\mathfrak{A}[B]$ and $\mathfrak{A}[C]$ are isomorphic. Let δ be the cardinality of B and let κ be a cardinal $\leqslant \delta$. Finally, for each i, let δ_i be the cardinality of $[a_i]_{\mathfrak{A}}$. Suppose that $\kappa \neq \delta_i$, for all i. Thus, $\mathbb{P}(\mathfrak{A}[B])(\kappa) = \mathbb{P}(\mathfrak{A})(\kappa)$ and $\mathbb{P}(\mathfrak{A}[C])(\kappa) = \mathbb{P}(\mathfrak{A})(\kappa)$. Suppose that there is i, $1 \leqslant i \leqslant n$, such that $\kappa = \delta_i$. Since h is an isomorphism, $\mathbb{P}(\mathfrak{A}[[a_1]_{\mathfrak{A}} \cup \cdots \cup [a_n]_{\mathfrak{A}}])(\kappa) = \mathbb{P}(\mathfrak{A}[[d_1]_{\mathfrak{A}} \cup \cdots \cup [d_n]_{\mathfrak{A}}])(\kappa)$. Further, $\mathbb{P}(\mathfrak{A}[B])(\kappa) =$

$\mathbb{P}(\mathfrak{A})(\kappa) - \mathbb{P}(\mathfrak{A}[[a_1]_{\mathfrak{A}} \cup \cdots \cup [a_n]_{\mathfrak{A}}])(\kappa)$ and $\mathbb{P}(\mathfrak{A}[C])(\kappa) = \mathbb{P}(\mathfrak{A})(\kappa) - \mathbb{P}(\mathfrak{A}[[d_1]_{\mathfrak{A}} \cup \cdots \cup [d_n]_{\mathfrak{A}}])(\kappa)$. Hence, $\mathbb{P}(\mathfrak{A}[B]) = \mathbb{P}(\mathfrak{A}[C])$.

Suppose that there are i and j, between 1 and n, such that $i \neq j$ and $[a_i]_{\mathfrak{A}}$ and $[a_j]_{\mathfrak{A}}$ are not disjoint. By condition 2(b), $[d_i]_{\mathfrak{A}}$ and $[d_j]_{\mathfrak{A}}$ are not disjoint. Thus, there is $t \geqslant 1$, $\{a'_1, \ldots, a'_t\} \subseteq \{a_1, \ldots, a_n\}$, and $\{d'_1, \ldots, d'_t\} \subseteq \{d_1, \ldots, d_n\}$ that satisfy the following:

1. if $i \neq j$, then $[a'_i]_{\mathfrak{A}}$ and $[a'_j]_{\mathfrak{A}}$ are disjoint, and $[d'_i]_{\mathfrak{A}}$ and $[d'_j]_{\mathfrak{A}}$ are disjoint;
2. for each a_i there is a'_j such that $(a_i, a'_j) \in \rho_{\mathfrak{A}}$ and for each d_i there is d'_j such that $(d_i, d'_j) \in \rho_{\mathfrak{A}}$;
3. $a'_1, \ldots, a'_t$ and $d'_1, \ldots, d'_t$ satisfy condition 2 of Lemma 7.

Proceeding as in the first case, for each i, between 1 and t, there is f_i such that f_i is a bijection from $[a'_i]_{\mathfrak{A}}$ to $[d'_i]_{\mathfrak{A}}$, $f_i(a'_i) = d'_i$; and, if $a_j \in [a'_i]_{\mathfrak{A}}$, then $f_i(a_j) = d_j$, for all j between 1 and n. h is then the union of the f_i and the automorphism f is constructed as in the first case. □

LEMMA 8 *Assume that ϕ is a sentence in L_n all of whose models are in $\mathbb{E}_n$ and that all of the models of ϕ of cardinality β are equivalent in L_n. Then, if $(\mathfrak{A}a_1 \cdots a_n)$ and $(\mathfrak{A}d_1 \cdots d_n)$ are models of ϕ of cardinality β, then $(\mathfrak{A}a_1 \cdots a_n)$ and $(\mathfrak{A}d_1 \cdots d_n)$ are isomorphic.*

Proof: Suppose that $(\mathfrak{A}a_1 \cdots a_n)$ and $(\mathfrak{A}d_1 \cdots d_n)$ are models of ϕ of cardinality β, but are not isomorphic. Then, there is no automorphism on $\mathfrak{A}$ taking a_i to d_i, for all i. By Lemma 7, there is some j such that $card[a_j]_{\mathfrak{A}} \neq card[d_j]_{\mathfrak{A}}$. Let $card[a_j]_{\mathfrak{A}} < card[d_j]_{\mathfrak{A}}$. Let Γ be the set of all cardinals α for which there are $b_1, \ldots, b_n$ in A such that $(\mathfrak{A}b_1 \cdots b_n)$ is a model of ϕ and $\alpha = card[b_j]_{\mathfrak{A}} < card[d_j]_{\mathfrak{A}}$. By supposition, Γ is non-empty. Let κ be the least member of Γ. Thus, there are $b_1, \ldots, b_n$ in A such that $(\mathfrak{A}b_1 \cdots b_n)$ is a model of ϕ and $card[b_j]_{\mathfrak{A}} = \kappa$. $(\mathfrak{A}b_1 \cdots b_n)$ and $(\mathfrak{A}d_1 \cdots d_n)$ are equivalent in L_n. Reasoning proceeds by finding a sentence true on one but false on the other. There is a pure formula $<(S, S')$ such that for all relational systems $\mathfrak{B}$ and all D and D', subsets of B, $\mathfrak{B} \models <(S, S')[D, D']$ iff $card\, D < card\, D'$. Let φ be as in the proof of Lemma 6. Consider the sentence

$$\exists S\, S' x_1 \cdots x_n (\varphi \mathbin{\&} \mathrm{Cell}(x_j, S) \mathbin{\&} \mathrm{Cell}(c_j, S') \mathbin{\&} <(S, S')). \tag{14}$$

This sentence is true on $(\mathfrak{A}d_1 \cdots d_n)$ but false on $(\mathfrak{A}b_1 \cdots b_n)$. □

The following is then immediate from Lemma 8 by the reasoning following the proof of Lemma 6

THEOREM 9 *Assume that ϕ is a sentence in L_n all of whose models are in $\mathbb{E}_n$ and that β is an infinite cardinal. Then, if all models of ϕ of cardinality β are equivalent in L_n, then all models of ϕ of cardinality β are isomorphic.*

Finally, Corollary 2 is immediate from Theorem 9 and the observation that L_n is homogeneous.

COROLLARY 2 *For all* n, $\mathbb{E}_n$ *satisfies both the quasi Fraenkel–Carnap property and the Fraenkel–Carnap property.*

4 AN APPLICATION

This section is devoted to an application of Corollary 1. If $\mathfrak{A}$ is an infinite equivalence, then certain properties of $\mathfrak{A}$ can be characterized in terms of the action of the partition function for $\mathfrak{A}$: $\mathfrak{A}$ is quasi-finitely characterizable iff $\mathbb{P}(\mathfrak{A})$ is describable in *card* A; and, $\mathfrak{A}$ is finitely characterizable iff $\mathbb{P}(\mathfrak{A})$ is describable in *card* A and *card* A is characterizable in L_Q. Thus, by Corollary 1, $\mathrm{Th}(\mathfrak{A})$ is quasi-finitely axiomatizable iff $\mathbb{P}(\mathfrak{A})$ is describable in *card* A; and, $\mathrm{Th}(\mathfrak{A})$ is finitely axiomatizable iff $\mathbb{P}(\mathfrak{A})$ is describable in *card* A and *card* A is characterizable in L_Q. The following Lemma plays an important role in the proofs of these results.

LEMMA 10 *Assume that* ϕ *in a sentence in* L_Q. *Then, there is a pure sentence* ψ *such that for all* $\mathfrak{C}$, *a binary relational system,* $\mathfrak{C}$ *is a model of* ψ *iff all models of* ϕ *of card* C *are isomorphic.*

Proof: It is assumed that the logical vocabulary of L_Q includes unary functional variables. While this assumption is not essential, it makes the construction of ψ more transparent. Let g be a unary functional variable. $\mathrm{bi}(g)$ is a pure formula such that if $\mathfrak{C}$ is a binary relational system and ρ is a unary function on C, then $\mathfrak{C} \models \mathrm{bi}(g)[\rho]$ iff ρ is a bijection on C.

Let R and R' be binary relational variables not occurring in ϕ or $\mathrm{bi}(g)$. $\mathrm{Iso}(g, R, R')$ is the pure formula

$$(\mathrm{bi}(g) \;\&\; \forall xy(R(x,y) \equiv R'(g(x), g(y)))). \tag{15}$$

ϕ^* is the pure sentence

$$\forall RR'((\phi(R) \;\&\; \phi(R')) \supset \exists g\mathrm{Iso}(g, R, R')). \tag{16}$$

Let $\mathfrak{C}$ be a binary relational system of cardinality β. Clearly, ϕ^* is true on $\mathfrak{C}$, iff all models of ϕ of cardinality β are isomorphic. □

Lemma 10 can be extended to any homogeneous second-order language whose non-logical vocabulary is finite. *The spectrum* of a sentence is the class of the cardinals of the models of the sentence. The spectrum of the sentence ϕ^* above is the collection of all cardinals κ such that all models of ϕ of cardinality κ are isomorphic. All models of $\{\phi, \phi^*\}$ of the same cardinality are isomorphic. Hence, if $\mathfrak{A}$ is a model of $\{\phi, \phi^*\}$, then $\mathfrak{A}$ is quasi-finitely characterizable.

LEMMA 11 *Assume that* $\mathfrak{A}$ *is an infinite equivalence of cardinality* β. *Then,* $\mathfrak{A}$ *is quasi-finitely characterizable iff* $\mathbb{P}(\mathfrak{A})$ *is describable in* β.

Proof: Assume that $\mathfrak{A}$ is an infinite equivalence of cardinality β. Suppose that $\mathbb{P}(\mathfrak{A})$ is describable in β. Let $\psi(S, S')$ be a pure formula that describes $\mathbb{P}(\mathfrak{A})$ in β. υ is a sentence in L_Q true on all and only equivalences. φ is the sentence

$$\forall SS'(\text{Witness}(S', S) \supset \psi(S, S')). \tag{17}$$

θ is the sentence

$$\forall S'S((\sim\exists S''(\text{Cell}(S'') \,\&\, \text{Eq}(S'', S) \,\&\, \forall x{\sim}S'(x)) \supset \psi(S, S')). \tag{18}$$

ϕ is the conjunction of υ, φ, and θ. Finally, ϕ^* is constructed from ϕ as above.

It suffices to show that $\mathfrak{A}$ is a model of ϕ^*. Suppose that ρ and η are binary relations on A such that (A, ρ) and (A, η) are models of ϕ. Thus, (A, ρ) and (A, η) are infinite equivalences of cardinality β; and φ and θ are true on these equivalences. Reasoning proceeds by establishing that (A, ρ) and (A, η) are isomorphic. Applying Lemma 2, it suffices to show that their partition functions are identical.

To that end, observe the following. Suppose that D and D' are subsets of A. Since $\psi(S, S')$ describes $\mathbb{P}(\mathfrak{A})$ in β, if D' is a witness set for D in (A, ρ), then $\mathbb{P}(\mathfrak{A})(\textit{card}\, D) = \textit{card}\, D'$. Further, since θ is true on $\mathfrak{A}$, if no cell in A/ρ has the same cardinality as D and D' is empty, then $\mathbb{P}(\mathfrak{A})(\textit{card}\, D) = \textit{card}\, D'$. Similar observations hold for (A, η).

Let κ be a cardinal $\leqslant \beta$. There are two cases to consider. Suppose that $\mathbb{P}((A, \rho))(\kappa) = 0$. Let D be a subset of A of cardinality κ and D' be the empty set. By the above observations, $\mathbb{P}(\mathfrak{A})(\kappa) = 0$. Notice that $\mathbb{P}((A, \eta))(\kappa) = 0$. Otherwise, there is C, a cell in A/η, of cardinality κ and there is C', a witness set for C in (A, η). Therefore, by the observation above, $\mathbb{P}(\mathfrak{A})(\kappa) \neq 0$. Suppose that $\mathbb{P}((A, \rho))(\kappa) \neq 0$. By reasoning as above, $\mathbb{P}((A, \eta))(\kappa) \neq 0$. Hence, $\mathbb{P}((A, \rho))(\kappa) = 0$ iff $\mathbb{P}((A, \eta))(\kappa) = 0$.

Now suppose that $\mathbb{P}((A, \rho))(\kappa) \neq 0$. Let D be a cell in A/ρ of cardinality κ and let D' be a witness set for D in (A, ρ). Hence, $\mathbb{P}(\mathfrak{A})(\kappa) = \textit{card}\, D'$. By the reasoning of the last paragraph, $\mathbb{P}((A, \eta))(\kappa) \neq 0$. There is C, a cell in A/η, such that $\textit{card}\, C = \kappa$. Let C' be a witness set for C in (A, η). Therefore, $\mathbb{P}(\mathfrak{A})(\kappa) = \textit{card}\, C'$. Hence, $\textit{card}\, D' = \textit{card}\, C'$ and $\mathbb{P}((A, \rho))(\kappa) = \mathbb{P}((A, \eta))(\kappa)$.

Suppose that $\mathfrak{A}$ is quasi-finitely characterizable. Choose ϕ, a sentence in L_Q, such that ϕ is true on $\mathfrak{A}$ and all models of ϕ of the same cardinality are isomorphic. Reasoning proceeds by constructing a pure formula, $\psi(S, S')$, that describes $\mathbb{P}(\mathfrak{A})$ in β. $\Theta(S'', S, S''', S', R)$ is the conjunction of the pure formulas

$$\text{Eq}(S'', S), \tag{19}$$

$$\text{Eq}(S''', S'), \tag{20}$$

$$\text{Witness}(S''', S'')(R). \tag{21}$$

$\Gamma(S, S', R)$ is the pure formula

$$\exists S'' S''' \Theta(S'', S, S''', S', R). \tag{22}$$

$\varphi(S, S', R)$ is the pure formula

$$(\sim\exists S''(\mathrm{Cell}(S'')(R) \,\&\, Eq(S, S'')) \,\&\, \forall x \sim S'(x)). \tag{23}$$

$\psi(S, S')$ is the pure formula

$$\exists R(\phi(R) \,\&\, (\Gamma(S, S', R) \vee \varphi(S, S', R))). \tag{24}$$

It remains to show that $\psi(S, S')$ describes $\mathbb{P}(\mathfrak{A})$ in β. Suppose that $\mathfrak{B}$ is a binary relational system of cardinality β and that D and D' are subsets of B. Reasoning proceeds by establishing that $\mathfrak{B} \models \psi(S, S')[D, D']$ iff $\mathbb{P}(\mathfrak{A})(\mathit{card}\, D) = \mathit{card}\, D'$. Suppose that $\mathfrak{B} \models \psi(S, S')[D, D']$. Thus, there is ρ, a binary relation on B, such that (B, ρ) is a model of ϕ. By the choice of ϕ, $\mathfrak{A}$ and (B, ρ) are isomorphic. Thus, (B, ρ) is an equivalence and, by Lemma 2, their partition functions are identical. Further, either $(B, \rho) \models \Gamma(S, S', R)[D, D', \rho]$ or $(B, \rho) \models \varphi(S, S', R)[D, D', \rho]$. In either case, $\mathbb{P}((B, \rho))(\mathit{card}\, D) = \mathit{card}\, D'$. Hence, $\mathbb{P}(\mathfrak{A})(\mathit{card}\, D) = \mathit{card}\, D'$.

Suppose that $\mathbb{P}(\mathfrak{A})(\mathit{card}\, D) = \mathit{card}\, D'$. There is ρ, a binary relation on B, such that (B, ρ) and $\mathfrak{A}$ are isomorphic. Thus, by Lemma 2, $\mathbb{P}((B, \rho))(\mathit{card}\, D) = \mathit{card}\, D'$. It follows that either $(B, \rho) \models \Gamma(S, S', R)[D, D', \rho]$ or we have $(B, \rho) \models \varphi(S, S', R)[D, D', \rho]$. In either case, $\mathfrak{B} \models \psi(S, S')[D, D']$.

Suppose that $\mathbb{P}(\mathfrak{A})$ is describable in β and that β is characterizable. By Lemma 10, there is ϕ, a sentence in L_Q true on $\mathfrak{A}$, such that all models of ϕ of the same cardinality are isomorphic. There is also a pure sentence, ψ, true on all and only binary relational systems of cardinality β. Hence, all models of $\{\phi, \psi\}$ are isomorphic to $\mathfrak{A}$, and $\mathfrak{A}$ is finitely characterizable. Conversely, if $\mathfrak{A}$ is finitely characterizable, it is quasi-finitely characterizable. Thus, by Lemma 10, $\mathbb{P}(\mathfrak{A})$ is describable in β. □

By Corollary 2.2 of [14], since $\mathrm{Th}(\mathfrak{A})$ is finitely axiomatizable, β is characterizable. This reasoning suffices to establish the following.

THEOREM 12 *Assume that $\mathfrak{A}$ is an infinite equivalence of cardinality β. Then, the following are equivalent:*

1. *$\mathfrak{A}$ is finitely characterizable; and*

2. *$\mathbb{P}(\mathfrak{A})$ is describable in β and β is characterizable in L_Q.*

Corollary 3 is immediate from Lemma 11, Theorem 12 and Corollary 1.

COROLLARY 3 *Assume that $\mathfrak{A}$ is an infinite equivalence of cardinality β. Then,*

1. *$\mathrm{Th}(\mathfrak{A})$ is quasi-finitely axiomatizable iff $\mathbb{P}(\mathfrak{A})$ is describable in β; and,*

2. Th($\mathfrak{A}$) *is finitely axiomatizable iff* $\mathbb{P}(\mathfrak{A})$ *is describable in* β *and* β *is characterizable in* L_Q.

Suppose that $\mathfrak{A}$ is an infinite equivalence of cardinality β and that β is characterizable in L_Q. By Corollary 3, one can establish that the second-order theory of an infinite equivalence of cardinality β is finitely axiomatizable by studying the action of its partition function, as opposed to finding a finite subset of its theory and showing that all of the models of this set are isomorphic. To the authors' knowledge there is no equivalence whose second-order theory can be shown to be finitely axiomatizable using Corollary 3 that could not be shown to be so merely by finding the appropriate finite set of sentences. However, there are some properties of the class of infinite equivalences whose second-order theories are finitely axiomatizable that follow easily from Corollary 3.

There are countably many infinite cardinals which are characterizable in L_Q [6]. Corollary 3 can be used to construct infinite equivalences whose second-order theories are finitely axiomatizable. For each $n \in \omega$ there is an equivalence of cardinality β whose partition function is of constant value n. Each of these partition functions is describable in β. Hence, there are a countable infinity of equivalences of cardinality β whose second-order theories are finitely axiomatizable. Further, there are uncountably many equivalences whose second-order theories are not finitely axiomatizable.

Corollary 3 can also be used to construct an infinite equivalence whose second-order theory is finitely axiomatizable and has simple expansions whose second-order theories are not finitely axiomatizable. Let β be the cardinal $\aleph_{\aleph_1}$. It follows from Proposition 4.3 of [6, page 437] and the fact that all finite cardinals are characterizable in L_Q, that β is characterizable in L_Q. Let $\mathfrak{A}$ be an equivalence of cardinality β having no finite cells and exactly one cell of cardinality $\aleph_\alpha$ for each ordinal $\alpha \leqslant \aleph_1$. It is easy to construct a pure formula $\phi(S, S')$ such that if $\mathfrak{B}$ is a binary relational system of cardinality β and D and D' are subsets of B, then $\mathfrak{A} \models \phi(S, S')[D, D']$ iff either D is finite and D' is empty or D is infinite and D' is a unit set. This formula describes $\mathbb{P}(\mathfrak{A})$ in β. By Corollary 3, Th($\mathfrak{A}$) is finitely axiomatizable. For for each ordinal $\alpha \leqslant \aleph_1$ there is $a(\alpha)$ in A such that $[a(\alpha)]_{\mathfrak{A}}$ is of cardinality $\aleph_\alpha$. By Lemma 7, if α and α' are different, then $(\mathfrak{A}\ a(\alpha))$ and $(\mathfrak{A}\ a(\alpha'))$ are not isomorphic. It follows that there is $\alpha \leqslant \aleph_1$ such that $(\mathfrak{A}\ a(\alpha))$ is not finitely axiomatizable. Otherwise, by Corollary 2, each $(\mathfrak{A}a(\alpha))$ is finitely characterizable. As L_1 is countable, this is impossible.

5 OPEN QUESTIONS

Partition functions for equivalences have played a major role in the above. There are some other questions about the second-order model theory of equivalences that can be answered using partition functions. For example, consider the question of whether or not all equivalences of cardinality $\aleph_0$ are isomorphic if they are equivalent in L_Q. It is relatively easy to show that if $\mathfrak{A}$ and $\mathfrak{B}$ are equivalences of cardinality $\aleph_0$ that are equivalent in L_Q, then their partition functions are identical (hence, the equivalences are isomorphic). Let $\mathfrak{A}$ be a countably infinite equivalence. $\aleph_0$ is describable in L_Q. Recall that not all of the partition functions of countably infinite equivalences are describable in $\aleph_0$. However, each of these partition functions is ***point-wise describable in*** $\aleph_0$ in the sense that for all n, $\{(n, \mathbb{P}(\mathfrak{A})(n))\}$ is describable in $\aleph_0$ and $\{(\aleph_0, \mathbb{P}(\mathfrak{A})(\aleph_0))\}$ is describable in $\aleph_0$. It follows that if $\mathfrak{B}$ is equivalent to $\mathfrak{A}$ in L_Q, then their partition functions are identical. Ajtai [1] attributes the general question of whether or not countably infinite interpretations with the same second-order theories are isomorphic to Pelikán and Kechris. Ajtai, among others, showed that the answer to the general question is independent of ZFC.

Partition functions appear to play much the same role played by configuration signatures for Dedekind algebras. It has been shown that many properties of and relations on Dedekind algebras can be characterized in terms of configuration signatures [10, 11]. Among these are properties familiar from universal algebra (e.g. homogeneous and universal algebras) and properties and relations from the model theory of first-order languages (e.g. elementary equivalence, saturated algebras, and both model complete and submodel complete algebras). To the authors' knowledge, it is open whether or not analogous results can be established for equivalences.

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