Adding to Relevant Restricted Quantification

Jc Beall*
UNIVERSITY OF CONNECTICUT, USA
UNIVERSITY OF OTAGO, NEW ZEALAND
http://entailments.net
jc.beall@uconn.edu

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ABSTRACT: This paper presents, in a more general setting, a simple approach to 'relevant restricted generalizations' advanced in previous work. After reviewing some desiderata for restricted generalizations, I present the target route towards achieving the desiderata. An objection to the approach, due to David Ripley, is presented, followed by three brief replies, one from a dialetheic perspective and the others more general.

This paper presents, in a more general setting, a simple approach to 'relevant restricted generalizations' advanced in [2, Ch. 5]. The paper presupposes familiarity with standard issues of restricted generalizations in relevant-logic settings [7, 9], and in particular with the issues as framed in [3] with special attention on applications to robustly contraction-free theories [12].[1] Brief discussion of target application-based issues is given in §1.2, but fuller and more leisurely discussion is confined to cited works.

*An objection to BXTT [2] by Vann McGee (in correspondence) occasioned the idea in this paper. The paper itself emerged in the very productive atmosphere of WCP4 (the 4th World Congress of Paraconsistency), and I'm grateful to Graham Priest for discussion, and particularly grateful to Greg Restall, who noticed, over dinner, a simplification of the ideas I was thinking about. I'm also very grateful to Dave Ripley, who raised the objection discussed in this paper, and who continues to be a lively and valuable philosophical interlocutor, and to Aaron Cotnoir and Michael Hughes for useful discussion. Finally, were it not for the very interesting ideas that Richard S. Anderson has recently had on the topic, this paper would not be what it is.

[1]One assumption throughout is that restricted generalizations involve a conditional. This is not an uncontroversial (or nonstandard) assumption. For other approaches, see Butchart [5], Belnap [4], Slaney [13, 14] and references therein. I should note that issues concerning contraction, contraposition, and so on (see §1) emerge even if the current assumption about a conditional is rejected; the problems simply take a slightly different—conditional-free—form.

The paper is structured as follows. §1 presents the desiderata advanced in [3], and §2 presents the target route towards achieving the desiderata. An objection, due to David Ripley, is presented in §3. §4 offers three brief replies to the Ripley objection, one from a dialetheic perspective and the others more general. §5 offers a few closing remarks.

1 MINIMAL DESIDERATA

Let $\mapsto$ be our target restricted conditional and $\rightarrow$ our given (detachable) relevant conditional.[2] The desiderata, put forth in [3], are as follows (keeping to the given numbering).[3]

1.1 GENERAL, A-LEVEL DESIDERATA

Some general desiderata are as follows.

- A1. $\alpha, \alpha \mapsto \beta \vdash \beta$ (Detachment).
- A2. $\beta \vdash \alpha \mapsto \beta$ (Conditional Weakening).
- A3. $\alpha \rightarrow \beta \vdash \alpha \mapsto \beta$.
- A6. $\vdash \alpha \wedge \beta \mapsto \alpha$ (Conditional Simplification).

Note that [3] doesn't list A6, but it—like Conditional Identity (viz., $\vdash \alpha \mapsto \alpha$)—is a desideratum, at least if, as I presume, we want it to be logically true that (e.g.) *all black cats are black* and so on.

1.2 APPLICATION-CENTERED, B-LEVEL DESIDERATA

Some less general desiderata are as follows.

- B1. $\alpha \mapsto \beta \nvdash \neg\beta \mapsto \neg\alpha$ (No Contraposition).
- B2. $\alpha \mapsto (\alpha \mapsto \beta) \nvdash \alpha \mapsto \beta$ (No Contraction).
- B2! More generally: robustly contraction-free in Restall's sense [12].

These desiderata emerge from target *applications* of the given logics, and in particular target 'semantic' theories as in [10] and [2].[4] Such theories have the resources to form Curry sentences, and so—given other logical features of the theories—the theories must remain 'robustly contraction-free' on pain of triviality. This motivates B2. Desideratum B1, in turn, is driven by target

[2] The assumption of detachability, made throughout, is rule detachability: $\alpha, \alpha \rightarrow \beta \vdash \beta$.

[3] I focus only on the A- and B-level desiderata listed in [3] that involve the target conditional. (Also: I herein skip the C-level desiderata, all of which—I think (though have not proved)—are achievable on this approach (given suitable choices of quantifiers).)

[4] And see [6]. In roughly the same family, though not paraconsistent, is Field's recent transparent truth theory [8].

'glutty' theories: theories containing α and $\neg\alpha$ for some α. In such theories we cannot ***non-trivially*** add a conditional $\Rightarrow$ that satisfies what might be called the evil trinity:

- $\alpha, \alpha \Rightarrow \beta \vdash \beta$ (Detachment).
- $\beta \vdash \alpha \Rightarrow \beta$ (Conditional Weakening).
- $\alpha \Rightarrow \beta \vdash \neg\beta \Rightarrow \neg\alpha$ (Contraposition).

Suppose that we add $\Rightarrow$ to a glutty (and otherwise non-trivial) theory in the ballpark (e.g., B-ish theories), and let β be a glut. By Conditional Weakening, β gives us $\alpha \Rightarrow \beta$. By Contraposition, we have $\neg\beta \Rightarrow \neg\alpha$. But β is a glut, and so we have $\neg\beta$, which, in concert with $\neg\beta \Rightarrow \neg\alpha$, yields $\neg\alpha$ via Detachment. And this is for all α, and so we have near-enough triviality: $\neg\alpha$ for all α. (If, as many of the target theories enjoy, we have $\neg\neg\alpha \vdash \alpha$, then we have triviality full stop.)

Of course, if one's application(s) of a given relevant logic are different from dialetheic or, more generally, robustly-contraction-free-demanding applications, then the B-level desiderata mightn't be pressing. Still, the general A-level desiderata are likely of interest.

2 PROPOSAL: SIMPLE ADDITION

Various proposals have been made towards achieving the given desiderata. For example, in [3], the Routley–Meyer ternary semantics (or simplified versions thereof) for one's given relevant conditional is assumed, and a new conditional is (semantically) defined via restrictions on the broader ternary relation. In [2, Ch. 5], an unsatisfiable sentence $\perp$ is invoked to define a candidate restricted conditional (among other discussed options). More famously, the Ackermann constant t, axiomatized via $\alpha \dashv\vdash \mathsf{t} \to \alpha$, is invoked to define a candidate restricted conditional (viz., $\alpha \land \mathsf{t} \to \beta$).[5] And there are other proposals.

I leave the details, virtues and potential vices of such proposals to debate. Moreover, whether such proposals are needed in addition to the following proposal I also leave for debate. What is notable—and my chief aim here to note—is that, at least in many target logics, one needn't introduce further machinery to achieve the A-level (or, if one wants, the B-level) desiderata. One already has a candidate restricted conditional. In particular, where $\lor$ is disjunction, define

$$\alpha \mapsto \beta \ := \ (\alpha \to \beta) \lor \beta$$

Assuming that disjunction and one's given relevant conditional (the arrow) behave properly, our restricted conditional, so defined, immediately delivers the given desiderata.

[5]Priest [10] endorses this sort of approach. For a non-triviality proof of adding the two-way axiom $\alpha \dashv\vdash \mathsf{t} \to \alpha$ to truth theory BXTT, see [2, Ch. 5].

2.1 A-LEVEL DESIDERATA

Details of particular logics are required for proofs, but 'semantic' considerations for B-vicinity logics point in the right direction.

- On A1: suppose that $\rightarrow$ Detaches and that α is true (in a suitable model). Then—assuming standard disjunction features—any model in which β is untrue is one in which $(\alpha \rightarrow \beta) \vee \beta$ is untrue. So, $\mapsto$ itself Detaches.
- On A6: suppose that $\rightarrow$ Conditionally Simplifies, that is, that $\alpha \wedge \beta \rightarrow \alpha$ is true in all models. Then—assuming standard disjunction features—every model is one in which $(\alpha \wedge \beta \rightarrow \alpha) \vee \alpha$, and hence $\alpha \wedge \beta \mapsto \alpha$, is true in all models. So, $\mapsto$ itself Conditionally Simplifies.

Moreover, as is plain, we immediately get A2 and A3 if we have Addition (after which the 'simple addition' proposal is named): $\alpha \vdash \alpha \vee \beta$ and $\beta \vdash \alpha \vee \beta$.

- On A2: Addition gives us $\beta \vdash (\alpha \rightarrow \beta) \vee \beta$.
- On A3: Addition gives us $(\alpha \rightarrow \beta) \vdash (\alpha \rightarrow \beta) \vee \beta$.

2.2 B-LEVEL DESIDERATA

Whether we have the B-level desiderata turns on our given relevant conditional in terms of which our restricted conditional is defined. As in §2.1, genuine proofs require details of the particular logics or model theory, but rough, 'semantic' considerations point in the right direction.

- On B1: there are two cases, one in which the background conditional contraposes (as in BX) and one in which it doesn't.

  Where $\rightarrow$ contraposes: take a model in which $\alpha \rightarrow \beta$ (and, hence, its contraposition) is untrue. Any such model in which β is true but $\neg\alpha$ untrue is a countermodel to Contraposition for our restricted conditional; such a model is one in which $(\alpha \rightarrow \beta) \vee \beta$ is true but $(\neg\beta \rightarrow \neg\alpha) \vee \neg\alpha$ untrue. (There are such models in the B-vicinity contraction-free logics.)

  Where $\rightarrow$ fails to contrapose: there are models in which $\alpha \rightarrow \beta$ is true but its contraposition untrue. Any such model in which $\neg\alpha$ is untrue is a countermodel to Contraposition for our restricted conditional; such a model is one in which $(\alpha \rightarrow \beta) \vee \beta$ is true but $(\neg\beta \rightarrow \neg\alpha) \vee \neg\alpha$ untrue. (There are such models in the B-vicinity contraction-free logics.)

- On B2: consider any model that invalidates Contraction for the background relevant conditional—that is, a model in which $\alpha \rightarrow (\alpha \rightarrow \beta)$ is true but $\alpha \rightarrow \beta$ untrue. Any such model in which β is untrue is a countermodel to Contraction for our restricted conditional. So, B2 holds.

- On B2!: whether we have robust freedom from contraction depends on the resources of the background logic (or target theory). Of course, if we have a non-triviality proof—as we do in some of the B-vicinity theories—we thereby have a proof of robust freedom.[6]

What the foregoing indicates is that, given an appropriate background logic (and many of the relevant logics are appropriate in the target sense), we need not look further for an A-level restricted conditional (or B-level if we have it). We have such a conditional.

3 RIPLEY OBJECTION

According to David Ripley (correspondence), the foregoing proposal faces a problem. In short, there are models in which, intuitively, all As are Bs, but our restricted conditional is untrue in such models.

To make the point, we simplify by focusing on a standard, B-vicinity 'worlds' model in which all worlds are perfectly classical.[7] For purposes of the example, we step up to the predicate-cum-quantifier level (with which level familiarity is assumed).[8] Here, @ is the base world, in terms of which ***truth in a model*** is defined, and I is the interpretation function. The essential features of the model are as follows.

- Let $\mathcal{W} = \{@, w\} = \mathcal{N}$.
- Let the domain be $\{a, b, c\}$.
- Let $I_@(\alpha(x)) = \{b\}$ and $I_@(\beta(x)) = \{a, b\}$.
- Let $I_w(\alpha(x)) = \{c\}$ and $I_w(\beta(x)) = \emptyset$.

The objection is that, intuitively, all As are Bs (at the actual world), since, at the actual world (viz., @), we have it that, for target sentences, $I_@(\alpha(x)) \subseteq I_@(\beta(x))$. On the other hand, the generalization using the proposed restricted conditional, namely,

$$\forall x(\alpha(x) \mapsto \beta(x))$$

is not true at the actual world. In particular, the c-instance of $\alpha(x) \mapsto \beta(x)$ is untrue at @. After all, $\beta(c)$ is not true at @. Moreover, since there's a point w at which $\alpha(c)$ is true but $\beta(c)$ untrue, $\alpha(c) \to \beta(c)$ is untrue at @ too. Hence,

$$(\alpha(c) \to \beta(c)) \vee \beta(c)$$

[6]Moreover, such a proof would immediately establish freedom from the evil trinity of §1.2. Hence, e.g., in BXTT, where we have a non-triviality proof, we immediately know that since $\mapsto$ Detaches and satisfies Conditional Weakening, it thereby doesn't contrapose.

[7]Hence, as Ripley (in correspondence) notes, the given model is independent of paraconsistency issues: if it shows a problem with our proposed restricted conditional, it equally shows a problem with taking (say) the strict hook as a restricted conditional.

[8]For elementary discussion of these frameworks, see Beall [1], and for broader and more detailed (though equally user-friendly) discussion see Dunn & Restall [7] or Priest [11].

is untrue at @. Hence, assuming features of the quantifiers (which features are not in question here), the proposed restricted conditional fails to get the facts right—or so the objection goes.

4 REMARKS ON THE RIPLEY OBJECTION

The objection, I think, does not show a defect in the proposal. I give three replies: two very general (and only sketched), the other peculiar to target dialetheic theories (including BXTT, which I endorse). I begin with the latter.

4.1 A DIALETHEIC REPLY

It is reasonable to expect that a candidate restricted conditional imply its hook counterpart. Indeed, this might well be made a desideratum—a D-level desideratum, we might say.

D1. $\alpha \mapsto \beta \vdash \alpha \supset \beta$

That our simple-addition conditional yields D1 may be seen along standard 'semantic' lines—for example, simplified semantics for B-vicinity logics. Suppose that $@ \not\models \alpha \supset \beta$, in which case $@ \models \alpha$ and $@ \not\models \neg\alpha$ but $@ \not\models \beta$. But, then, $@ \models (\alpha \rightarrow \beta) \vee \beta$ only if $@ \models \alpha \rightarrow \beta$, in which case, because $\rightarrow$ detaches, $@ \models \beta$. But this is impossible since $@ \not\models \beta$. Hence, there's no model in which $(\alpha \rightarrow \beta) \vee \beta$ is true and $\alpha \supset \beta$ not. Hence, by definition of our simple-addition conditional, there's no model in which $\alpha \mapsto \beta$ is true but $\alpha \supset \beta$ not.

On the other hand, the converse of D1, namely,

ID. $\alpha \supset \beta \vdash \alpha \mapsto \beta$

is dubious for various reasons, particularly in a dialetheic context. In standard dialetheic theories [2, 10], ID yields triviality if the restricted conditional $\mapsto$ detaches. (Proof: let α be a glut, in which case $\alpha \supset \beta$ is true, and so, via ID, $\alpha \mapsto \beta$ is true. But $\mapsto$ detaches, and so β is true. Triviality.) So, ID is not an option for non-trivial dialetheic theories. Hence, in addition to D1, another D-level desideratum is failure of D1's converse:

D2. $\alpha \supset \beta \nvdash \alpha \mapsto \beta$

And this brings us back to Ripley's objection.

4.1.1 RIPLEY THE GOOD

A positive way of seeing Ripley's 'objection' is not so much as an objection as a concrete confirmation that we have what we want: Ripley's model is confirmation of D2 (i.e., a countermodel to ID). We have $@ \models \forall x(A(x) \supset B(x))$ but $@ \not\models \forall x(A(x) \mapsto B(x))$. This is good.

4.1.2 RIPLEY THE BAD

A negative way of seeing Ripley's objection is that it simply demands too much.[9] The objection demands a constraint on the truth-at-a-point conditions (the 'truth conditions', if you want) of our restricted conditional. The demand amounts to one of the following constraints, where $I_w(E)$ and $I_w^{-1}(E)$ are, respectively, the extension and antiextension of expression E at point w.

- R1. $w \models \forall x(\alpha(x) \mapsto \beta(x))$ iff $I_w(\alpha(x)) \subseteq I_w(\beta(x))$.
- R2. $w \models \forall x(\alpha(x) \mapsto \beta(x))$ iff $y \in I_w^{-1}(\alpha(x))$ or $y \in I_w(\beta(x))$ for any y in the domain.

R1 maintains that what's necessary and sufficient for the truth of a restricted generalization at a point w is that the extension of $\alpha(x)$ at w be a subset of the extension of $\beta(x)$ at w, while R2 invokes the condition that something's either in the antiextension of $\alpha(x)$ at w or in the extension of $\beta(x)$ at w. Without at least one of R1 and R2, the Ripley objection does not go through. The trouble, however, is that R1 and R2 demand too much if, as I have assumed, subsets are understood along a classical (say, ZFC) theory. In short, either condition results in a *contracting* conditional, thereby undermining one of the main desiderata.

That a contracting conditional results from R1, where subsethood is understood along the assumed ZFC lines, may be seen by noting that R1 simply imposes the logical behaviour of our (classical) subset relation $\subseteq$ on the target restricted conditional. Since the former itself contracts (and detaches) so too does the latter. So, this demand is too much. (But see §4.2 for further comment.)

That R2 is equally problematic may be seen by noticing that, at least for target logics, it requires our restricted conditional to have precisely the truth-at-a-point conditions that the hook $\alpha \supset \beta$, defined as $\neg\alpha \vee \beta$, 'enjoys'.[10] But, then, this will fail to detach in target frameworks (or, in some, contract), and so fail at least some of the application-driven desiderata.

4.2 A GENERAL REPLY

A very brief but more general reply is available.[11] Ripley's objection charges that from $A \subseteq B$ we ought to have a true restricted generalization that *all* A*s*

[9] I am grateful to both David Ripley and, in particular, Richard S. Anderson for correspondence that led to a clearer formulation of this section.

[10] A Routley-Star approach to negation makes the conditions more involved, but the point goes through for such frameworks.

[11] In fact, I think that there are quite a few replies that turn on rather big issues concerning truth, meaning, model theory, and more. (E.g., if, as some deflationists might think, we are using the model theory as a mere heuristic guide to the logic, or at any rate not taking it to model 'real truth conditions', it is not obvious that the objection raises any problems for the *logic* of the proposed conditional.) The issues are both interesting and important, but more attention to them is demanded than I can give in this paper.

are B*s*. In turn, the objection highlights cases in which the subset relation holds but in which 'All As are Bs' fails to hold, where the given restricted generalization is understood along the proposed simple-addition restricted conditional.

The question to ask is why *all* A*s are* B*s* ought to follow from the corresponding subset claim. The answer, of course, seems obvious: this follows by definition. And so it does. The question, however, concerns the definition in question. And this is where the Ripley objection falters. If our set theory is classical, then we should expect—when talking only about classical sets (say, the ZFC universe)—that *all* A*s are* B*s* is a *material* (i.e., hook) restricted generalization that detaches in the given 'domain'.[12] But in this case, there's no reason to think that the target B-vicinity theories cannot enjoy as much;[13] the target generalizations simply aren't the simple-addition generalizations (and needn't be, since the hook suffices in the given context). On the other hand, if our set theory is non-classical, then only details will tell whether Ripley's objection gets off the ground. After all, it may well be that, in the non-classical 'set theory', the simple-addition conditional is used to define 'subset', in which case the model theory might deliver the results that Ripley's objection claims are absent. But, again, only details will tell. Either way, the objection fails to establish a problem with the simple-addition proposal (at least as one among various detachable restricted conditionals).

4.3 ONE MORE REPLY

One more reply points to an ambiguity in restricted generalizations. In short, we have two restricted conditionals, one that detaches and one that doesn't. In addition to the simple-addition conditional, which I've proposed for the former role, we have the hook: $\alpha \supset \beta$ or, in primitive notation, $\neg\alpha \vee \beta$. This is not detachable; however, it is true in the sort of models to which Ripley points. On this reply, what the Ripley objection highlights is that neither of the two restricted conditionals (the simple-addition and the hook) is in general adequate for all purposes; however, it may be that for any purpose, one or the other conditional suits. This is for future work to tell.

5 CLOSING REMARKS

Relevant logics, and the theories closed under them, have long faced an issue concerning restricted generalizations. A recent discussion of the problem is in [3], wherein one solution is given. This paper reports the good news that, by using 'simple addition', we meet the desiderata laid out in [3], and so achieve a restricted conditional in a simple, fairly theory-neutral fashion. Moreover, and more good news, the chief objection to the proposal (viz., Ripley's objec-

[12] My 'domain' talk is a sloppy way of saying that, in effect, the hook is detachable in the theory ZFC: you won't find α and $\alpha \supset \beta$ in the given theory without finding β.

[13] In fact, some of them do [2, 10].

tion) may be answered along one of various lines. The task now is to push the proposal further. Such is future work.[14]

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[14] I'd like to thank an anonymous referee whose comments not only improved this paper, but also contained very interesting suggestions that, due to schedule pressures, I regrettably could not incorporate. Again, I look forward to future work.

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