Complement-Topoi and Dual Intuitionistic Logic

Luis Estrada-González*
Facultad de Humanidades
Universidad Autónoma del Estado de Morelos
loisayaxsegrob@gmail.com

Received by Greg Restall
Published November 26, 2010
http://www.philosophy.unimelb.edu.au/ajl/2010
© 2010 Luis Estrada-González

Abstract: Mortensen in [11] studies dual intuitionistic logic by dualizing topos internal logic, but he did not study a sequent calculus. In this paper I present a sequent calculus for complement-topos logic, which throws some light on the problem of giving a dualization for LJ.

1 Introductory Remarks

"Dual intuitionistic logic" has been investigated to varying degrees of success through different perspectives. McKinsey and Tarski in [8] studied it from an algebraic point of view through the algebraic properties of "closure" or Brouwerian algebras, the algebraic duals to Heyting algebras.[1] Curry in [2] presented what he called "Skolem lattices", which comprise "absolute implicational lattices" and "absolute subtractive lattices". Rauszer (cf. [14], [15]) used algebraic, Hilbert-style and relational methods, but not Gentzen calculi, to investigate "intuitionistic logic with dual operators", "pseudo-difference" being the dual to intuitionistic implication.

From then the problem of dualizing intuitionistic logic has been carried out mostly in a proof-theoretic setting. Czermak in [3] investigated dual intuitionistic logic by restricting Gentzen's LK to "singletons (at most) on the left", which is the natural dual notion to Gentzen's "singletons (at most) on the right" restriction for LJ. Goodman in [5] used Brouwerian algebras to investigate the "logic of contradictions", with special emphasis on the appropriate notion of a conditional for dual intuitionistic logic. He also gave a sequent calculus for his

*I wish to thank Prof. Chris Mortensen, Claudia Olmedo-García, Prof. Ivonne Pallares-Vega as well as an anonymous referee for their helpful comments and suggestions. Diagrams were drawn using Paul Taylor's diagrams package, v. 3.92.

[1] It is worth noting that they did that just for dualize them and study algebraic models of intuitionistic logic.

logic but did not investigate cut-elimination at all. Urbas [19] highlights several deficiencies of Goodman's and Czermak's analyses and defines several Gentzen calculi with the "singletons on the left" restriction, but added rules for incorporating within the same logic both implication and its dual and also proved cut-elimination for his logics. Goré's work [6] makes some remarks on Urbas' proposals. Thus, the issue of dualizing LJ is far from settled and occasionally new subtleties to deal with appear.

In [11, Ch. 11], Chris Mortensen, jointly with Peter Lavers, studies dual intuitionistic logic by dualizing topos internal logic, which is an interesting approach to the issue of dualizing intuitionistic logic, although he did not study a sequent calculus. In this paper I present a sequent calculus for complement-topos logic, which throws some light on the dualization of LJ. In order to make the paper as self-contained as possible, sections 2 and 3 are a fair bit of survey. First, in section 2 I will present the essentials of the topos-theoretical characterization of logical connectives.[2] In section 3 I will describe briefly the theory of complement-topoi, where logical connectives are characterized dually w.r.t "standard" topoi, and present a concrete example, namely the dualization in the category of graphs. Finally, in section 4 I will present a sequent calculus for complement-topoi internal logic as well as one dual to LJ, which I compare with other proposals of dualization of intuitionistic logic, especially Urbas'. My aims are to show that Urbas' criticisms to Goodman's and Czermak's attempts of dualization are not conclusive, and to show that the dualization of the internal logic of a topos, based on a deep topological duality, favors dualizations like that of Goodman.

2 LOGICAL CONNECTIVES IN TOPOS THEORY

An (elementary) topos is a category with initial (0) and terminal (1) objects, pullbacks, pushouts, exponentiation, and a *subobject classifier*, which is an object Ω together with a morphism true such that for every monic m there is a unique morphism χ_m which makes the following diagram a pullback:

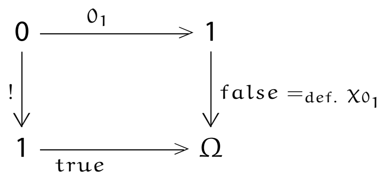
$$\begin{array}{ccc} S & \overset{m}{\rightarrowtail} & A \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \chi_m} \\ 1 & \underset{\mathsf{true}}{\longrightarrow} & \Omega \end{array}$$

χ_m is called "the characteristic or classifying morphism of m" and $!$ is the unique morphism from S to 1 in $\mathcal{C}$. A subobject classifier is unique up to

[2] I assume the reader is familiar with the essentials of category theory. One of the best introductions is [9].

isomorphism, and so the morphism χ_m.[3] *Propositions* or *truth-values* are morphisms $\varphi : 1 \longrightarrow \Omega$.

$\mathsf{false} : 1 \longrightarrow \Omega$ is the character of the initial object 0:



where 0_1 is the only morphism from an initial object to a terminal one.

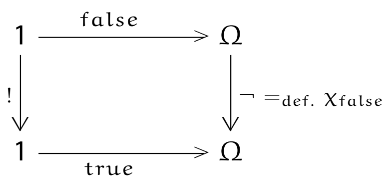
Any proposition φ has the following properties

$$\mathsf{false} \leqslant \varphi \qquad \varphi \leqslant \mathsf{true}$$

(if '$\leqslant$' is interpreted as a deducibility relation then the above properties say that every proposition is deducible from false, and that true is deducible from every proposition).

In a topos, a n-ary connective k is a morphism $k : \Omega^n \longrightarrow \Omega$. I will consider just the more usual connectives, three binary ($\wedge$, conjunction; $\vee$, disjunction; $\Rightarrow$, implication) and a unary one ($\neg$, negation) defined as follows:

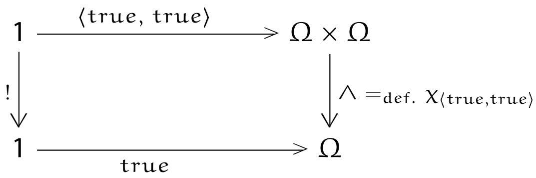
Negation. Let be $\mathsf{false} : 1 \longrightarrow \Omega$. Then $\neg : \Omega \longrightarrow \Omega$ is the only morphism that makes the following diagram a pullback:



That is, $\neg$ is the characteristic morphism of false. The negation of a proposition φ, denoted $\neg\varphi$, is defined as the composition $\neg \circ \varphi : 1 \longrightarrow \Omega$. This definition implies that $\neg \circ \varphi$ is the same morphism as true if and only if $\varphi : 1 \longrightarrow \Omega$ is the same morphism as false, said more briefly, $\neg\varphi = \mathsf{true}$ if and only if $\varphi = \mathsf{false}$.

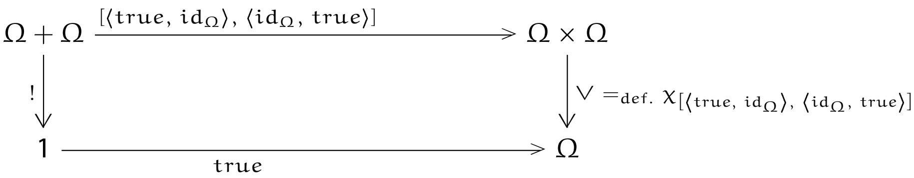
Conjunction. Conjunction $\wedge : \Omega \times \Omega \longrightarrow \Omega$ is defined as the characteristic morphism of $\langle \mathsf{true}, \mathsf{true} \rangle : 1 \longrightarrow \Omega \times \Omega$, i.e. $\wedge : \Omega \times \Omega \longrightarrow \Omega$ is the only morphism that makes the following diagram a pullback:

[3] In the axiomatization of topoi, the requirement of the existence of such subobject classifier is usually called "Ω-axiom".



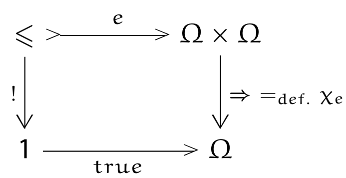
Given two propositions $\varphi : 1 \longrightarrow \Omega$ and $\psi : 1 \longrightarrow \Omega$ in a topos, the conjunction of φ and ψ, denoted $\varphi \wedge \psi$, is defined as $\wedge \circ \langle \mathtt{true}, \mathtt{true} \rangle : 1 \longrightarrow \Omega$. This implies that $\varphi \wedge \psi$ is the same morphism as $\mathtt{true}$ if and only if both $\varphi = \mathtt{true}$ and $\psi = \mathtt{true}$. Said more briefly, $(\varphi \wedge \psi) = \mathtt{true}$ if and only if $\varphi = \mathtt{true}$ and $\psi = \mathtt{true}$.

Disjunction. Disjunction $\vee : \Omega \times \Omega \longrightarrow \Omega$ is defined as the characteristic morphism of $[\langle \mathtt{true}, \mathtt{id}_\Omega \rangle, \langle \mathtt{id}_\Omega, \mathtt{true} \rangle]$, i.e. $\vee : \Omega \times \Omega \longrightarrow \Omega$ is the only morphism that makes the following diagram a pullback:



For any propositions φ and ψ, the proposition $\varphi \vee \psi$ is the composite morphism $\vee \circ \langle \varphi, \psi \rangle : 1 \longrightarrow \Omega$, where $\langle \varphi, \psi \rangle$ is the usual product morphism $\langle \varphi, \psi \rangle : 1 \longrightarrow \Omega \times \Omega$. Unlike negation or conjunction, a disjunction may be the same morphism as $\mathtt{true}$ in several distinct ways, and the morphism $[\langle \mathtt{true}, \mathtt{id}_\Omega \rangle, \langle \mathtt{id}_\Omega, \mathtt{true} \rangle] : \Omega + \Omega \longrightarrow \Omega \times \Omega$ synthesizes all those cases (i.e. in order to get $\vee \circ \langle \varphi, \psi \rangle = \mathtt{true}$ at least one of φ and ψ must be the same morphism as $\mathtt{true}$).

Implication. Implication $\Rightarrow: \Omega \times \Omega \longrightarrow \Omega$ is defined as the characteristic morphism of $e : \leqslant \longrightarrow \Omega \times \Omega$, the equalizer of $\wedge : \Omega \times \Omega \longrightarrow \Omega$ and the first projection p_1, so it makes the following diagram a pullback:



This definition implies that $(\varphi \Rightarrow \psi) = \mathtt{true}$ if and only if $(\varphi \wedge \psi) = \varphi$ (as can be noted, the equalizer e expresses the condition on the right: It equals conjunction and the first projection). For any propositions φ and ψ, the proposition $\varphi \Rightarrow \psi$ is the composite morphism $\Rightarrow \circ \langle \varphi, \psi \rangle : 1 \longrightarrow \Omega$, where $\langle \varphi, \psi \rangle$

is the usual product morphism $\langle \varphi, \psi \rangle : 1 \longrightarrow \Omega \times \Omega$. As in the case of disjunction, the characterization of implication synthetizes all the cases in which $(\varphi \Rightarrow \psi) = \mathtt{true}$.

The zero-order *internal logic* of a given topos $\mathcal{E}$ is the 'algebra' of morphisms from a terminal object to the subobject classifier, collection of morphisms usually denoted as '$\mathcal{E}[1, \Omega]$'. There is a theorem establishing necessary and sufficient conditions for a proposition φ being the same morphism as $\mathtt{true}$ in a given topos $\mathcal{E}$. Let $\varphi \models_{\mathcal{E}} \psi$ denote that whenever the morphism φ is the same morphism as $\mathtt{true}$ in $\mathcal{E}$, so is ψ ($\models_{\mathcal{E}} \varphi$ means that φ is the same morphism as $\mathtt{true}$ in $\mathcal{E}$), and let $\models_{I}$ be intuitionistic consequence relation. Then the following theorem holds:

THEOREM 1 *For every topos $\mathcal{E}$ and proposition φ, $\models_{\mathcal{E}} \varphi$ if and only if $\models_{I} \varphi$.*

i.e. in general $\mathcal{E}[1, \Omega]$ is a Heyting algebra. Quantifiers can also be dualized, as is showed in [11, p. 112f] but for my present purposes it will suffice to consider just zero-order logic.

Sound rules of inference can be given to characterize topos logic. A *sequent* is an expression $\Gamma : \varphi$, where Γ is a finite (possibly empty) set of formulae and φ is a formula. A sequent is *true* if and only if Γ does imply φ. When a sequent $\Gamma : \varphi$ is true we write

$$\Gamma \vdash \varphi$$

(for example, one has $\neg\neg\varphi \vdash \varphi$ in classical logic, but not in intuitionistic logic, and one has $\varphi, \neg\varphi \vdash \psi$ in both classical and intuitionistic logic, but not in an inconsistency-tolerant logic).

First are the *structural rules*. From the sequent(s) above the line one can infer the one below. An asterisk shows that the sequent below follows from an empty set of assumptions:

$$\text{Trivial sequent:} \quad \frac{*}{\varphi : \varphi}$$

$$\text{True and false:} \quad \frac{*}{: \mathtt{true}} \qquad \frac{*}{\mathtt{false} : \varphi}$$

$$\text{Thinning:} \quad \frac{\Gamma : \varphi}{\Gamma, \psi : \varphi}$$

$$\text{Cut:} \quad \frac{\Gamma, \psi : \varphi \quad \text{and} \quad \Gamma : \psi}{\Gamma : \varphi}$$

There is one reversible *connective rule* for each connective. From the sequent(s) above the double line one can infer the sequent below, and from the one below one can infer the either of the two above:

$$\frac{\Gamma, \varphi : \mathtt{false}}{\overline{\Gamma : \neg\varphi}}$$

$$\frac{\Gamma : \varphi \quad \text{and} \quad \Gamma : \psi}{\Gamma \,:\, \varphi \wedge \psi}$$

$$\frac{\Gamma, \varphi \,:\, \theta \quad \text{and} \quad \Gamma, \psi \,:\, \theta}{\Gamma,\ \varphi \vee \psi \,:\, \theta}$$

$$\frac{\Gamma, \varphi \,:\, \psi}{\Gamma \,:\, \varphi \Rightarrow \psi}$$

It is worth noting that the structural rule for false and the connective rules for negation and disjunction are derived, just as the corresponding morphisms false, $\neg$ and $\vee$ needed the other ones in order to be defined. The proof is straightforward; see [9, Chapter 15].

Colin McLarty has rightly pointed out that the internal logic of a topos coincides with no intuitionistic logic studied before topoi (cf. [9, p. vii] and [10, pp. 153ff]). So, if the internal logic is different to intuitionistic logic, how does its dualization have something to do with the issue of dual intuitionistic logic? The internal logic strikingly resembles intuitionistic logic, indeed there is no difference at the zero-order level. Differences lie at the higher-order level, where traditional intuitionistic principles like the existence property ($\exists x Fx$ is accepted only if for some constant c Fc is accepted) or the disjunction property (accept a disjunctive statement $\varphi \vee \psi$ only if either φ or ψ is accepted) do not hold. Thus, there would be no difference between zero-order dual internal logic and zero-order dual intuitionistic logic. Nonetheless, there is an actual difference in the rules since the internal logic of topoi lacks of some of the structural rules and some connective rules do not match any connective rule in LJ. However, if the internal logic of a topos can be dualized and a collection of rules of inference for characterizing it can be given, it will throw some light on how LJ should be dualized. Mortensen, in a joint work with Lavers, (cf. [11], [12]) has showed that the internal logic of a topos can be presented in a dual manner to the standard one. In the next section I will introduce complement-topoi as suggested by Mortensen and Lavers.

3 COMPLEMENT-TOPOI

A Heyting algebra can be thought of as a distributive lattice, with a bottom element, $\perp$, and an operator, $\Rightarrow$, satisfying the condition

$$(a \wedge b) \leqslant c \;\; \textit{iff} \;\; a \leqslant (b \Rightarrow c)$$

(thus $\perp \Rightarrow \perp$ defines the top element). $\neg a$ is defined as $a \Rightarrow \perp$.

Let $\mathcal{T}$ be a topological space. Then a standard example of a Heyting algebra is the topological Heyting algebra $\langle X, \wedge, \vee, \Rightarrow, \perp \rangle$, where X is the set of open sets in $\mathcal{T}$, $\wedge$ and $\vee$ represent intersection and union, respectively, $\perp$ is $\emptyset$ and

$a \Rightarrow b$ is $(\overline{a} \vee b)^{\circ}$ (overlining denotes complementation and $^{\circ}$ the interior operator of the topology). $\neg a$ is $\overline{a}^{\circ}$.

For finite sets of premises intuitionistic logic is sound and complete with respect to the class of Heyting algebras. That is, $\alpha_1, \ldots, \alpha_n \models \beta$ iff for every homomorphism h into such an algebra, $h(\alpha_1, \ldots, \alpha_n) \leqslant h(\beta)$.

The whole construction can be dualized in a natural way to give rise to an inconsistency-tolerant or paraconsistent logic instead of a paracomplete one. A *dual Heyting algebra*, or *Brouwerian algebra*, is a distributive lattice with a top element, $\top$, and an operation, $-$, which satisfies the following condition:

$$c \leqslant (a \vee b) \ \textit{iff} \ (c - b) \leqslant a$$

(which makes $\top - \top$ the bottom element). $\neg a$ is defined as $\top - a$. If $\mathcal{T}$ is a topological space, then $\langle X, \vee, \wedge, -, \top \rangle$, where X is the set of closed sets in $\mathcal{T}$, again $\wedge$ and $\vee$ represent intersection and union, respectively, $\top$ is the whole space and $a - b$ is $(a \wedge \overline{b})^{c}$ (again overlining denotes complementation and c is the closure operator of the topology). $\neg a$ is $\overline{a}^{c}$.

The logic generated by Brouwerian algebras is dual to intuitionistic logic. In particular, in intuitionistic logic one has $(\alpha \wedge \neg\alpha) \models \beta$ and $\alpha \models \neg\neg\alpha$, but not $\beta \models (\alpha \vee \neg\alpha)$ nor $\neg\neg\alpha \models \alpha$. Thus, in dual intuitionistic logic, or CSL, "closed set logic", as Mortensen ([11], [12]) calls it, one has $\beta \models (\alpha \vee \neg\alpha)$ and $\neg\neg\alpha \models \alpha$, but not $(\alpha \wedge \neg\alpha) \models \beta$ nor $\alpha \models \neg\neg\alpha$. As it is well-known, a logic in which $(\alpha \wedge \neg\alpha) \models \beta$ does not hold is called "inconsistency-tolerant" or, more usually, "paraconsistent".

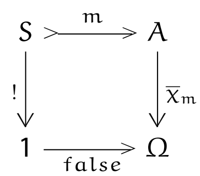
Roughly, to dualize a Heyting algebra any ocurrence of $\leqslant, \geqslant, \wedge, \vee, \top, \bot$ on a formula must be replaced by $\geqslant, \leqslant, \vee, \wedge, \bot, \top$, respectively. For dualizing $\Rightarrow$ things are slightly more complicated. In order to dualize a given formula $a \Rightarrow b$, change the antecedent by the consequent and vice versa and then replace $\Rightarrow$ by $-$.

Mortensen's argument for developing an inconsistency-tolerant approach to category theory does not rest on a sophisticated philosophical position, but in that given that every topological space gives a topos (the category of presheaves on the space), mathematically

> (...) specifying a topological space by its closed sets is as natural as specifying it by its open sets. So it would seem odd that topos theory should be associated with open sets rather than closed sets. Yet this is what would be the case if open set logic were the natural propositional logic of toposes. At any rate, there should be a simple 'topological' transformation of the theory of toposes, which stands to closed sets and their logic [i.e. inconsistency-tolerant], as topos theory does to open sets and intuitionism. [11, p. 102]

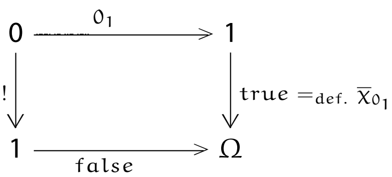
If the duality between intuitionistic and CSL is as deep as topological, then a representation of CSL as the internal logic of a topos should be equally natural. In what follows I expound Mortensen and Lavers's dualization of logical connectives in a topos.[4]

A *complement-classifier* for a category C with terminal object is an object Ω together with a morphism $\mathtt{false}$ such that for every monic m there is a unique morphism $\overline{\chi}_m$ which makes the following diagram a pullback:



$\overline{\chi}$ is called "the complement-characteristic or complement-classifying morphism of m" and ! is the unique morphism from S to 1 in C. A subobject classifier is unique up to isomorphism, and so the morphism $\overline{\chi}$.

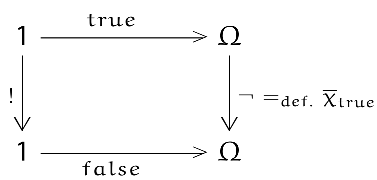
A *complement-topos* is a category with initial (0) and terminal (1) objects, pullbacks, pushouts, exponentiation, and a complement-classifier. Now, the morphism $\mathtt{true} : 1 \longrightarrow \Omega$ is the character of the initial object 0:



where 0_1 is the only morphism from an initial object to a terminal one.

Negation, conjunction, disjunction and implication are dualized as follows[5]:

Negation. Let be $\mathtt{true} : 1 \longrightarrow \Omega$. Then $\neg : \Omega \longrightarrow \Omega$ is the only morphism that makes the following diagram a pullback:



[4] Is is important to set their individual contributions. Of the ten diagrams in [11, Ch. 11], Mortensen drew the first one and the final five, while Lavers drew the remaining four. In terms of this paper, the first diagram of this section is due to Mortensen, those for $\mathtt{true}$, negation, conjunction, and disjunction are due to Lavers. The diagram for dual-implication never was explicitly drawn, but it was discussed in [11, p. 109].

[5] Intuitively, the usual category-theoretic characterization of connectives says when a proposition is true, while the dual says when it is false.

That is, $\neg$ is the complement-characteristic morphism of $\mathtt{true}$. The negation of a proposition φ, denoted $\neg\varphi$, is defined as the composition $\neg \circ \varphi : 1 \longrightarrow \Omega$. This definition implies that $\neg \circ \varphi$ is the same morphism as $\mathtt{false}$ if and only if $\varphi : 1 \longrightarrow \Omega$ is the same morphism as $\mathtt{true}$, said more briefly, $\neg\varphi = \mathtt{false}$ if and only if $\varphi = \mathtt{true}$.

Disjunction. Disjunction $\vee : \Omega \times \Omega \longrightarrow \Omega$ is defined as the complement-characteristic morphism of $\langle \mathtt{false}, \mathtt{false}\rangle : 1 \longrightarrow \Omega \times \Omega$, i.e. $\vee : \Omega \times \Omega \longrightarrow \Omega$ is the only morphism that makes the following diagram a pullback:

$$\begin{array}{ccc} 1 & \xrightarrow{\langle \mathtt{false},\ \mathtt{false}\rangle} & \Omega \times \Omega \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \vee\ =_{\text{def.}}\ \overline{\chi}_{\langle \mathtt{false},\mathtt{false}\rangle}} \\ 1 & \xrightarrow[\mathtt{false}]{} & \Omega \end{array}$$

Given two propositions $\varphi : 1 \longrightarrow \Omega$ and $\psi : 1 \longrightarrow \Omega$ in a topos, the disjunction of φ and ψ, denoted $\varphi \vee \psi$, is defined as $\vee \circ \langle \mathtt{false}, \mathtt{false}\rangle : 1 \longrightarrow \Omega$. This implies that $\varphi \vee \psi$ is the same morphism as $\mathtt{false}$ if and only if both $\varphi = \mathtt{false}$ and $\psi = \mathtt{false}$. Said more briefly, $(\varphi \vee \psi) = \mathtt{false}$ if and only if $\varphi = \mathtt{false}$ and $\psi = \mathtt{false}$. Compare with Set, where the complement of $\{(0,0)\}$ in 2×2 is $\{(1,1), (1,0), (0,1)\}$.

Conjunction. Conjunction $\wedge : \Omega \times \Omega \longrightarrow \Omega$ is defined as the complement-characteristic morphism of $[\langle \mathtt{false},\ \mathtt{id}_\Omega\rangle, \langle \mathtt{id}_\Omega,\ \mathtt{false}\rangle]$, i.e. $\wedge : \Omega \times \Omega \longrightarrow \Omega$ is the only morphism that makes the following diagram a pullback:

$$\begin{array}{ccc} \Omega + \Omega & \xrightarrow{[\langle \mathtt{false},\ \mathtt{id}_\Omega\rangle,\ \langle \mathtt{id}_\Omega,\ \mathtt{false}\rangle]} & \Omega \times \Omega \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \wedge\ =_{\text{def.}}\ \overline{\chi}_{[\langle \mathtt{false},\ \mathtt{id}_\Omega\rangle,\ \langle \mathtt{id}_\Omega,\ \mathtt{false}\rangle]}} \\ 1 & \xrightarrow[\mathtt{false}]{} & \Omega \end{array}$$

A conjunction may be the same morphism as $\mathtt{false}$ in several ways, unlike negation or disjunction and the morphism

$$[\langle \mathtt{false},\ \mathtt{id}_\Omega\rangle, \langle \mathtt{id}_\Omega,\ \mathtt{false}\rangle] : \Omega + \Omega \longrightarrow \Omega \times \Omega$$

synthetizes all those cases. The composition

$$\wedge \circ [\langle \mathtt{false},\ \mathtt{id}_\Omega\rangle, \langle \mathtt{id}_\Omega,\ \mathtt{false}\rangle] : \Omega + \Omega \longrightarrow \Omega$$

expresses in a condensed form all those combinations of propositions φ, ψ such that their conjunction is the same morphism as $\mathtt{false}$ (for a conjunction being the same morphism as $\mathtt{false}$ it suffices that at least one of φ or ψ be the same morphism as $\mathtt{false}$). Compare again with Set, where the complement of conjunction is $\{(1,0),(0,1),(0,0)\}$.

Dual-implication, *sustraction* or *pseudo-difference*. Pseudo-difference

$$- : \Omega \times \Omega \longrightarrow \Omega$$

is defined as the characteristic morphism of $\overline{e} : \geqslant \longrightarrow \Omega \times \Omega$, where $\overline{e}$ is the equalizer of $\vee : \Omega \times \Omega \longrightarrow \Omega$ and the first projection p_1, so it makes the following diagram a pullback:

$$\begin{array}{ccc} \geqslant & \overset{\overline{e}}{\rightarrowtail} & \Omega \times \Omega \\ {\scriptstyle !}\downarrow & & \downarrow {\scriptstyle - =_{def.} \overline{\chi}_e} \\ 1 & \underset{\mathtt{false}}{\longrightarrow} & \Omega \end{array}$$

This is the right topos-theoretic dualization of the definition of implication given above. Remember that $(\varphi \Rightarrow \psi) = \mathtt{true}$ if and only if $(\varphi \wedge \psi) = \varphi$, and it is dualized $(\psi - \varphi) = \mathtt{false}$ if and only if $(\varphi \vee \psi) = \varphi$, which is expressed by the pullback above.

Dualization of implication is a delicate matter, though. To begin with, implication in usual topos theory may be defined in several ways, for example by considering it the characteristic morphism of $e' : \leqslant \longrightarrow \Omega \times \Omega$, the equalizer of disjunction and the second projection, which would lead to several different dualizations. Goodman proved that in CSL no connective definable in terms of the connectives $\wedge, \vee, \neg, -$ has $\top$ as semantic assignment iff the assignment of its antecedent is less or equal than the assignment of its consequent. Some authors argue that a connective like $-$ cannot be regarded as an implication at all, since $(\varphi - \varphi) = \mathtt{false}$ for every φ and does not satisfy *modus ponens*, that CSL lacks of a reasonable implication and therefore it is not a serious logic and much less a logic strong enough for developing some mathematics based on it. Certainly $-$ might not be regarded as an implication, in the same way that the dual of conjunction is not even a kind of conjunction. $-$ should be regarded rather as an "anti-implication", as Popper once suggested (cf. [13]).

Mortensen has argued against this alleged deficiency of CSL. He points out that it is not clear how much of mathematics depends on an object-language implication. What mathematics needs, he says, is a deducibility relation, but that is provided by ordering and an adequate proof theory; Goodman himself proved that derivability in CSL respects the natural semantic ordering of set inclusion.[6] Moreover, nothing in the above rules out the possibility of defining a reasonable implication in CSL or in complement-topos theory. That an

[6]This is a controversial point. Mortensen thinks that functionality is mathematically prior to, and a more important matter than some logical notions. Someone might object to this by saying that ordinary math books use implication constantly, and for example use definitions

implication cannot be defined in terms of the other connectives is not a strong argument; after all connectives in, e.g. intuitionistic logic are not interdefinable and it is not thought of as defective. Mortensen proposed an implication for complement-topoi which, however, should not be regarded necessarily as a dualization of usual implication, but a more general case. I will discuss it in the following section.

It is clear that if $\mathcal{E}$ is a topos and $\mathcal{E}'$ is the category obtained by renaming $\mathtt{true}$ as $\mathtt{false}$ and defining dually the connectives then $\mathcal{E}'$ is again a topos, since terminal and initial objects, pullbacks, pushouts, and exponentials are notions and constructions prior to the characterization of subobject classifiers and connectives. Moreover, Mortensen proved the following

THEOREM 2 (Duality Theorem) *S is true in $\mathcal{E}$ if and only if S' is true in $\mathcal{E}'$.*

Let me present a simple example on how the dualization works in a concrete topos. $S^{\downarrow\downarrow}$ is the category of (irreflexive directed multi-)***graphs***.[7] Its subobject classifier $\Omega_{S^{\downarrow\downarrow}}$ has three truth-values with the order $0_A \prec \binom{s}{t} \prec A$ (0_A and A are $\mathtt{false}$ and $\mathtt{true}$ in $S^{\downarrow\downarrow}$, respectively) and negation gives the following identities of morphisms:

$$\neg A = 0_A, \quad \neg\binom{s}{t} = 0_A, \quad \neg 0_A = A$$

Since $(\varphi \Rightarrow \psi) = \mathtt{true}$ if and only if $(\varphi \wedge \psi) = \varphi$, in general $(\neg\neg\varphi \Rightarrow \varphi) \neq A$ in $S^{\downarrow\downarrow}$ because even though $(\neg\neg\varphi \Rightarrow \varphi) = A$ when $\varphi = A$ or $\varphi = 0_A$, $(\neg\neg\varphi \wedge \varphi) \neq \neg\neg\varphi$ when $\varphi = \binom{s}{t}$. Given that $(\neg\neg\varphi \Rightarrow \varphi) \neq A$ but there is no formula Φ such that $\Phi = \mathtt{true}$ in classical logic and $\Phi = \mathtt{false}$ in intuitionistic logic, $(\neg\neg\varphi \Rightarrow \varphi) = \binom{s}{t}$ when $\varphi = \binom{s}{t}$. Moreover, $\varphi \vee \neg\varphi$ fails to be the same morphism as A, since $(\varphi \vee \psi) = \mathtt{true}$ if and only if either $\varphi = \mathtt{true}$ or $\psi = \mathtt{true}$. If $\varphi = \binom{s}{t}$, $\neg\varphi = 0_A$, so neither $\varphi = A$ nor $\neg\varphi = A$ and hence $(\varphi \vee \neg\varphi) \neq A$. Thus, the internal logic of $S^{\downarrow\downarrow}$ is not classical.

Complement-$S^{\downarrow\downarrow}$ has the same three truth-values with its original order, but (complement) negation gives now the following identities of morphisms:

$$\neg 0_A = A, \quad \neg\binom{s}{t} = A, \quad \neg A = 0_A$$

In $S^{\downarrow\downarrow}$ one has $(\varphi \vee \neg\varphi) \neq A$, and dualizing one obtains $(\varphi \wedge \neg\varphi) \neq 0_A$. Remember that in a complement-topos $(\varphi \wedge \psi) = \mathtt{false}$ if and only if either $\varphi = \mathtt{false}$ or $\psi = \mathtt{false}$. If $\varphi = \binom{s}{t}$, $\neg\varphi = A$, so neither $\neg\varphi = 0_A$ nor $\varphi \neq 0_A$ and hence $(\varphi \wedge \neg\varphi) \neq 0_A$. Besides, in a Heyting algebra (like the algebra $\mathcal{E}[1, \Omega]$) in general it is not the case that $\psi \leqslant (\varphi \vee \neg\varphi)$, which dualized gives that in general it is not the case that $(\varphi \wedge \neg\varphi) \leqslant \psi$. So, the internal logic of complement-$S^{\downarrow\downarrow}$ is not classical, but inconsistency-tolerant. Moreover, in

stated as conditionals so that one constantly quantifies into conditional contexts, and every time one proves some object does not have a defined property one is negating a conditional. Seemingly this cannot all be pushed into the metalanguage without severe contortions.

[7] Nice introductions to this category can be found in [21] and [7].

complement-$S^{\downarrow\downarrow}$ both $\varphi \vee \neg\varphi$ and $\neg(\varphi \wedge \neg\varphi)$ are the same morphism as A. In $S^{\downarrow\downarrow}$ $(\varphi \wedge \neg\varphi) = 0_A$, which dualized is $(\varphi \vee \neg\varphi) = A$. In $S^{\downarrow\downarrow}$ $\neg(\varphi \vee \neg\varphi) = 0_A$ (since in intuitionistic logic the negation of a classical theorem is always false), and dualization gives $\neg(\varphi \wedge \neg\varphi) = A$.

Since classical logic is its own dual, the internal logic of e.g. Set is not modified by this dualization, so the complement-classifier is indistinguishable (via categorial methods) from a standard subobject classifier. Thus, as Vasyukov ([20] p. 292) points out: "(...) in Set we always have paraconsistency because of the presence of both types of subobject classifiers (...)" just as we always have in it (at least) intuitionistic logic.[8]

4 Rules of inference for complement-topos logic and the issue of dual intuitionistic logic

Just as the dualization of connectives in a topos, the rules for complement-topoi logic also mirrors the "topologico-algebraic" dualization. That is, any ocurrence of $\wedge$, $\vee$, $\mathtt{true}$, $\mathtt{false}$ on a formula must be replaced by $\vee$, $\wedge$, $\mathtt{false}$, $\mathtt{true}$, respectively. To dualize a given formula $a \Rightarrow b$, replace the antecedent by the consequent and *vice versa* and then replace $\Rightarrow$ by $-$. Since $a \leqslant b$ can be interpreted as a sequent $a : b$ and the dual of $a \leqslant b$ is $b \leqslant a$ (or $a \geqslant b$), the dual of $a : b$ is $b : a$.[9] Thus, the corresponding rules for complement-topoi logic are the following ones:

Structural Rules:

Trivial sequent: $$\frac{*}{\varphi : \varphi}$$

False and true: $$\frac{*}{\mathtt{false} :} \qquad \frac{*}{\varphi : \mathtt{true}}$$

Thinning: $$\frac{\varphi : \Gamma}{\varphi : \Gamma, \psi}$$

Cut: $$\frac{\varphi : \Gamma, \psi \ \text{ and } \ \psi : \Gamma}{\varphi : \Gamma}$$

Connective rules:

$$\frac{\mathtt{true} : \Gamma, \varphi}{\neg\varphi : \Gamma}$$ (double line)

[8] The presence of paraconsistency within classical logic is not news. See for example [1] where some paraconsistent negations in S5 and first-order classical logic are defined.

[9] Such dualization is (the zero-order part of) the mapping $\star$ discussed in [19, p. 444], which builds upon one described in [3].

$$\frac{\varphi : \Gamma \quad \texttt{and} \quad \psi : \Gamma}{\varphi \vee \psi : \Gamma}$$

$$\frac{\theta : \Gamma, \varphi \quad \texttt{and} \quad \theta : \Gamma, \psi}{\theta : \Gamma, \varphi \wedge \psi}$$

$$\frac{\psi : \Gamma, \varphi}{\psi - \varphi : \Gamma}$$

Dually to the case of topoi, the structural rule for true and the connective rules for negation and conjunction are derived, just as the corresponding morphisms true, $\neg$ and $\wedge$ needed false and $\vee$ in order to be defined.

Finally, let us compare this dualization with other attempts, especially that of Urbas (cf. [19]). Urbas tries to "rectify" previous works by Czermak and Goodman on the formulation of dual intuitionistic logic. These authors offered sequent calculi which are (at most) singular in the left, but neither uses Gentzen's original connectives (negation, conjunction, disjunction and implication). Czermak's logic lacks dual rules for implication, while Goodman did not use negation and employs pseudo-difference and the constant $\top$, and for this Urbas says that "[c]onsequently, it is not immediately clear in what exact sense each is dual to intuitionistic logic LJ." His proposal of dualization of LJ is the following set of rules, which Urbas calls "LDJ":

Structural Rules

Identity $\dfrac{*}{\varphi : \varphi}$

Thinning : $\dfrac{: \Gamma}{\varphi : \Gamma}$: Thinning $\dfrac{\varphi : \Gamma}{\varphi : \Gamma, \psi}$

: Exchange $\dfrac{\varphi : \Gamma, \psi, \theta, \Delta}{\varphi : \Gamma, \theta, \psi, \Delta}$

: Contraction $\dfrac{\varphi : \Gamma, \psi, \psi}{\varphi : \Gamma, \psi}$

Cut $\dfrac{\varphi : \Gamma, \psi \quad \texttt{and} \quad \psi : \Delta}{\varphi : \Gamma, \Delta}$

Connective rules

$$\neg : \quad \frac{\ : \Gamma, \varphi}{\neg\varphi \ : \ \Gamma} \qquad : \neg \quad \frac{\varphi \ : \ \Gamma}{\ : \ \Gamma, \neg\varphi}$$

$$\wedge : \quad \frac{\psi \ : \ \Gamma}{\varphi \wedge \psi \ : \ \Gamma} \quad \frac{\varphi \ : \ \Gamma}{\varphi \wedge \psi \ : \ \Gamma} \qquad : \wedge \quad \frac{\theta \ : \Gamma, \varphi \ \texttt{ and } \ \theta \ : \Gamma, \psi}{\theta \ : \ \Gamma, \ \varphi \wedge \psi}$$

$$\vee : \quad \frac{\varphi : \Gamma \ \texttt{ and } \ \psi : \Gamma}{\varphi \vee \psi \ : \ \Gamma} \qquad : \vee \quad \frac{\theta \ : \ \Gamma, \ \varphi}{\theta \ : \ \Gamma, \ \varphi \vee \psi} \quad \frac{\theta \ : \ \Gamma, \ \psi}{\theta \ : \ \Gamma, \ \varphi \vee \psi}$$

$$\Rightarrow : \quad \frac{: \Gamma, \ \varphi \ \texttt{ and } \ \psi : \Delta}{\varphi \Rightarrow \psi \ : \ \Gamma, \ \Delta} \qquad : \Rightarrow \quad \frac{\varphi \ : \ \Gamma}{\ : \ \Gamma, \ \varphi \Rightarrow \psi} \quad \frac{\theta \ : \ \Gamma, \ \psi}{\theta \ : \ \Gamma, \ \varphi \Rightarrow \psi}$$

All these rules satisfy the "singletons in the left" restriction, a necessary condition to dualize intuitionistic logic. Urbas says:

> Most of the rules of LDJ require no comment, as they are simply the result of restricting the rules of Gentzen's classical sequent system LK to being (at most) singular in the antecedent. The only exception is the pair of rules [(:⇒)]. [19, p. 442]

I think that one of the most natural ways to obtain a dual intuitionistic logic is by means of the "topologico-algebraic dualization" mentioned above. Urbas himself regards that dualization as "(...) a more precise way of establishing the correspondences between intuitionistic and dual-intuitionistic logics and their fragments."[19, p. 445] Nonetheless, he defends LDJ as a more accurate dualization and gives three main reasons for that:

1. First, LDJ can be regarded as dual, he says, since it satisfies the "singletons (at most) in the antecedent" restriction;
2. it is more accurate in that a dual Glivenko theorem holds for it[10], and
3. it is more accurate for it is formulated with the same connectives as LJ.

But the topologico-algebraic dualization does provide an account of both why and how to obtain those particular dual rules as well as an account of the restriction itself: The singletons in the left restriction arises since the dual of an intuitionistically admissible sequent $\Gamma : \Delta$, which always will be at most singleton in the right, is $\Delta : \Gamma$, which in its turn will always be at most singleton in the left. On the other hand, there seems to be nothing in the notion of dualization which forces the dual of a connective to be one of the same kind, i.e. nothing in the notion of dualization seems to force, for example, a dual conjunction be also a conjunction or a dual implication an implication. Finally, the

[10]That is, LDJ shares all sentential theorems but not counter-theorems with LK.

fact that LDJ satisfies the dual Glivenko property of having the same sentential theorems as LK does not make it better than other dualizations, since it can be proved that they have properties that can be also regarded as dual Glivenko properties; see for example [6]. Now, let us take a closer look at the *desideratum* "you shall dualize using the same connectives".

Urbas criticizes Goodman's and Czermak's proposals because it is not immediately clear in what exact sense each is dual to intuitionistic logic LJ." But Urbas' LDJ also lacks "immediate clarity" as dual to LJ on looking carefully at the rule $:\Rightarrow$, since according to the method employed with the other connectives it should be

$$:\Rightarrow \quad \frac{\varphi \ : \ \Gamma,\ \psi}{:\ \Gamma,\ \varphi \Rightarrow \psi}$$

and not the more complex pair of Urbas' $:\Rightarrow$. Urbas tries to explain his formulation by appealling to the differences of deductive strength whether "Ketonen" or "non-Ketonen" rules are used. A rule for the introduction of a given binary connective is called *non-Ketonen* if above the line appears just one of the formulae connected. Classical logic LK can be formulated indifferently using Ketonen or non-Ketonen rules for introducing conjunction in the antecedent and disjunction or conjunction in the consequent. The difference becomes crucial only when, as in LJ and its dual, restrictions to singularity are considered. "Accordingly, Gentzen formulated LK using [non-Ketonen] so as to be able to obtain LJ by imposing this restriction." For example, only non-Ketonen rules for introducing disjunction in the consequent can be restricted to singularity as required for intuitionistic logic. Similarly, only non-Ketonen rules for introducing conjunction in the antecedent can be restricted to singularity as required for LDJ. "(...) happily, Gentzen also used these rules in formulating LK."[19, p. 442]

But Gentzen did not use non-Ketonen rules for introducing implication, since the usual Ketonen rules for implication are easily restricted to singularity in the consequent as required in order to formulate intuitionistic logic, and are also easily restricted to singularity in the consequent as required in order to formulate dual intuitionistic logic. Why the usage of the same connectives is required, but not of the same kind of rules for each connective? Urbas claims that dualization is involved here too. LJ is formulated using a Ketonen rule for implication, and the non-Ketonen rules can be derived. Nonetheless, if LJ is formulated using the non-Ketonen rules then the Ketonen rule cannot be derived. Dually, if LDJ is formulated using the non-Ketonen rules for implication, the Ketonen rules can be derived, but if LDJ is formulated using the Ketonen rule the non-Ketonen rules cannot be derived. Thus, the use of non-Ketonen rules for implication is a matter of deductive strength. But all this is too much problematic for a defense of LDJ as a more accurate and immediately clearer dualization than Czermak's and Goodman's. Urbas does not provide a single account of how dualization is going to be done. He says that the "singletons (at most) in the left" restriction in LK dualizes LJ, but he does not merely apply this method to implication. He asks for usage of Gentzen's original connectives and he is happy with Gentzen's use of non-Ketonen rules for conjunction, which facilitates dualization, but he is lead to alter Gentzen's presentation since dualizing the original Ketonen rule for implication gives lesser deductive power to dual intuitionistic logic than expected. Moreover, Urbas does not provide any reason in order to connect dualization, in one hand, and the "necessity" to modify the kind of rule for implication, on the other. I do not claim that LDJ is not the correct dualization of LJ, if such a thing exists. I am only claiming that Urbas' arguments to this effect are not very compelling, and much less as criticisms to previous attempts.

The topologico-algebraic dualization underlying the dualization of the internal logic of a topos gives an important hint on how to dualize clearly and directly LJ. Following that dualization, dual LJ would be especified through the following rules:

Structural Rules

$$\text{Identity} \quad \frac{*}{\varphi\ :\ \varphi}$$

$$\text{Thinning :} \quad \frac{:\ \Gamma}{\varphi\ :\ \Gamma} \qquad :\text{Thinning} \quad \frac{\varphi\ :\ \Gamma}{\varphi\ :\ \Gamma,\ \psi}$$

$$:\text{Exchange} \quad \frac{\varphi\ :\ \Gamma,\ \psi,\ \theta,\ \Delta}{\varphi\ :\ \Gamma,\ \theta,\ \psi,\ \Delta}$$

$$:\text{Contraction} \quad \frac{\varphi\ :\ \Gamma,\ \psi,\ \psi}{\varphi\ :\ \Gamma,\ \psi}$$

$$\text{Cut} \quad \frac{\varphi\ :\Gamma,\ \psi \quad \texttt{and} \quad \psi\ :\Delta}{\varphi\ :\ \Gamma,\ \Delta}$$

Connective rules

$$\neg : \quad \frac{:\ \Gamma, \varphi}{\neg\varphi\ :\ \Gamma} \qquad :\neg \quad \frac{\varphi\ :\ \Gamma}{:\ \Gamma,\ \neg\varphi}$$

$$\vee : \quad \frac{\varphi : \Gamma \quad \texttt{and} \quad \psi : \Gamma}{\varphi \vee \psi\ :\ \Gamma} \qquad :\vee \quad \frac{\theta\ :\ \Gamma,\ \varphi}{\theta\ :\ \Gamma,\ \varphi \vee \psi} \quad \frac{\theta\ :\ \Gamma,\ \psi}{\theta\ :\ \Gamma,\ \varphi \vee \psi}$$

$$\wedge : \quad \frac{\psi\ :\ \Gamma}{\varphi \wedge \psi\ :\ \Gamma} \quad \frac{\varphi\ :\ \Gamma}{\varphi \wedge \psi\ :\ \Gamma} \qquad :\wedge \quad \frac{\theta\ :\Gamma, \varphi \quad \texttt{and} \quad \theta\ :\Gamma, \psi}{\theta\ :\ \Gamma,\ \varphi \wedge \psi}$$

$$-:\quad \frac{\psi\ :\ \varphi,\ \Gamma}{\psi - \varphi\ :\ \Gamma} \qquad :-\quad \frac{\varphi : \Gamma,\ \ \text{and}\ \ \theta : \psi,\ \Delta}{\theta\ :\ \psi - \varphi,\ \Gamma,\ \Delta}$$

It remains problematic whether dual intuitionistic logic has a reasonable implication. The evidence is overwhelming against the existence of an implication in dual intuitionistic logic, but no argument seems to be conclusive. However, I think that this issue has received more attention than it deserves. As Mortensen has pointed out, the deducibility relation, so important for a logic, is provided in this case by the proof theory and/or by an order relation. Nonetheless, Mortensen thinks that there is a "simple and reasonable" implication on any (bounded) lattice, namely $(\varphi \Rightarrow_M \psi) = \mathtt{true}$ if $\varphi \leqslant \psi$ and $(\varphi \Rightarrow_M \psi) = \mathtt{false}$ otherwise, and gives a categorial representation of it (cf. [11, pp. 10ff]). This implication is a different connective to standard implication in a topos. As is well known, $\neg\neg\varphi \Rightarrow \varphi$ is not an intuitionistic theorem. Let us consider again the category $S^{\downarrow\downarrow}$ of (irreflexive directed multi-)*graphs*. Since $\neg\neg\varphi$ is not generally lesser or equal than φ, $(\neg\neg\varphi \Rightarrow \varphi) = \mathtt{false}$. But this cannot be, since there is no (zero-order) formula Φ such that $\Phi = \mathtt{true}$ in classical logic and $\Phi = \mathtt{false}$ in intuitionistic logic. So, standard topos implication is not Mortensen's S5-style implication. Moreover, there seems to be other reasonable implications on any lattice, for example

$(\varphi \Rightarrow_A \psi) = \mathtt{false}$ iff φ is greater than ψ;

$(\varphi \Rightarrow_B \psi) = \mathtt{false}$ iff $\varphi = \mathtt{true}$ and $\psi = \mathtt{false}$.

I leave the discussion of these implications and their categorial description as well as a more detailed discussion of implication in dual intuitionistic logic for further work.

5 CONCLUSIONS

I have expounded the essentials of theory of complement-topoi, whose internal logic is dual to that of standard topoi. I introduced a sequent calculus for complement-topos logic, which follows Mortensen's topologico-algebraic inspired dualization of truth morphisms. Even though the sequent calculus corresponding to the internal logic of a topos is not exactly LJ, its dualization can be applied straightforwardly to LJ. On this ground, I examined Urbas' arguments against other attempts of dualization of LJ, quite similar to the topologico-algebraic based dualization expounded here, and I showed that Urbas' LDJ has not clear advantages over the proposals he criticized.

There are several open problems concerning dualization in topos theory, but there are two which deserve especial attention. First, even though truth-morphisms, including quantifiers, can be dualized, there is no obvious way on how does it sit with the Cartesian closedness of a topos, i.e. it seems that both type theory and higher-order logic remain unchanged for complement-topoi since Cartesian closedness is independent of the classifier and dualization. Second, there seems to be other reasonable implications besides the standard one present in any topos, be it a standard or a complement-topos. What is their categorial representation, through diagrams for example?

References

[1] Jean-Yves Béziau. S5 is a Paraconsistent Logic and so is First-Order Classical Logic. *Logical Studies*, 9:301–309, 2002.

[2] Herbert B. Curry. *Foundations of Mathematical Logic*. Dover, New York, 1976.

[3] J. Czermak. A Remark on Gentzen's Calculus of Sequents. *Notre Dame Journal of Formal Logic*, 18(3):471–474, 1977.

[4] Robert Goldblatt. *Topoi: The Categorial Analysis of Logic*. North-Holland, The Netherlands, 1984. Revised edition.

[5] Nicolas Goodman. The Logic of Contradiction. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 27(8–10):119–126, 1981.

[6] R. Goré. Dual Intuitionistic Logic Revisited. *Proceedings of the International Conference on Automated Reasoning with Analytic Tableaux and Related Methods*, Lecture Notes In Computer Science, Vol. 1847, Springer-Verlag, London, 252–267, 2000.

[7] William Lawvere and Stephen Schanuel. *Conceptual Mathematics: A First Introduction to Categories* Cambridge University Press, Cambridge, 1997.

[8] J. C. C. McKinsey and Alfred Tarski. On Closed Elements in Closure Algebras. *Annals of Mathematics*, 47(2):122–162, 1946.

[9] Colin McLarty. *Elementary Categories, Elementary Toposes*. Oxford Clarendon Press, Toronto, 1988.

[10] Colin McLarty. Review of Bell's *Toposes and Local Set Theories*. *Notre Dame Journal of Formal Logic*, 31(1):150–161, 1990.

[11] Chris Mortensen. *Inconsistent Mathematics*. Kluwer Mathematics and Its Applications Series, Kluwer, Dordrecht.

[12] Chris Mortensen. Closed Set Logic. In R. T. Brady, editor, *Relevant Logics and Their Rivals, Vol. II*, pages 254–262. Ashgate Publishing, Aldershot, 2003.

[13] Karl Popper. On the Theory of Deduction. Parts I and II, *Indagationes Mathematicae*, 10:173–183 and 322–331, 1948.

[14] Cecilia Rauszer. Semi-Boolean Algebras and their Applications to Intuitionistic Logic with Dual Operation. *Fundamenta Mathematicae*, 83:219–249, 1974.

[15] Cecilia Rauszer. An Algebraic and Kripke-Style Approach to a Certain Extension of Intuitionistic Logic. *Dissertationes Mathematicae*, 167:1–62, 1980.

[16] Gonzalo Reyes and M. Zawadowski. Formal Systems for Modal Operators on Locales. *Studia Logica*, 52(4):595–613, 1993.

[17] Gonzalo Reyes and H. Zolfaghari. Topos-Theoretic Approaches to Modality. In Carboni,. A., M. C. Pedicchio, and G. Rosolini, editors, *Proceedings of the 1990 Meeting on Category Theory held in Como, Italy*, Lecture Notes in Mathematics 1488, 359–378, 1991.

[18] Gonzalo Reyes and H. Zolfaghari. Bi-Heyting Algebras, Toposes and Modalities. *Journal of Philosophical Logic*, 25(1):25–43, 1996.

[19] Igor Urbas. Dual Intuitionistic Logic. *Notre Dame Journal of Formal Logic*, 37(3):440–451, 1996.

[20] Vladimir Vasyukov. Structuring the Universe of Universal Logic. *Logica Universalis*, 1(2):277–294, 2007.

[21] Sebastiano Vigna. A Guided Tour in the Topos of Graphs. Technical Report 199-97, Università di Milano, Dipartimento di Scienze dell'Informazione, Milano, 1997.

The *Australasian Journal of Logic* (ISSN 1448-5052) disseminates articles that significantly advance the study of logic, in its mathematical, philosophical or computational guises. The scope of the journal includes all areas of logic, both pure and applied to topics in philosophy, mathematics, computation, linguistics and the other sciences.

Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

The journal is freely available at the journal website at

`http://www.philosophy.unimelb.edu.au/ajl/`.

All issues of the journal are archived electronically at the journal website.

SUBSCRIPTIONS Individuals may subscribe to the journal by sending an email, including a full name, an institutional affiliation and an email address to the managing editor at ajl-editors@unimelb.edu.au. Subscribers will receive email abstracts of accepted papers to an address of their choice. For institutional subscription, please email the managing editor at ajl-editors@unimelb.edu.au.

Complete published papers may be downloaded at the journal's website at `http://www.philosophy.unimelb.edu.au/ajl/`. The journal currently publishes in pdf format.

SUBMISSION The journal accepts submissions of papers electronically. To submit an article for publication, send the LaTeX source of a submission to a member of the editorial board. For a current list of the editorial board, consult the website.

The copyright of each article remains with the author or authors of that article.