Merge

IN HONOUR OF ROBERT K. MEYER.

Chris Mortensen
Department of Philosophy
The University of Adelaide
North Terrace, South Australia 5005
Australia

Chris.Mortensen@adelaide.edu.au

Received by Greg Restall
Published April 29, 2011
http://www.philosophy.unimelb.edu.au/ajl/2011

Abstract: Methods for unifying inconsistent pairs of theories, which we call collectively MERGE, are defined and their properties outlined.

This paper is dedicated to the memory of Bob Meyer, Maximum Leader of the Logician's Liberation League, and one of the great logicians of his generation.

1 INTRODUCTION

Among his many discoveries, Meyer (1976) constructed the first finite inconsistent arithmetic modulo 2, which we will here call RM3mod2. This is an inconsistent theory, for example $0 = 0$ and $\sim 0 = 0$ are theorems. But it is also *nontrivial*, that is not every sentence is a theorem. In particular, $0 = 1$ is not a theorem. It is well-known that Meyer showed that RM3mod2 arithmetic is an extension of his consistent incomplete relevant arithmetic, which he called $R^{\#}$. The # nomenclature refers to the axiomatic specification of the arithmetic by means of Peano's axioms, with R as its background logic. Meyer's argument for the inclusion of $R^{\#}$ in RM3mod2 used ordinary mathematical induction on the length of sentences. That is, his proof used only methods that are finitistically acceptable, in Hilbert's sense. The fact that $0 = 1$ is not a theorem of RM3mod2 likewise does not need a complicated argument, it is seen immediately from the classical integers modulo 2. It follows from these facts that $R^{\#}$ is also non-trivial. But, as Meyer observed, this means that $R^{\#}$ can be shown to be nontrivial by finitistic means that can be represented in $R^{\#}$ itself, despite the fact that the Gödel incompleteness theorems can be proved of $R^\#$. This contrasts with Gödel's second incompleteness theorem for *Boolean* Peano arithmetic, which he called $P^\#$, which shows that the nontriviality of $P^\#$, and equivalently its negation-consistency, cannot be proved by means representable in $P^\#$ itself. Meyer's result does not refute Gödel's, since they pertain to different arithmetics. But it does show that in comparing relevant arithmetic with classical arithmetic, the former has at least one salient advantage.

One way to look at what Meyer did, is to pull apart RM3mod2 into two consistent arithmetics, namely (a) classical Zmod2 or Z_2, and (b) $RM3^\#$, where $R \subset RM3$, so that $R^\# \subset RM3^\#$. Then we can say that he united or merged these two consistent theories into a single inconsistent theory which could be seen to be nontrivial. Inconsistency was inevitable, since $0 = 2$ holds in Z_2, whereas $\neg 0 = 2$ holds in $R^\#$ and $RM3^\#$. This suggests a general strategy or strategies, which we can collectively call *Merge*, to take pairs of consistent theories and put them together into a common inconsistent extension.

The question raised here is: what is the best way to describe this strategy? There is an obvious answer to this question, but it emerges that there are reasonably-motivated alternatives. In order to appreciate the alternatives, it is necessary to detour through some results about the *Routley Functor*, also known as the *Routley Star Operator*. We begin with some known results in the next section, then proceed to make some additional applications.

2 SEMITHEORIES, THEORIES AND THE ROUTLEY STAR

A theory is a set of sentences closed under the deducibility relation $\vdash_L$ of a logic L. To be more precise, we need to distinguish single-premiss deducibility from multi-premiss deducibility. Thus:

DEFINITION 1 (1) T is an L-*semitheory* iff whenever $A \in T$ and $A \vdash_L B$, it follows that $B \in T$. (2) T is an L-*theory* iff whenever $A_1, \ldots, A_n \in T$ and $A_1, \ldots, A_n \vdash_L B$, it follows that $B \in T$.

The deducibility relation in clause (2) is to be understood as meaning that the premisses form a set in the usual way, not ordered nor with repeats. We will generally drop the prefix "L-" except to specify a particular logic. For any logic which contains a conjunction operator & which obeys the rule of adjunction: from premisses A and B to deduce $A \& B$, that is $A, B \vdash A \& B$, we have an obvious equivalent to the definition of a theory: *a theory is a semitheory closed under conjunctions*. Of course a theory need not have conjunction in its language, when it would be necessary to use the more general definition.

The aim of this exercise is to extract results in the theory of theories about *Merge*, and the *Routley Star*, by appeal to minimal properties of logics. In particular, as we see below, these properties are:

(i) *Contraposition*, in the form: if $A \vdash B$ then $\sim B \vdash \sim A$.

(ii) *Double Negation* (DN), in the form: A is interdeducible with $\sim\sim A$.

(iii) *De Morgan's Laws* (DEM), in the form: $\sim(A \vee B)$ is interdeducible with $\sim A \,\&\, \sim B$; and $\sim(A \,\&\, B)$ is interdeducible with $\sim A \vee \sim B$.

Logics having these properties include those in the well-known Anderson–Belnap family of relevant logics, such as E, R, RM, RM3, and first degree entailment, as well as many others. These logics contain conjunction &, disjunction $\vee$, negation $\sim$ and implication $\rightarrow$ operators. In fact we do not appeal to the properties of implication, just &, $\vee$ and $\sim$.

We now recall the definition of the *Routley Star*.

DEFINITION 2 If S is any set of sentences, then $S^* := \{A : \sim A \notin S\}$.

The *Star* plays an essential role in the Routley–Meyer semantics of relevant logics, see e.g. Routley and Meyer (1972, 1973).

Now an important property following from these definitions is:

THEOREM 3 (Routley and Routley 1972) If T is a semitheory with respect to a L obeying *Contraposition*, then T^* is also semitheory.

Proof: Let T be a semitheory, and suppose $A \in T^*$ and $A \vdash_L B$. From the latter, assuming contraposition, $\sim B \vdash_L \sim A$. From the former, $\sim A \notin T$. Hence $\sim B \notin T$, that is $B \in T^*$. □

This can be described as the invariance of the property of semitheoryhood under the action of *Star*. However, being a *theory* is not generally *Star*-invariant. Instead we have the following definition:

DEFINITION 4 A semitheory is PRIME iff for any disjunction in the semitheory, at least one disjunct is in the semitheory, too.

Then, given weak properties for the background logic (DEM, DN and *contraposition*, see above), we have:

THEOREM 5 (Routley and Routley, Meyer, 1972, 1973)

(1) T is a theory iff T^* is a prime semitheory.

(2) T is a prime theory iff T^* is,

(3) If T is consistent then T^* is complete and $T \subseteq T^*$

(4) If T is complete then T^* is consistent and $T^* \subseteq T$

(5) $T = T^{**}$.

(6) T is inconsistent iff T^* is incomplete.

(7) T is consistent and complete iff $T = T^*$.

From clause (1), semitheories that are not theories are to be found by starring any non-prime theory. Interesting examples are afforded by starring Meyer's $\mathsf{R}^{\#}$, or $\mathsf{P}^{\#}$. Both contain all instances of $A \vee {\sim}A$; however, the first Gödel incompleteness theorem implies (as long as they are consistent) that neither G nor ${\sim}G$ is a theorem, where G is the Gödel sentence. Hence $\mathsf{R}^{\#}$ and $\mathsf{P}^{\#}$ are non-prime, and so their stars are not closed under adjunction. This example is discussed further in Mortensen (2011).

From clause (6), the role of the Star is quite fine-grained. It snips out pairs $A, {\sim}A$ from inconsistent theories, or adds them both to incomplete theories. Further, clauses (2) and (7) say: being a prime theory, and being consistent and complete, are both Star-invariant. Clause (5) is clearly equivalent to DN.

3 MULTIPLE THEORIES

With these preliminaries, we move on to consider the prospects for putting two or more theories of the same logic together. To begin with a simple result:

THEOREM 6 (1) *If* V, W *are semitheories, then so are* $V \cap W$ *and* $V \cup W$.
(2) *If* V, W *are theories, then so is* $V \cap W$.

Proof: (1) If $A \in V \cap W$ then $A \in V$ and $A \in W$. Hence if $A \vdash B$ then $B \in V$ and $B \in W$, that is $B \in V \cap W$. Also, if $A \in V \cup W$ then $A \in V$ or $A \in W$, hence $B \in V$ or $B \in W$, that is $B \in V \cup W$. (2) If in addition V, W are closed under conjunctions, then if A, B are in $V \cap W$ then each is in both V, W so that their conjunction is in both also. □

That is, semitheoryhood, and closure under conjunctions, are both preserved under intersections. However, closure under conjunctions is not generally preserved under unions. Dually, primeness is not in general preserved under intersections. A disjunction might hold in both theories though different disjuncts hold in the different theories.

Now we need further results involving the *Star*. First, there is order-reversal: the *Star* is contravariant.

THEOREM 7 *If* V, W *are any semitheories of any logic satisfying double negation, then* $V \subseteq W$ *iff* $W^* \subseteq V^*$

Proof: (L to R): Suppose $A \in W^*$. Then ${\sim}A \notin W$, so by $V \subseteq W$, we have ${\sim}A \notin V$, so $A \in V^*$. (R to L): Suppose $A \in V$. Then by Theorem 5(5), that is double negation, $A \in V^{**}$, so ${\sim}A \notin V^*$, so by $W^* \subseteq V^*$, we have ${\sim}A \notin W^*$, so $A \in W^{**} = W$. □

Also, in line with clause (5) of Theorem 5, the Routley Functor has a de Morgan character.

THEOREM 8 *If* V, W *are any sets of sentences, then* $(V \cap W)^* = V^* \cup W^*$.

Proof: $A \in (V \cap W)^*$ iff $\sim A \notin (V \cap W)$, iff $\sim A \notin V$ or $\sim A \notin W$, iff $A \in V^*$ or $A \in W^*$, iff $A \in V^* \cup W^*$. □

THEOREM 9 *If* V, W *are any sets of sentences, then* $(V \cup W)^* = V^* \cap W^*$.

Proof: $A \in (V \cup W)^*$ iff $\sim A \notin (V \cup W)$, iff $\sim A \notin V$ and $\sim A \notin W$, iff $A \in V^*$ and $A \in W^*$, iff $A \in V^* \cap W^*$. □

4 MERGE

With these necessary preliminaries, we finally get to the main results. We want to see how one takes two theories V, W, incompatible with one another, and merges their deductive resources into a single inconsistent theory. It seems to this author that there are several reasonable ways to come at this. First, there is the obvious:

DEFINITION 10 The deductive closure of $V \cup W$ under single-premiss deductions and conjunctions, written $(V \cup W)^{\vdash\&}$, that is the intersection of all theories containing $V \cup W$, which we call MERGE 1.

But there is also:

DEFINITION 11 The deductive closure of $V \cup W$ under single-premiss deductions but not conjunctions, written $(V \cup W)^{\vdash}$, that is the intersection of all semitheories containing $V \cup W$, which we call MERGE 2.

It is worth making the distinction between MERGE 1 and MERGE 2, because MERGE 2 is generally not closed under conjunctions, and therefore allows for the possibility of non-adjunctive strategies. These have been popular among paraconsistent logicians in recent decades. But Merge1 is no smaller than MERGE 2; this is because all theories are semitheories, so that the intersection of all theories is no less than the intersection of all semitheories.

And then again, there is also:

DEFINITION 12 $(V \cap W)^*$, which we call MERGE 3.

This is another way of taking account jointly of what two overlapping theories say. MERGE 3 takes what is in common in the two theories, and fills in with inconsistency the background where the theories disagree. If V, W are theories, then so is $V \cap W$ (Theorem 6), but MERGE 3 is generally only a prime semitheory, not conjunctively closed for all formulae. An advantage of MERGE 3 is that there is a rapid non-triviality proof for it: it is only trivial if $V \cap W$ is null, which does not generally hold.

Now we have relationships between these MERGE concepts:

THEOREM 13 *If* V, W *are complete semitheories, then* $(V \cap W)^* \subseteq (V \cup W)^{\vdash} \subseteq (V \cup W)^{\vdash\&}$, *that is,* MERGE 3 $\subseteq$ MERGE 2 $\subseteq$ MERGE 1.

Proof: $(V \cap W)^* = V^* \cup W^*$ by Theorem 7. By Theorem 5(4), $V^* \subseteq V$ and $W^* \subseteq W$, hence $V^* \cup W^* \subseteq V \cup W$. But $V \cup W \subseteq (V \cup W)^{\vdash}$, hence MERGE 3 $\subseteq$ MERGE 2. That MERGE 2 $\subseteq$ MERGE 1 follows by the remark after Definition 11. □

In short, MERGE 3, starring the intersecton, makes for the least commitment when merging complete semitheories.

However, there is also:

THEOREM 14 *If* V, W *are consistent semitheories, then* $(V \cup W)^{\vdash} \subseteq (V \cap W)^*$*, that is,* MERGE 2 $\subseteq$ MERGE 3.

Proof: $V \cap W \subseteq V \subseteq V^*$ by Theorem 5(3). Ditto for W. So by Theorem 7, reversing the order we have that $V, W \subseteq (V \cap W)^*$. Hence $V \cup W \subseteq (V \cap W)^*$. But since V, W are semitheories, so is $V \cap W$ by Theorem 6, and so is $(V \cap W)^*$ by Theorem 3. But by definition $(V \cup W)^{\vdash}$ is the least semitheory containing $(V \cup W)$, hence the theorem follows. □

We can therefore also conclude:

COROLLARY 15 *If* V, W *are consistent complete semitheories,* MERGE 2 $=$ MERGE 3.

If V, W are consistent complete semitheories of classical logic then if $V \neq W$ then $(V \cup W)^{\vdash \&}$ is trivial, but if the background logic is paraconsistent then $(V \cup W)^{\vdash \&}$ is not generally trivial. However, $(V \cap W)^*$ is not generally trivial, except in the special case where $V \cap W = \{\}$.

Consistent complete theories are common, but not the sole possibility. For example, Meyer's $\mathsf{RM3mod2}$ has as a subtheory the MERGE 1 of consistent complete Z_2, and consistent complete standard arithmetic, but also includes the consistent incomplete relevant arithmetic $\mathsf{R}^{\#}$. So there is a mix of possibilities here.

It also follows that:

THEOREM 16 *If* V, W *are consistent and complete, that is if* $V = V^*$ *and* $W = W^*$*, then* $(V \cap W)^* = V^* \cup W^* = V \cup W = (V \cup W)^{\vdash}$.

Unfortunately, MERGE 3 has a drawback in the inconsistent case (I owe this point to Greg Restall). It seems like a reasonable desideratum for a MERGE concept, is that both V and W are included in MERGE(V, W). It is clear from the above that this is satisfied by MERGE 2, and by MERGE 3 as long as $V \cap W$ is consistent. But if both V and W contain some A and also $\sim A$, then MERGE 3 lacks them both. Thus MERGE 3 is incomplete, and does not include V nor W.

The simplest remedy is to introduce a further MERGE.

DEFINITION 17 MERGE 4 $:=$ MERGE 2 $\cup$ MERGE 3.

Then:

THEOREM 18 MERGE 4 *is complete.*

Proof: Let $A \notin$ MERGE 4. Then $A \notin V, W$ and $A \notin V \cap W$. It follows that $\sim A \in (V \cap W)^* =$ MERGE 3 $\subseteq$ MERGE 4. □

Clearly MERGE 4 is a semitheory if MERGE 2 and MERGE 3 are. Moreover, MERGE 4(V, W) includes both V and W, since MERGE 2 does. But MERGE 4 is still generally weaker than MERGE 1, since MERGE 4 is generally not closed under conjunctions, whereas MERGE 1 always is.

5 CONCLUSION

These constructions give us different ways to put mutually inconsistent theories together into non-trivial packages. However, it must be warned that lurking under the surface there are other threats for anyone in the paraconsistency project. These amount to the observation that the functional structure of *mathematical* theories may well generate *mathematical triviality*: a theory is mathematically trivial when every logic-free sentence (that is lacking $\&, \vee, \sim, \rightarrow, \leftrightarrow, \exists, \forall$) holds. The diagnosis of this phenomenon is that the behaviour of functions such as addition and multiplication is not controlled by the strictly logical properties of theories, and so closure under these functions can, and sometimes does, spread contradictions everywhere in the strictly mathematical parts of the theory (see e.g. Mortensen 2000). If only we had Star-invariance for such properties we would have a way to demonstrate mathematical non-triviality, but we do not. Functionality, that is the capacity to carry out calculations, seems to be a semantic matter, determined by features of the domains of models, rather than by properties of deductive logic.

In a sequel it is planned to explore the prospects for strengthening these methods to cover such cases.

6 BIBLIOGRAPHY

Meyer, Robert K. (1976) "Relevant Arithmetic" *Bulletin of the Section of Logic of the Polish Academy of Science* 5, 133–137.

Mortensen, Chris (2000), "Prospects for Inconsistency," in D. Batens et. al., eds., *Frontiers of Paraconsistency*, London, Research Studies Press, 203–208.

— (2011). "Arithmetic Starred," to appear in K.Tanaka et. al. eds., *Proceedings of the Fourth World Congress on Paraconsistency*.

Routley, R. and Meyer, R. K., (1973) "The Semantics of Entailment 1," in H. Leblanc (ed.), *Truth, Syntax and Modality*, Amsterdam, North-Holland.

— (1972), "The Semantics of Entailment II," *Journal of Philosophical Logic*, 1, 53–73.

Routley, R. and V., (1972), "The Semantics of First Degree Entailment," *Noûs* 6, 335–359.

The *Australasian Journal of Logic* (ISSN 1448-5052) disseminates articles that significantly advance the study of logic, in its mathematical, philosophical or computational guises. The scope of the journal includes all areas of logic, both pure and applied to topics in philosophy, mathematics, computation, linguistics and the other sciences.

Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

The journal is freely available at the journal website at

`http://www.philosophy.unimelb.edu.au/ajl/`.

All issues of the journal are archived electronically at the journal website.

SUBSCRIPTIONS Individuals may subscribe to the journal by sending an email, including a full name, an institutional affiliation and an email address to the managing editor at `ajl-editors@unimelb.edu.au`. Subscribers will receive email abstracts of accepted papers to an address of their choice. For institutional subscription, please email the managing editor at `ajl-editors@unimelb.edu.au`.

Complete published papers may be downloaded at the journal's website at `http://www.philosophy.unimelb.edu.au/ajl/`. The journal currently publishes in pdf format.

SUBMISSION The journal accepts submissions of papers electronically. To submit an article for publication, send the LaTeX source of a submission to a member of the editorial board. For a current list of the editorial board, consult the website.