Extending Metacompleteness to Systems with Classical Formulae.

IN HONOUR OF ROBERT K. MEYER.

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Received by Greg Restall
Published September 22, 2010
http://www.philosophy.unimelb.edu.au/ajl/2010
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Abstract: In honour of Bob Meyer, the paper extends the use of his concept of metacompleteness to include various classical systems, as much as we are able. To do this for the classical sentential calculus, we add extra axioms so as to treat the variables like constants. Further, we use a one-sorted and a two-sorted approach to add classical sentential constants to the logic DJ of my book, *Universal Logic*. It is appropriate to use rejection to represent classicality in the one-sorted case. We then extend these methods to the quantified logics, but we use a finite domain of individual constants to do this.

In [5], Meyer introduced the notion of coherence for logics that "can be plausibly interpreted in their own metalogic" (p. 658). Meyer set up this idea by introducing METAVALUATIONS v for a modal logic L such that $v(\Box A) = T$ iff $\vdash \Box A$ in L, with classical-style valuations for the connectives '$\supset$' and '$\sim$'. He defined a formula A to be METAVALID iff $v(A) = T$, for all metavaluations v of L, and the logic L to be COHERENT iff each theorem of L is metavalid. He went on to show that a wide range of modal logics are coherent, and that, in particular, the property 'if $\vdash \Box A \vee \Box B$ then $\vdash \Box A$ or $\vdash \Box B$' holds for these logics. He also showed that the relevant logics NR and E are coherent, with NR satisfying the above property and with E satisfying a similar disjunctive property, 'if $\vdash (A \to B) \vee \cdots \vee (A_n \to B_n) \vee C$, for C '$\to$'-free, then $\vdash A_i \to B_i$, for some i, or $\vdash C$, where C is then a tautology.'

Meyer in [6] used a preferred metavaluation for a logic L (defined below), which essentially has the effect of expressing the theorems of L in an inductive semantic-style form. Meyer then introduced the notion of METACOMPLETENESS for L to mean that L is sound and complete with respect to this metavaluation. He went on to prove meta-completeness for a wide range of positive relevant logics, including their quantified logics, thus showing the priming property,

α if $\vdash A \vee B$ then $\vdash A$ or $\vdash B$,

for the theorems of such logics. For the quantified logics, the additional property,

β if $\vdash \exists xA$ then $\vdash A^t/x$, for some term t,

was shown to hold.

Meyer set up the (preferred) METAVALUATION v (called the canonical quasi-valuation v' in [6]) on the formulae of a quantified relevant logic L as follows :

(i) $v(p) = F$, for all sentential variables p.

(ii) $v(A \,\&\, B) = T$ iff $v(A) = T$ and $v(B) = T$.

(iii) $v(A \vee B) = T$ iff $v(A) = T$ or $v(B) = T$.

(iv) $v(A \to B) = T$ iff $\vdash_L A \to B$ and, if $v(A) = T$ then $v(B) = T$.

(v) $v(\forall xA) = T$ iff $v(A^t/x) = T$, for all terms t, i.e. all individual variables and constants.

(vi) $v(\exists xA) = T$ iff $v(A^t/x) = T$, for some term t.

Meyer used a simple induction on formulae to prove completeness, i.e. if $v(A) = T$ then $\vdash_L A$, and he proved soundness, i.e. if $\vdash_L A$ then $v(A) = T$, using the usual induction on proof steps, thus establishing metacompleteness for the quantified relevant logic L.

Before going on, we present some of the main relevant logics and their quantified forms, including the ones explicitly referred to in this paper. Primitives: $\sim$, &, $\vee$, $\to$, $\forall$, $\exists$ (connectives and quantifiers); p, q, r, ... (sentential variables); f, g, h, ... (predicate variables); x, y, z, ... (individual variables).

AXIOMS.

1. $A \to A$.
2. $A \,\&\, B \to A$.
3. $A \,\&\, B \to B$.
4. $(A \to B) \,\&\, (A \to C) \to .A \to B \,\&\, C$.

5. $A \to A \vee B$.
6. $B \to A \vee B$.
7. $(A \to C) \,\&\, (B \to C) \to .A \vee B \to C$.
8. $A \,\&\, (B \vee C) \to (A \,\&\, B) \vee (A \,\&\, C)$.
9. $\sim\sim A \to A$.
10. $A \to \sim B \to .B \to \sim A$.
11. $A \to B \to .B \to C \to .A \to C$.
12. $A \to B \to .C \to A \to .C \to B$.
13. $A \to .A \to B \to B$.
14. $(A \to B) \,\&\, (B \to C) \to .A \to C$.
15. $A \to .A \to A$.
16. $A \to .B \to A$.
17. $(A \to B \vee C) \,\&\, (A \,\&\, B \to C) \to .A \to C$.
18. $(A \to .B \to C) \to .B \to .A \to C$.

Rules.

1. $A, A \to B \Rightarrow B$.
2. $A, B \Rightarrow A \,\&\, B$.
3. $A \to B, C \to D \Rightarrow B \to C \to .A \to D$.
4. $A \to \sim B \Rightarrow B \to \sim A$.
5. $A \Rightarrow A \to B \to B$.
6. $A \Rightarrow B \to A$.

Sentential Systems.

$B = A1\text{-}9, R1\text{-}4$.

$DW = A1\text{-}10, R1\text{-}3$.

$DJ = A1\text{-}10, A14, R1\text{-}3$.

$TW = A1\text{-}12, R1\text{-}2$.

$EW = TW + R5$.

$RW = TW + A13$.

Quantificational Axioms.

1. $\forall xA \to A^t/x$, where t is free for x in A.
2. $A^t/x \to \exists xA$, where t is free for x in A.
3. $\forall x(A \to B) \to .A \to \forall xB$, where x is not free in A.
4. $\forall x(A \to B) \to .\exists xA \to B$, where x is not free in B.
5. $\forall x(A \vee B) \to A \vee \forall xB$, where x is not free in A.
6. $A \mathbin{\&} \exists xB \to \exists x(A \mathbin{\&} B)$, where x is not free in A.

Quantificational Rule.

1. $A \Rightarrow \forall xA$.

Quantified Systems.

Any of the above sentential systems L can be extended to their corresponding quantified system LQ by the addition of all the quantificational axioms and the rule. However, despite the wide range of relevant logics covered, Meyer's metavaluation did not work for logics with the usual De Morgan negation, though he did show that a constructive negation could be added. Subsequently, Slaney, in [8], established metacompleteness for the major contraction-less logics, TW, EW and RW, all with De Morgan negation, using an additional metavaluation v^*, shown as follows with L being one of these three logics (Slaney used predicates M and M^* in [8]):

(i) $v(p) = F$, for all sentential variables p.

$v^*(p) = T$, for all sentential variables p.

(ii) $v(A \mathbin{\&} B) = T$ iff $v(A) = T$ and $v(B) = T$.

$v^*(A \mathbin{\&} B) = T$ iff $v^*(A) = T$ and $v^*(B) = T$.

(iii) $v(A \vee B) = T$ iff $v(A) = T$ or $v(B) = T$.

$v^*(A \vee B) = T$ iff $v^*(A) = T$ or $v^*(B) = T$.

(iv) $v({\sim}A) = T$ iff $\vdash_L {\sim}A$ and $v^*(A) = F$.

$v^*({\sim}A) = T$ iff $v(A) = F$.

(v) $v(A \to B) = T$ iff $\vdash_L A \to B$, if $v(A) = T$ then $v(B) = T$, and if $v^*(A) = T$ then $v^*(B) = T$.

$v^*(A \to B) = T$ iff, if $v(A) = T$ then $v^*(B) = T$. (for RW and EW)

$v^*(A \to B) = T$. (for TW)

As a result, he was able to prove that TW, EW and RW are prime, i.e. they satisfy (α) above. He also showed that TW has no theorems of the form $\sim(A \to B)$ and that, for EW and RW, $\vdash \sim(A \to B)$ iff $\vdash A$ and $\vdash \sim B$. However, the proof can be simplified slightly by dropping $\vdash \sim A$ from the evaluation of $v(\sim A)$ and completeness shown by proving 'if $v(A) = T$ then $\vdash A$' and 'if $v^*(A) = F$ then $\vdash \sim A$' together by induction on formulae. Then, Slaney, in [9], provided reduced semantics for two broad types of logics, with a completeness argument made simpler by the use of the priming property (α) for these logics. In the process, Slaney established metacompleteness for two classes of logics, M1 logics and M2 logics, M1 logics generalizing upon the TW case above, and M2 logics generalizing upon the RW and EW case. Slaney defines M1 and M2 logics as follows:

> An M1 logic is any system that can be axiomatized as B plus any of the axioms 10–12, 14–17, with or without rule 6.[1]
>
> An M2 logic is one axiomatized as B plus rule 5, plus any of the axioms 10–13, 15–16, 18, with or without rule 6.

However, none of the M1 logics nor the M2 logics contain as a theorem the Law of Excluded Middle (LEM), this being easily explained by noting that neither p nor $\sim p$ are provable in any of these logics, the priming property (α) is easily derivable from metacompleteness, and thus that $p \vee \sim p$ is not provable for any p. Thus, no tautology is derivable in any of these logics, since all the normal-forming operations can be performed in systems containing B and any derivable tautology would be equivalent to a conjunctive normal form, each conjunct of which would be a derivable disjunction containing an excluded middle. The main object of this paper is to find a way of extending metacompleteness to classical systems with the LEM, in particular. Indeed, there is some concern expressed in Universal Logic [1] that the full advantage of the metacompleteness of the logic DJ^dQ (theorem-wise equivalent to the logic DJQ) cannot be realized because of the subsequent addition of classical sentences which, in particular, satisfy the LEM. As a result, Slaney's work in [9] has been bypassed in [1] because it is argued that a broad range of sentences ought to be classically evaluated, which would then invoke the addition of the LEM $(A' \vee \sim A')$ and the Disjunctive Syllogism Rule, DS, $(\sim A', A' \vee B \Rightarrow B)$ for these classical sentences A', schematically represented within the logic by a separate sort: A', B', C', ... So, we wish to find some way of extending metacompleteness to include classical sentences and, thus, to re-establish Slaney's reduced semantics with its simpler completeness argument.

[1] Giambrone notes in [2] that Axiom 17 does not have a postulate in the Routley-Meyer semantics. This was originally noted in [7], p. 345. However, the metacompleteness argument still goes through for this axiom.

1 The Sentential Calculus.

We start by considering sentential calculus itself. Metacompleteness ought to be good for a logic as it yields the intuitive property (α), requiring that for each provable disjunction one of its disjuncts be provable. Were metacompleteness to hold, what does this require of the logic? In particular, since the LEM is a theorem, it would follow that either A or ~A is a theorem, for each formula A. Clearly not both are theorems since the DS would then yield all formulae B and the system would trivialize. So, exactly one of A and ~A would be theorems. In particular, for each sentential "variable" p, one of p and ~p would be a "theorem".

This raises two points, firstly in connection with p being a variable and secondly with p or ~p being a theorem. If one of p and ~p is a theorem, for each p, then this is somewhat like a truth-value assignment to each variable, as one would give for an interpretation in the semantics. As far as p is concerned, it would behave in the syntax more like a constant than a variable, leaving the metavaluation looking more like an ordinary valuation. This we will have to see. On the second point, neither p nor ~p are provable as theorems in sentential calculus, for any p, and so, if the normal axioms and rule of sentential calculus are kept, one of p and ~p would have to be added as an extra axiom, for each of the denumerably many variables p. However, this would have to be done recursively as part of a recursive axiomatization, and so not all the non-denumerably many corresponding valuations would be realizable in such axiomatizations. In any case, any non-recursive specification of valuations will not represent a set of valuations needed for any practical purpose. We are now ready to check the metacompleteness.

So, basing the axiomatization of sentential calculus on that of Whitehead and Russell [10], we have the following schematic system, but with additional axioms:

PRIMITIVES: $\sim, \vee$.

DEFINITIONS: $\&, \supset$.

Axioms.

1. $A \vee A \supset A$.
2. $B \supset A \vee B$.
3. $A \vee B \supset B \vee A$.
4. $A \vee (B \vee C) \supset B \vee (A \vee C)$.
5. $B \supset C \supset .A \vee B \supset A \vee C$.

RULE.

1. $A, A \supset B \Rightarrow B$.

ADDITIONAL AXIOMS.

For each sentential variable p, exactly one of p and $\sim p$ is added, in accordance with some recursive specification r. We will call the system SC_r, for this specification r. We inductively set up the metavaluation v, without its associated v^*, as it is superfluous here.

(i) $v(p) = T$ or F, according as p or $\sim p$ is an additional axiom, for all sentential variables p.

(ii) $v(\sim A) = T$ iff $v(A) = F$.

(iii) $v(A \vee B) = T$ iff $v(A) = T$ or $v(B) = T$.

We proceed, similarly to that of Slaney in [8], through the following completeness and soundness lemmas, embracing the simplification mentioned above, and of course there is no '$\rightarrow$'.

LEMMA 1 *If* $v(A) = T$ *then* A *is a theorem of* SC_r*, and if* $v(A) = F$ *then* $\sim A$ *is a theorem of* SC_r*.*

Proof: We prove both together by induction on formula construction. Use is made of axiom 2, the theorems, $A \supset \sim\sim A$, $A \supset A \vee B$ and $\sim A \& \sim B \supset \sim(A \vee B)$, rule 1 and the derived rule, $A, B \Rightarrow A \& B$. □

LEMMA 2 *If* A *is a theorem of* SC_r *then* $v(A) = T$.

Proof: By the usual induction on proof steps, using the definition of '$\supset$'. □

Thus, we have established:

THEOREM 1 *The sentential calculus* SC_r *is metacomplete.*

COROLLARY *The priming property* (α) *holds for* SC_r*, i.e. if* $\vdash A \vee B$ *then* $\vdash A$ *or* $\vdash B$.

Here, the metacompleteness amounts to soundness and completeness with respect to a recursive valuation of the ordinary sentential calculus semantics. One should note the ease of this proof in comparison to the usual completeness arguments in Hunter [3]. Note also that Lemma 2 shows that the axiomatization is consistent, and in particular that if $p(\sim p)$ is added then $\sim p$ (p) is not derivable.

Another way of looking at this metacompleteness result is to consider the sentential variables as constants, which, if they are to be treated properly as constants, would have properties represented as extra axioms. Here, however, because of structural limitations, all that can be said about sentential constants is that they are true or false, which can then be represented as axioms consisting of the respective unnegated or negated forms of the sentential variable. This must be a recursive specification since it would involve usage of sentential constants in practice.

This is then another way of doing sentential calculus: use constants, do not include variables. However, formula-schemes can be used to replace the usage of variables, which they generally do anyway in formalizations without a uniform substitution rule. Variables would only be necessary if there is sentential quantification, and even then one can probably just use bound variables.

2 Adding Classical Sentences to DJ, and to Slaney's M1 and M2 logics.

In *Universal Logic* [1], classical sentences are added by using a second sort of sentential variables, given by p', q', r', ... However, these are better seen as constants, not only because of the considerations above in Section 1 but also in the light of the view of classicality given in the book [1], where sentences have to be individually examined to see whether they are indeed classical. So, we will proceed to use sentential constants instead of these variables. Then we can follow the treatment in Section 1 by adding additional axioms and corresponding metavaluations for the classical sentential constants, p', q', ... in accordance to their truth or falsity. (We will keep the same symbolism as in [1].) For the sake of uniformity, we will put the general sentential variables, ranging over both classical and non-classical sentences, as constants, metacompleteness still applying, as we will see below. However, we will also consider leaving the general sentential variables as they were, and adding classical sentential constants. Further, we will consider the prospect of having just the one sort of sentential constants, with classicality of individual sentences having to be derived within the system.

2.1 The Two-sorted Approach.

Universal Logic is based on the logic DJ^d, which is DJ with the addition of the meta-rule:

MR1. If $A \Rightarrow B$ then $C \vee A \Rightarrow C \vee B$.

Due to the metacompleteness of DJ, they have the same theorems and so we will just use DJ. We set up DJ with general sentential constants, p, q, r, ..., instead of variables, adding the classical sentential constants, p', q', r', ..., with additional axioms as described in Section 1. For the general constants, we add the further axioms: For each sentential constant p, none, one or both of p and $\sim p$ is added, in accordance with some recursive specification. We still use the schemes, $A, B, C, \ldots$, to represent formulae and $A', B', C', \ldots$, to represent classical formulae, which are built up entirely from classical sentential constants, using only $\sim$, & and $\vee$. As in [1], for the classical formulae A', we add the axiom and rule:

CA1. $A' \vee \sim A'$. (LEM)

CR1. $\sim A', A' \vee B \Rightarrow B$. (DS)

These enable all the tautologies of sentential calculus to be derived with help from DJ. Let us call this combined system DJ^cC, the superscript 'c' being for the constants and the capital 'C' for the classical extension. The metavaluations v and v^* are then as follows:

(i) $v(p) = T$ or F, according as p is an additional axiom or not, for all sentential constants p.

$v^*(p) = F$ or T, according as $\sim p$ is an additional axiom or not, for all sentential constants p.

$v(p') = T$ or F, according as p' or $\sim p'$ is an additional axiom, for all classical sentential constants p'.

$v^*(p') = v(p')$.

(ii) $v(A \,\&\, B) = T$ iff $v(A) = T$ and $v(B) = T$.

$v^*(A \,\&\, B) = T$ iff $v^*(A) = T$ and $v^*(B) = T$.

(iii) $v(A \vee B) = T$ iff $v(A) = T$ or $v(B) = T$.

$v^*(A \vee B) = T$ iff $v^*(A) = T$or $v^*(B) = T$.

(iv) $v(\sim A) = T$ iff $v^*(A) = F$.

$v^*(\sim A) = T$ iff $v(A) = F$.

(v) $v(A \to B) = T$ iff $A \to B$ is a theorem, if $v(A) = T$ then $v(B) = T$, and if $v^*(A) = T$ then $v^*(B) = T$.

$v^*(A \to B) = T$.

We follow Slaney [8] and Section 1 in establishing metacompleteness.

LEMMA 3 *If* $v(A) = T$ *then* A *is a theorem of* DJ^cC*, and if* $v^*(A) = F$ *then* $\sim A$ *is a theorem of* DJ^cC.

Proof: As before, we prove both together by induction on formula construction, making use of some simple derived rules of DJ^cC. □

We also need the following lemma:

LEMMA 4 *For all classical formulae* A', $v(A') = v^*(A')$.

Proof: By induction on classical formulae, constructed from classical sentential constants using $\sim$, & and $\vee$. □

LEMMA 5 *If* A *is a theorem of* DJ^cC *then* $v(A) = T$.

Proof: By the usual induction on proof steps, using Lemma 4 for CA1 and CR1. □

Thus:

THEOREM 2 *The system* DJ^cC *is metacomplete.*

COROLLARY *For* DJ^cC, *if* $\vdash A \vee B$ *then* $\vdash A$ *or* $\vdash B$.

Metacompleteness still holds with the general sentential variables left as they were in [8] and in [1], and the classical sentential constants introduced as above, i.e. with the base case of the metavaluations as follows:

(i) $v(p) = F$, for all sentential variables p.

$v^*(p) = T$, for all sentential variables p.

$v(p') = T$ or F, according as p' or $\sim p'$ is an additional axiom, for all classical sentential constants p'.

$v^*(p') = v(p')$.

Metacompleteness can also be shown for Slaney's M1 and M2 logics, with either general sentential constants or variables, by checking Lemma 5 for the further axioms and rules. Note that for M2 logics, $v^*(A \to B) = T$ iff, if $v(A) = T$ then $v^*(B) = T$.

2.2 The One-sorted Approach.

It may be thought that a single sort of general sentential constants would be neater and philosophically preferable in that the classical sentences would then naturally occur as a subset of the general sentences. To do this, we remove the additional axioms for classical sentential constants from DJ^cC, leaving just the ones for the general sentential constants, and also remove the axiom CA1 and the rule CR1. We can then define p as a CLASSICAL SENTENTIAL CONSTANT iff exactly one of p and $\sim p$ is added as an axiom. We call this system DJ^c, its metavaluation being as for DJ^cC, but with the base case as follows:

(i) $v(p) = T$ or F, according as p is an additional axiom or not, for all sentential constants p.

$v^*(p) = F$ or T, according as ~p is an additional axiom or not, for all sentential constants p.

Then, if we build up classical formulae A entirely from classical sentential constants using ~, & and ∨, as before, we can show that $v(A) = v^*(A)$, as we did in Lemma 4 for DJ^cC. The other lemmas and metacompleteness also follow for DJ^c. From these, we can then show that exactly one of A and ~A are derivable in DJ^c, for all classical formulae A. Thus, CA1 is provable but CR1 is only an admissible rule since it relies on the non-derivability of A, given the theoremhood of ~A.

However, to define classicality in this way loses one of the main benefits of the single sort, that is, to allow any formula A, such that exactly one of A and ~A is a theorem, to be a classical formula. Thus, classicality would be derived within the system rather than be formed using formation rules. This definition, which we will call T-CLASSICALITY, will include the earlier one, which we will now call F-CLASSICALITY, and enable further formulae to be classical that were not so before.

By using the second sort, we were essentially able to say that the constant p' was an axiom whilst $\sim p'$ was not, or that $\sim p'$ was an axiom whilst p' was not. Such constraints were not possible using general constants since if p were an axiom then either ~p might also be an axiom or maybe not. So the sort of classical sentences and formulae not only enables us to say that such formulae A' or their negations $\sim A'$ are derivable but also that their respective negated forms $\sim A'$ or unnegated forms A' are not derivable. So, with a single sort, to capture classicality fully within the syntax one would need some method of representing non-derivability.

Standard Hilbert-style axiomatizations (of a single sort) only represent derivability and, in order to represent non-derivability, we would need to add additional apparatus. Such a thing would be REJECTION, usually symbolized as '⊣' whilst derivability is symbolized as '⊢ '. The idea of rejection goes back to Aristotle, but it was Łukasiewicz in [4] (see pp. 67–72, 94–8, and Ch. v, esp. p. 109.), who included rejection in his ~, ⊃ sentential calculus axiomatization by adding the rejection axiom, $\dashv p$, the rejection substitution rule, $\dashv A(^{B}/p, \ldots, B_n/p_n) \Rightarrow \dashv A(p, \ldots, p_n)$, and the rejection modus ponens rule, $\vdash A \supset B, \dashv B \Rightarrow \dashv A$. Using just these, he was able to prove that all non-theorems were rejectable.[2]

We can also use rejection here to represent one half of classicality, as follows. We set up the additional axioms and rules, to be added to DJ^c, to obtain the system DJ^cR.

[2]I thank Assoc. Prof. David Londey for including this material in his M.A. subject at the University of New England, in 1966.

Additional Axioms.

Given that, for each sentential constant p, none, one or both of $\vdash p$ and $\vdash {\sim}p$ are added, in accordance with some recursive specification, we add $\dashv p$ or $\dashv{\sim}p$ (or both), whenever the corresponding $\vdash p$ or $\vdash {\sim}p$ is not included.

Additional Rules.

AR1. $\vdash A \to B, \dashv B \Rightarrow \dashv A$.

AR2. $\dashv A, \dashv B \Rightarrow \dashv A \vee B$.

AR3. $\vdash A \vee B, \dashv A \Rightarrow \vdash B$.

A formula A is T-classical iff $\vdash A$ and $\dashv{\sim}A$, or $\vdash {\sim}A$ and $\dashv A$.

The previous axiom CA1 follows immediately for classical A, whilst for for the previous rule CR1, let $\vdash {\sim}A$ and $\vdash A \vee B$. Since A is classical, $\dashv A$ and, by AR3, $\vdash B$. (We will shortly show that it is not the case that both $\vdash A$ and $\dashv A$, for any A.)

We set up the base case of the metavaluations v and v^* as follows:

(i) $v(p) = T$ or F, according as p is an additional axiom or a rejection axiom, for all sentential constants p.

$v^*(p) = F$ or T, according as ${\sim}p$ is an additional axiom or a rejection axiom, for all sentential constants p.

The remainder of the metavaluations follows those of DJ^c and DJ^cC.

lemma 6 *If* $v(A) = T$ *then* $\vdash A$*, and if* $v^*(A) = F$ *then* $\vdash {\sim}A$.

Proof: As for Lemma 3. □

lemma 7 *If* $\vdash A$ *then* $v(A) = T$*, and if* $\dashv A$ *then* $v(A) = F$.

Proof: We prove both of these together by induction on the combined proof steps for both the theorems and the rejection theorems. We only need to consider the new axioms and rules involving rejection, all of which are clear. □

corollary 1 *It is not the case that both* $\vdash A$ *and* $\dashv A$*, for any* A.

corollary 2 (i) *If* $\vdash A$ *and* $\dashv {\sim}A$ *then* $v(A) = v^*(A) = T$.

(ii) *If* $\vdash {\sim}A$ *and* $\dashv A$ *then* $v(A) = v^*(A) = F$.

(iii) *Hence, if* A *is a* T*-classical formula then* $v(A) = v^*(A)$.

theorem 3 *The logic* DJ^cR *is metacomplete, i.e.* $\vdash A$*, iff* $v(A) = T$*, and hence if* $\vdash A \vee B$ *then* $\vdash A$ *or* $\vdash B$.

Corollary 2 yields a soundness result for T-classical formulae. To obtain the corresponding completeness result, i.e. 'if $v(A) = v^*(A)$ then A is T-classical', one would generally try to prove 'if $v(A) = F$ then $\dashv A'$, and 'if $v^*(A) = T$ then $\dashv \sim A$'. However, by formula induction, these are provable for all →-free formulae, yielding:

LEMMA 8 *If* A *is an →-free formula then* :

(*i*) *if* $v(A) = v^*(A) = T$ *then* $\vdash A$ *and* $\dashv \sim A$, *and*

(*ii*) *if* $v(A) = v^*(A) = F$ *then* $\vdash \sim A$ *and* $\dashv A$. *Hence, if* A *is an* F*-classical formula then:*

(*iii*) A *is T-classical, since* $v(A) = v^*(A)$.

There are also some T-classical formulae that are not F-classical , since (α) if $\vdash A$, and $\dashv \sim A$ then $\vdash A \vee B$ and $\dashv \sim(A \vee B)$, regardless of B, and (β) if $\vdash \sim A$ and $\dashv A$ then $\vdash \sim(A \,\&\, B)$ and $\dashv A \,\&\, B$, regardless of B.

To try to continue with the induction, if we were to prove 'if $v^*(A) = T$ then $\dashv \sim A$' for formulae of form $A \to B$, we would need to add $\dashv \sim(A \to B)$ as an extra axiom, for all formulae A and B, which would be fine, since $v(\sim(A \to B)) = F$ and hence there are no theorems of the form $\sim(A \to B)$. However, to prove 'if $v(A \to B) = F$ then $\dashv A \to B$' one would need to establish a complete rejection system for DJ which included the rejection of all non-theorems of the form $A \to B$. We would have to leave this onerous task for another occasion.

The single sort can also be used for M1 and M2 logics, with all the above results applying. The only difference for M2 logics would be that in the above continued induction to prove 'if $v^*(A \to B) = T$ then $\dashv \sim(A \to B)$' and 'if $v(A \to B) = F$ then $\dashv A \to B$', we seem to require both $\vdash B \to .A \to B$ and $\vdash \sim A \to .A \to B$ as axioms of the M2 logic.

3 The Predicate Calculus.

In order to establish metacompleteness of predicate calculus, there must be no free individual variables, since either $A(x)$ or $\sim A(x)$ must be provable in a metacomplete system with a free variable x. Similarly to sentential calculus, we also replace sentential and predicate variables by corresponding constants. With the standard assignment to the metavaluation v for ∀, the usual metacompleteness argument fails to go through with an infinite domain of individual constants. There appears to be a need for the infinitary rule, $A(a), A(a_2), \ldots, A(a_n), \ldots \Rightarrow \forall x A(x)$, where $\{a, a_2, \ldots, a_n, \ldots\}$, is the infinite set of constants, in order to establish the completeness half of the proof.[3] However, as

[3] If predicate calculus with an infinite domain of individuals were metacomplete then it is likely that Peano Arithmetic would follow suit, with the metacompleteness yielding completeness (in Godel's sense). Note that the infinitary rule used to establish metacompleteness has the same form as the ω-rule required for completing Peano Arithmetic.

we will prove below, the argument is fine when there are a finite number of constants making up the domain, and the rule, $A(a), A(a_2), \ldots, A(a_n) \Rightarrow \forall x A(x)$, where $\{a, a_2, \ldots, a_n, \ldots\}$ is the set of constants, is added. Thus, we proceed to re-axiomatize the predicate calculus along these lines.

Primitives.

$\{a, a_2, \ldots, a_n, \ldots\}$, . (the finite set of individual constants)

$f, g, h, \ldots$ (predicate constants)

$x, y, z, \ldots$ (bound individual variables)

$p, q, r, \ldots$ (sentential constants)

$\sim, \vee, \forall$ (connectives and quantifier)

Formation Rules.

1. Each sentential constant p is an atomic formula.
2. If f is a predicate constant and $b, \ldots, b_m$ are individual constants then $fb \ldots b_m$ is an atomic formula.
3. If A and B are formulae then $\sim A$ and $A \vee B$ are formulae.
4. If A is a formula, x is a bound individual variable and a is an individual constant then $\forall x A^x/a$ is a formula, where A^x/a is A with all occurrences of a (if any) replaced by x.

definitions. $A \& B, A \supset B, \exists x A$, as usual.

Axioms.

1. $A \vee A \supset A$.
2. $B \supset A \vee B$.
3. $A \vee B \supset B \vee A$.
4. $A \vee (B \vee C) \supset B \vee (A \vee C)$.
5. $B \supset C \supset .A \vee B \supset A \vee C$.
6. $\forall x A \supset Aa/x$.
7. $\forall x(A \supset B) \supset .A \supset \forall x B$. [Note that x does not occur free in A.]

Rules.

1. $A, A \supset B \Rightarrow B$.
2. $A(a), A(a_2), \ldots, A(a_n) \Rightarrow \forall x A(x)$, where the a_i's and the x occur in the same places within A.

Additional Axioms.

In accordance with some recursive specification r, we add exactly one of p and $\sim p$, for each sentential constant p, and exactly one of $fb \ldots b_m$ and $\sim fb \ldots b_m$, for each atomic formula of form $fb \ldots b_m$. We will call the system PC_r, for this specification r. PC_r is an extension of the usual predicate calculus PC, with the each of the free variables in any theorem replacable by each of the constants, $\{a, a_2, \ldots, a_n, \ldots\}$.

We inductively set up the metavaluation v, with the addition of $\forall$, in the manner of Meyer [6], but without free variables.

(i) $v(p) = T$ or F, according as p or $\sim p$ is an additional axiom.

$v(fa \ldots a_n) = T$ or F, according as $fa \ldots a_n$ or $\sim fa \ldots a_n$ is an additional axiom.

(ii) $v(\sim A) = T$ iff $v(A) = F$.

(iii) $v(A \vee B) = T$ iff $v(A) = T$ or $v(B) = T$.

(iv) $v(\forall x A) = T$ iff $v(A^b/x) = T$, for all individual constants b.

lemma 9 *If* $v(A) = T$ *then* A *is a theorem of* PC_r*, and if* $v(A) = F$ *then* $\sim A$ *is a theorem of* PC_r*.*

Proof: We prove both together by induction on formula construction. Use is made of rule 2 and the theorem, $\sim A^b/x \supset \sim\forall\, xA$. □

lemma 10 *If* A *is a theorem of* PC_r *then* $v(A) = T$.

Proof: By the usual induction on proof steps. □

Thus, we have established:

theorem 4 *The predicate calculus* PC_r *is metacomplete, and thus if* $\vdash A \vee B$ *then* $\vdash A$*, or* $\vdash B$*, and if* $\vdash \exists x A$ *then* $\vdash A^b/x$*, for some* b.

4 Adding Classical Sentences to DJQ.

4.1 The Two-sorted Approach.

For DJQ^cC, we add to DJ^cC in Section 2, $\{a, a_2, \ldots, a_n, \ldots\}$, (the finite set of individual constants), f, g, h, …(general predicate constants), f', g', h', …(classical predicate constants), and x, y, z, …(bound individual variables). The classical predicate constants f' yield classical atomic formulae $f'b \ldots b_m$ when combined with any string of individual constants $b \ldots b_m$ of appropriate length. F-classical formulae, which are built up entirely from classical atomic formulae, of forms p' and $f'b \ldots b_m$, using only $\sim$, & and $\vee$, $\vee$, $\forall$ and $\exists$, are again symbolized: A', B', C', ….

Additional axioms, as described in Section 3 for each classical atomic formula of form $f'b \ldots b_m$, are added to DJ^cC. For the general atomic formulae of form $fb \ldots b_m$, we add the further axioms:

> For each formula $fb \ldots b_m$, none, one or both of $fb \ldots b_m$ and $\sim fb \ldots b_m$, are added, in accordance with some recursive specification.

The axiom CA1 and rule CR1 are again included for classical formulae. For general formulae, we add to DJ^cC the axioms QA1–6 of DJQ, but the conditions on them are deleted, not being relevant to the current system based on constants. The rule QR1 is replaced by R2 of Section 3, which will suffice for classical formulae as well.

We add the following to the metavaluations v and v^* of DJ^cC:

(i) $v(fb \ldots b_m) = T$ or F, according as $fb \ldots b_m$ is an additional axiom or not.

$v^*(fb \ldots b_m) = F$ or T, according as $\sim fb \ldots b_m$ is an additional axiom or not.

$v(f'b \ldots b_m) = T$or F, according as $f'b \ldots b_m$ or $\sim f'\ b \ldots b_m$ is an additional axiom.

$v^*(f'b \ldots b_m) = v(f'b \ldots b_m)$.

(vi) $v(\forall xA) = T$ iff $v(A^b/x) = T$, for all individual constants b.

$v^*(\forall xA) = T$ iff $v^*(A^b/x) = T$, for all individual constants b.

(vii) $v(\exists xA) = T$ iff $v(A^b/x) = T$, for some individual constant b.

$v^*(\exists xA) = T$ iff $v^*(A^b/x) = T$, for some individual constant b.

The following lemmas and theorem then present no difficulty.

LEMMA 11 *If* $v(A) = T$ *then* A *is a theorem of* DJQ^cC, *and if* $v^*(A) = F$ *then* $\sim A$ *is a theorem of* DJQ^cC.

LEMMA 12 *For all classical formulae* A', $v(A') = v^*(A')$.

LEMMA 13 *If* A *is a theorem of* DJQ^cC *then* $v(A) = T$.

THEOREM 5 *The system* DJQ^cC *is metacomplete, if* $\vdash A \vee B$ *then* $\vdash A$, *or* $\vdash B$, *and if* $\vdash \exists xA$ *then* $\vdash A^b/x$, *for some* b.

Metacompleteness still holds with general sentential variables replacing the general sentential constants, general predicate variables replacing general predicate constants and free or bound individual variables replacing bound individual variables. However, one maintains the finite set of individual constants ranging over the domain. The general axioms and rules R1–3 of DJQ are used, with terms t representing free individual variables or individual constants. We add the rule R2 of Section 3 to apply to general formulae. We note that QR1, $A \Rightarrow \forall xA$, is then an admissible rule, as each individual constant a_i can be substituted for x in the proof of A, and then R2 (of Section 3) can be applied to yield $\forall xA$. F-classical formulae are constructed as before, using bound variables only. The axiom CA1 and the rules CR1 and also R2 (of Section 3) apply to F-classical formulae A'.

Case (i) of the metavaluations v and v^* is as follows:

(i) $v(p) = F$, for all sentential variables p.

$v^*(p) = T$, for all sentential variables p.

$v(p') = T$ or F, according as p' or $\sim p'$ is an additional axiom.

$v^*(p') = v(p')$.

$v(ft \ldots t_m) = F$, for all general atomic formulae of form $ft \ldots t_m$.

$v^*(ft \ldots t_m) = T$, for all general atomic formulae of form $ft \ldots t_m$.

$v(f'b \ldots b_m) = T$or F, according as $f'b \ldots b_m$ or $\sim f'\ b \ldots b_m$ is an additional axiom.

$v^*(f'b \ldots b_m) = v(f'b \ldots b_m)$.

Cases (vi) and (vii) for $\forall xA$ and $\exists xA$ are as above. Metacompleteness can also be shown for Slaney's M1 and M2 logics, with the standard quantificational extension, with either general constants or general variables, as elucidated above.

4.2 The One-sorted Approach.

Like the sentential case, we have a single sort of general formulae, removing the axioms for classical formulae from DJQ^cC, just leaving the ones for the general formulae (including R2 of Section 3). Let A be an atomic formula of form p or $fb \ldots b_m$. We define A as an F-classical atomic formula iff exactly one of A and $\sim A$ is added as an axiom. We call this system DJQ^c, its metavaluation being as for DJQ^cC, but with the following base case:

(i) $v(p) = T$ or F, according as p is an additional axiom or not.

$v^*(p) = F$ or T, according as $\sim p$ is an additional axiom or not.

$v(fb \ldots b_m) = T$ or F, according as $fb \ldots b_m$ is an additional axiom or not.

$v^*(fb \ldots b_m) = F$ or T, according as $\sim fb \ldots b_m$ is an additional axiom or not.

We build up F-classical formulae A entirely from F-classical atomic formulae using $\sim$, & and $\vee$, $\forall$ and $\exists$, where $\forall xA$ and $\exists xA$ are F-classical iff A^b/x is F-classical, for all individual constants b. We can then show that $v(A) = v^*(A)$, for F-classical A. The other lemmas and metacompleteness also follow for DJQ^c, and exactly one of A and $\sim A$ are derivable in DJQ^c, for all F-classical formulae A.

As previously, we also set up DJQ^cR incorporating rejection. We have the following additional axioms and rules, to be added to DJQ^c, to obtain DJQ^cR.

Additional Axioms.

Let A be an atomic formula, i.e. of form p or $fb \ldots b_m$. Given that, for each atomic formula A, none, one or both of $\vdash A$, and $\vdash \sim A$ are added, in accordance with some recursive specification, we add $\dashv A$ or $\dashv \sim A$ (or both), whenever the corresponding $\vdash A$, or $\vdash \sim A$ is not included.

Additional Rules.

AR1. $\vdash A \to B, \dashv B \Rightarrow \dashv A$.

AR2. $\dashv A, \dashv B \Rightarrow \dashv A \vee B$.

AR3. $\vdash A \vee B, \dashv A \Rightarrow \vdash B$.

AR4. $\dashv A(a), \dashv A(a_2), \ldots, \dashv A(a_n) \Rightarrow \dashv \exists xA(x)$.

Again, a formula A is T-classical iff $\vdash A$, and $\dashv \sim A$, or $\vdash \sim A$ and $\dashv A$. We set up the base case of the metavaluations v and v^* as follows:

(i) $v(A) = T$ or F, according as A is an additional axiom or a rejection axiom, for all atomic formulae A.

$v^*(A) = F$ or T, according as $\sim A$ is an additional axiom or a rejection axiom, for all atomic formulae A.

The remainder of the metavaluations follows those of DJQ^c and DJQ^cC.

LEMMA 14 *If* $v(A) = T$ *then* $\vdash A$, *and if* $v^*(A) = F$ *then* $\vdash \sim A$.

Proof: As for Lemmas 3, 6 and 11. □

LEMMA 15 *If* $\vdash A$ *then* $v(A) = T$*, and if* $\dashv A$ *then* $v(A) = F$.

Proof: As for Lemmas 7 and 13. We only need to consider the quantificational axioms and rule involving rejection, all of which are clear. □

COROLLARY 1 *It is not the case that both* $\vdash A$ *and* $\dashv A$*, for any* A.

COROLLARY 2 (*i*) *If* $\vdash A$*, and* $\dashv \sim A$ *then* $v(A) = v^*(A) = T$.

(*ii*) *If* $\vdash \sim A$ *and* $\dashv A$ *then* $v(A) = v^*(A) = F$.

(*iii*) *Hence, if* A *is a* T*-classical formula then* $v(A) = v^*(A)$.

THEOREM 6 *The logic* DJQ^cR *is metacomplete, i.e.* $\vdash A$*, iff* $v(A) = T$.

To obtain the converse of Corollary 2, i.e. 'if $v(A) = v^*(A)$ then A is T-classical', we try to prove 'if $v(A) = F$ then $\dashv A$' and 'if $v^*(A) = T$ then $\dashv \sim A$'. However, by formula induction, these are provable for all $\rightarrow$-free formulae, yielding:

LEMMA 16 *If* A *is an* $\rightarrow$*-free formula then:*

(*i*) *if* $v(A) = v^*(A) = T$ *then* $\vdash A$*, and* $\dashv \sim A$*, and*

(*ii*) *if* $v(A) = v^*(A) = F$ *then* $\vdash \sim A$ *and* $\dashv A$*. Hence, if* A *is an* F*-classical formula then:*

(*iii*) A *is T-classical, since* $v(A) = v^*(A)$.

We can also take advantage of the rejection mechanism to drop the requirement of a finite domain of individuals, leaving it open for us to explore the extent of T-classicality of formulae, interpreted over an infinite domain. So, we replace the finite set $\{a, a_2, \ldots, a_n, \ldots\}$ of individual constants by a recursive sequence $\{a, a_2, \ldots, a_n, \ldots\}$, . Since we can no longer use R2, as it would become infinitary, we need some way of establishing universally quantified formulae, not only for the proof theory but also for metacompleteness. The only recourse seems to be to re-introduce individual variables and the associated rule QR1 of DJQ.

So, we set up the system $DJQ^{cv}R$, which is DJQ^cR with the above individual constants and with individual variables: $x, y, z, \ldots$, and with the following associated changes. We symbolize terms as t_1 , $t_2, \ldots$, representing individual constants or variables. We construct formulae in a similar manner to that for DJQ.

For all atomic formulae A of forms p and $ft_1t_2 \ldots t_m$, where the t_i's are terms, we add as axioms none, one or both of A or $\sim A$, subject to the conditions that if A is added then A^t/x must also be added, for all terms t, and if $\sim A$ is added then $\sim A^t$ $/x$ must also be added, for all terms t. We use the original axioms and rules of DJQ, including QA1–6 and QR1. We add the rejection axioms and rules, where these axioms ensure that each atomic formula is either an axiom or a rejected one, and not both, and the rules are AR1–3.

The rule, $\dashv A \Rightarrow \dashv \exists xA$, is not added as $\dashv A$ does not imply $\dashv A^c/x$, for any constant c, and we need $\dashv A^c/x$, for all c, in order to justify $\dashv \exists xA$. Unfortunately, the absence of this rule will weaken our treatment of classicality, taking away some of the advantage of adding QR1.

For formulae A, that can contain free variables, we define a T-classical formula A, where $\vdash A$, and $\dashv \sim A$, or $\vdash \sim A$ and $\dashv A$. An F-classical formula is built up from T-classical atomic formula using ~, & and ∨, ∀ and ∃ only, where $\forall xA$ and $\exists xA$ are F-classical iff A^t/x is F-classical , for all terms t.

We set up the cases (i), (vi) and (vii) of the metavaluations v and v^* as follows:

(i) $v(A) = T$ or F, according as A is an additional axiom or a rejection axiom, for all atomic formulae A.

$v^*(A) = F$ or T, according as ~A is an additional axiom or a rejection axiom, for all atomic formulae A.

(vi) $v(\forall xA) = T$ iff $v(A^t/x) = T$, for all terms t.

$v^*(\forall xA) = T$ iff $v^*(A^t/x) = T$, for all terms t.

(vii) $v(\exists xA) = T$ iff $v(A^t/x) = T$, for some term t.

$v^*(\exists xA) = T$ iff $v^*(A^t/x) = T$, for some term t.

In (vi) and (vii), we follow Meyer [6], p. 510, in using all the terms, but with bound variable rewriting in A when t is bound upon substitution into A.

LEMMA 17 *If* $v(A) = T$ *then* $\vdash A$, *and if* $v^*(A) = F$ *then* $\vdash \sim A$.

Proof: Use QR1. □

LEMMA 18 *If* $\vdash A$ *then* $v(A) = T$, *and if* $\dashv A$ *then* $v(A) = F$.

Proof: We follow Meyer in [6], pp. 512-3, in verifying the quantificational axioms QA1–6 and rule QR1. □

COROLLARY 1 *It is not the case that both* $\vdash A$ *and* $\dashv A$, *for any* A.

COROLLARY 2 (*i*) *If* $\vdash A$, *and* $\dashv \sim A$ *then* $v(A) = v^*(A) = T$.

(*ii*) *If* $\vdash \sim A$ *and* $\dashv A$ *then* $v(A) = v^*(A) = F$.

(*iii*) *Hence, if* A *is a* T-*classical formula then* $v(A) = v^*(A)$.

THEOREM 7 *The logic* $DJQ^{cv}R$ *is metacomplete, i.e.* $\vdash A$ *iff* $v(A) = T$.

If we try to prove 'if $v(A) = F$ then $\dashv A$' and 'if $v^*(A) = T$ then $\dashv \sim A$', these are provable for all formulae built using only ~, & and ∨ . Let an SF-classical formula be a quantifier-free F-classical formula. Hence:

LEMMA 19 *If* A *is an* $\to$*-free quantifier-free formula then:*

(*i*) *if* $v(A) = v^*(A) = T$ *then* $\vdash A$, *and* $\dashv {\sim}A$,

(*ii*) *if* $v(A) = v^*(A) = F$ *then* $\vdash {\sim}A$ *and* $\dashv A$, *Hence, if* A *is an S*F*-classical formula then:*

(*iii*) A *is T-classical, since* $v(A) = v^*(A)$.

However, there are some T-classical formulae with quantifiers. If $\vdash A^t/x$ and $\dashv {\sim}A^t/x$, for some term t, then $\vdash \exists xA$ and $\dashv {\sim}\exists xA$. If $\vdash {\sim}A^t/x$ and $\dashv A^t/x$, for some term t, then $\vdash {\sim}\forall xA$ and $\dashv \forall xA$. Unfortunately, we are unable to prove $\dashv \exists xA$ and $\dashv {\sim}\forall xA$ from $\dashv A^t/x$, for all terms t, and $\dashv {\sim}A^t/x$, for all terms t, respectively, without the use of an infinitary rule. This is the problem with establishing the T-classicality of all F-classical formulae. Thus, this is the casualty of dropping the finite domain. We can still, however, establish $v(A) = v^*(A)$, for all F-classical formulae A, which means that exactly one of A and ~A is provable in $DJQ^{cv}R$, though, as we said before, this relies on a meta-theoretic notion of non-provability.

All these results for the single sort can be shown for M1 and M2 logics.

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