

Partial Confirmation of a Conjecture on the Boxdot Translation in Modal Logic

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Abstract: The purpose of the present note is to advertise an interesting conjecture concerning a well-known translation in modal logic, by confirming a (highly restricted) special case of the conjecture.

I THE CONJECTURE

We work with the conventional language of (mono)modal logic, taking \Box and some functionally complete set of boolean connectives as primitive, to generate the set of formulas from the set of propositional variables, p_1, \dots, p_n, \dots . For p_1 and p_2 we generally write p and q . Given a formula A , $\Box A$ abbreviates the formula $\Box A \wedge A$, and a formula in which all occurrences of \Box are in subformulas of the latter form will be called a \Box -formula. The *boxdot translation*, τ_{\Box} , from modal formulas to modal formulas replaces all occurrences of \Box with \Box . According to a well-known result, this translation embeds KT faithfully in (or ‘into’) K in the sense that for all formulas A :

$$\vdash_{KT} A \text{ if and only if } \vdash_K \tau_{\Box}(A).$$

Because of its simplicity, the boxdot translation has been a traditional favourite in the philosophical literature on the significance of such embeddings, for example, [7], [4] (and more briefly, by the same authors, in Example 2.9 of [8] and §2 of [5]). Recent technical studies in which it figures prominently include Goris [3] and Litak [6]; see also Zolin [10], Lemma 5.9. ([3] and [10] include references to the literature on modal provability logic, in which again \Box and τ_{\Box} have enjoyed considerable currency.)

Now, experience with a considerable number of cases prompts the conjecture that the range of (normal: see below) modal logics in which τ_{\Box} faithfully embeds KT—in the sense that reference to them can replace that to K in the biconditional inset above—comprises precisely the logics S for which $K \subseteq S \subseteq KT$.¹ While it is not hard to see that τ_{\Box} embeds KT faithfully into *all* the logics between K and KT, and also (since $\vdash_{KT} A \leftrightarrow \tau_{\Box}(A)$ for all A) that no proper extension of KT enjoys this status as a candidate ‘target’ for the embedding (with KT as ‘source’), it is not so easy to see that *only* logics between K and KT are suitable candidates. (We have informally circulated this problem amongst several logicians with a special interest in modal translations, but none has so far been able to provide a solution.) What, in particular, is there to rule out the existence of a logic \subseteq -incomparable with KT, in which KT is nonetheless embedded by τ_{\Box} ? According to the conjecture just outlined, no such logic exists: τ_{\Box} embeds KT only into sublogics of KT. “Logic”, here, means *normal modal logic*;² for these we use the labels of Chellas [1] where they exist, and in general for a set of formulas Γ , $S \oplus \Gamma$ is the smallest normal extension of S containing all formulas in Γ . When $\Gamma = \{A\}$ for some formula A, we write “ $S \oplus A$ ” rather than “ $S \oplus \{A\}$ ”. In particular, then, $K \oplus A$ is the smallest normal modal logic containing the formula A. In view of the facts about τ_{\Box} cited above, the conjecture amounts to the conjecture that for any non-theorem A, of KT (= $K \oplus T$, in the notation just introduced), there is a K-unprovable \Box -formula B with $\vdash_{K \oplus A} B$. As we may put it, the normal extension of K by any non-theorem

¹Considerations from the Kripke semantics point in a similar direction, but since these have proved suggestive rather than conclusive for us to date, they will not be in play explicitly in what follows. The idea is that whenever A is not KT-provable, the reflexive closures of frames on which A is valid will validate some formulas—the A^* of the proof of Corollary 4 below—not valid on every reflexive frame. Note that we do not say that A is valid on a frame if and only if A^* is valid on its reflexive closure—by analogy with the well-known fact that $\tau_{\Box}(A)$ is valid on a frame iff A is valid on its reflexive closure. In fact, there can be no function $g(\cdot)$ from formulas to formulas with the property that for all formulas A, A is valid on a frame iff $g(A)$ is valid on its reflexive closure, since different frames with the same reflexive closure need not validate the same formulas.

²We are taking a normal modal logic to be a set of formulas in the language described above, containing all the theorems of K and closed under Necessitation, Modus Ponens, and Uniform Substitution. For such a logic S we write “ $\vdash_S A$ ” for “ $A \in S$ ” and read this as: “A is provable in (or: is a theorem of) S”. Because of the Uniform Substitution condition, it doesn’t matter whether we think of KT as the smallest normal modal logic containing the formula $\Box p \rightarrow p$ or as the smallest such logic containing all instances of the schema $\Box A \rightarrow A$. Whereas Chellas [1] uses T as the name of the schema, we use it as the name for the formula just mentioned.

of KT is \Box -nonconservative. (Note that the \Box -formulas comprise precisely the image of the translation τ_{\Box} .)

Massaged into the form just given, the conjecture clearly admits of no counterexample in which A is of modal degree 0, since for such an $A \notin KT$, $K \oplus A$ is the inconsistent logic (as A is then a \Box -free formula which is not a two-valued tautology). In the following section, we shall extend this to cover the case of formulas of modal degree 1—formulas, that is, in which no occurrence \Box lies within the scope of another occurrence of \Box . (On this convention a formula of modal degree n is also of modal degree $n + 1$.) Evidently this is a considerable limitation. The normal proper extensions of K axiomatizable by one-variable first degree formulas are precisely KD , KD_c , $KD!$, KT , KT_c , $KT!$, $K \oplus \Box \perp$ and the inconsistent logic. Amongst first-degree formulas considered as axioms by which to extend K without restriction as to the number of variables, the most famous are perhaps those from the infinite sequence of formulas Alt_n ($n \in \mathbb{N}$) from Segerberg [9], p. 52. This section concludes with some remarks on the general case, without any restriction as to modal degree.

Sometimes when $A \notin KT$, we can see that $K \oplus A$ is not \Box -conservative because its theorems include $\tau_{\Box}(A)$, and $\not\vdash_K \tau_{\Box}(A)$. Examples include the cases in which A is 4 ($= \Box p \rightarrow \Box \Box p$) or B ($= p \rightarrow \Box \Diamond p$, where $\Diamond p$ is $\neg \Box \neg p$), or T_c ($= p \rightarrow \Box p$). In other cases, $\not\vdash_{K \oplus A} \tau_{\Box}(A)$, such as when A is $\Box \perp$, $\Box \perp \vee \Diamond \perp$, or D_c ($= \Diamond p \rightarrow \Box p$, a minor variant of Segerberg's Alt_1); but here it is not hard to find \Box -formulas other than $\tau_{\Box}(A)$ which are not K -provable but which are of the form $\tau_{\Box}(B)$ for KT -unprovable B . (In the cases just mentioned, take B as T_c , $\Diamond(\Box p \vee \Box \neg p)$, and $p \vee \Box(p \rightarrow q) \vee \Box(p \rightarrow \neg q)$, respectively.) The function f described in the following section applies to any first degree A to provide a B with the desired behaviour.

2 RESOLVING THE FIRST DEGREE CASE

In any normal modal logic, a formula A is provably equivalent to a conjunction of disjunctions of formulas of the form $\neg B \vee \neg \Box C \vee \Box D_1 \vee \dots \vee \Box D_n$ in which B is \Box -free, and for the case of A of modal degree 1, the formulas C, D_1, \dots, D_n are also \Box -free;³ we will write such 'basic disjunctions' in the implicational form:

$$(B \wedge \Box C) \rightarrow (\Box D_1 \vee \dots \vee \Box D_n),$$

in which the consequent is understood as \perp when $n = 0$ and either or both of the conjuncts in the antecedent may be absent.⁴ (If both are, we may think

³Fine [2] is the *locus classicus* for normal forms in (normal) modal logic. Fine takes \Diamond as primitive rather than \Box , and (essentially) works with disjunctive rather than (as here) conjunctive normal forms.

⁴We don't need to write " $\Box C_1 \wedge \dots \wedge \Box C_m$ " in the antecedent, since by normality this simplifies to $\Box(C_1 \wedge \dots \wedge C_m)$. In terms of the disjunctive formulation, this is why we have just $\neg \Box C$ rather than $\neg \Box C_1 \vee \dots \vee \neg \Box C_m$.

of the antecedent as \top or, equivalently, identify the conditional with its consequent.)

Whether or not a given A is a first degree modal formula, if A is written as a conjunction $A_1 \wedge \cdots \wedge A_k$ with each A_i ($1 \leq i \leq k$) a basic formula (as inset above), we define $f(A)$ to be the conjunction of the formulas $f(A_i)$ where f maps A_i , to the formula:

$$(s \wedge B \wedge \Box C) \rightarrow (\Box(D_1 \vee s) \vee \cdots \vee \Box(D_n \vee s)),$$

in which s is a propositional variable not occurring in A_i (i.e. not occurring in the given B, C, D_1, \dots, D_n), and the \Box -notation is as explained in Section 1.

So defined, $f(A)$ is not unique, since various different conjunctive normal forms $\bigwedge_{i=1}^k A_i$ of A can make a difference, as well as the choice of s from the countable list of propositional variables (see the opening sentence of Section 1), though the first difference is inconsequential and the second could be ironed out by choosing s for A_i as the first p_j in that enumeration which does not occur in A_i . Rather than making any such moves, however, we can simply take $f(A)$ to denote an arbitrarily selected formula satisfying the defining conditions.

LEMMA 1 *For any formula A : $\vdash_K A \rightarrow f(A)$.*

Proof: It suffices to observe that where $\bigwedge_{i=1}^k A_i$ is a normal form for A , we have $\vdash_K A_i \rightarrow f(A_i)$, for $1 \leq i \leq k$. \square

LEMMA 2 *For any formula A of modal degree 1, if $\vdash_K f(A)$ then $\vdash_{KT} A$.*

Proof: As in the proof of Lemma 1, it suffices to check this for each A_i . Take A_i as above, i.e., as $(B \wedge \Box C) \rightarrow (\Box D_1 \vee \cdots \vee \Box D_n)$, so that $f(A)$ is $(s \wedge B \wedge \Box C) \rightarrow (\Box(D_1 \vee s) \vee \cdots \vee \Box(D_n \vee s))$. Our hypothesis is that this is provable in K , so weakening the “ \Box ”s in the consequent to simple “ \Box ”s, and unpacking the “ \Box ” in the antecedent (together with some re-arranging), we have:

$$\vdash_K ((s \wedge B \wedge C) \wedge \Box C) \rightarrow (\Box(D_1 \vee s) \vee \cdots \vee \Box(D_n \vee s)).$$

Since A , and therefore A_i , is of modal degree (at most) 1, $s \wedge B \wedge C$ is a \Box -free formula, so, using the fact that K is a (fully) modalized logic in the sense of [10], we can infer that either (a) $\vdash_K ((s \wedge B \wedge C) \rightarrow \perp)$ or (b) $\vdash_K \Box C \rightarrow (\Box(D_1 \vee s) \vee \cdots \vee \Box(D_n \vee s))$. In case (a) we have $\vdash_K B \rightarrow \neg C$ (after substituting B or \top for s), and therefore (since $\vdash_{KT} \Box C \rightarrow C$) $\vdash_{KT} B \rightarrow \neg \Box C$, from which we conclude that $\vdash_{KT} A_i$. In case (b) we have, substituting $\bigwedge_{i=1}^n D_i$ for s and simplifying: $\vdash_K \Box C \rightarrow (\Box D_1 \vee \cdots \vee \Box D_n)$, so $\vdash_K A_i$ and therefore again $\vdash_{KT} A_i$. \square

Putting these ingredients together settles the conjecture of Section 1 insofar as it bears on first degree formulas:

THEOREM 3 *Suppose A is a first degree modal formula not provable in KT . Then $K \oplus A$ proves some \Box -formula not provable in K .*

Proof: Let A be as described. We may choose $f(A)$ as the desired \Box -formula, since it is provable in $K \oplus A$, by Lemma 1 (and Modus Ponens), but not provable in K , by Lemma 2 (contraposed). \square

For the first degree KT-unprovable formulas A listed at the end of Section 1, the formulas B given there as having B provable in $K \oplus A$ without $\tau_{\Box}(B)$ provable in K are minor variations on what the definition of $f(A)$ would deliver. For example, for the case of A as $\Box\perp$ we gave as a candidate B the formula $p \rightarrow \Box p$. Since this A contains no propositional variables, taking s as the first variable not occurring in A means that s is p (alias p_1), and $f(A)$ is therefore $p \rightarrow \Box(\perp \vee p)$, which is provably equivalent in K to $p \rightarrow \Box p$. More generally, since we may have to make substitutions, we can say for the proffered examples of B that B and $f(A)$ are K -interducible in the sense that $K \oplus B = K \oplus f(A)$.⁵ This happens in the case of $A = p \rightarrow \Box p$ in which again we gave B as $p \rightarrow \Box p$, while $f(A)$ is $(p \wedge q) \rightarrow \Box(p \vee q)$ and we need to substitute p for q before proceeding further.

Returning to the conjecture as first formulated in Section 1, we have the following:

COROLLARY 4 If $S = K \oplus \Gamma$ for a set Γ of first degree formulas and $S \not\subseteq \text{KT}$, then τ_{\Box} does not embed KT faithfully in S .

Proof: Given that $S \not\subseteq \text{KT}$, pick $A \in \Gamma \setminus \text{KT}$, and the proof of Theorem 3 shows that $f(A)$ is a \Box -formula in $K \oplus A \setminus K$. Thus we have (1) $\not\vdash_K f(A)$ while (2) $\vdash_S f(A)$. Since $f(A)$ is a \Box -formula, replacing every \Box in $f(A)$ with \square gives a formula A^* for which $f(A) = \tau_{\Box}(A^*)$, and (1) and (2) can be reformulated as (1'): $\not\vdash_K \tau_{\Box}(A^*)$, (2'): $\vdash_S \tau_{\Box}(A^*)$. From the fact (recalled in Section 1) that τ_{\Box} embeds KT in K , we infer from (1') that $\not\vdash_{\text{KT}} A^*$. If it were also the case that τ_{\Box} faithfully embedded KT in S , (2') would imply that $\vdash_{\text{KT}} A^*$: a contradiction. \square

What is wanted, then, is a way of lifting the restriction to first degree formulas from Theorem 3 (and Corollary 4)—or else of showing, with a counterexample, that it cannot be lifted. Should the former turn out to be possible, perhaps the proof of Theorem 3 can be turned into the inductive step of a proof of the desired general result (by induction on the modal degree of formulas); however, we do not currently see how the details of such an argument would go.

⁵Indeed, since the rule of necessitation is not required here, we could say more informatively that $K + B = K + f(A)$, where $S + \Gamma$ is the smallest (not necessarily normal) modal logic extending S and containing all formulas in Γ . In fact, everything said in Lemma 2 about KT applies to the quasi-normal modal logic $K + T (= K[\top]^0$ from [9], p. 179), suggesting a variant on Corollary 4 below in which the references to $K \oplus \Gamma$ and KT are replaced by references to $K + \Gamma$ and $K + T$, the qualification “first degree” is dropped, and reference to τ_{\Box} is replaced by a one to the translation τ_0 , where $\tau_0(A)$ is the result of replacing only the outermost (i.e., not modally embedded) occurrences of \Box in A by \square . We do not go into this in any detail because of the marginal interest of τ_0 by comparison with τ_{\Box} .

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