Church-Rosser property and intersection types

George Koletsos
Department of Computer Science
National Technical Univ. of Athens
GR-15773 Zografou, Greece
koletsos@math.ntua.gr

George Stavrinos*
MPLA, Department of Mathematics
Univ. of Athens
GR-15784 Zografou, Greece
g.stavrinos@math.ntua.gr

Received by Greg Restall
Published August 4, 2008
http://www.philosophy.unimelb.edu.au/ajl/2008

Abstract: We give a proof via reducibility of the Church-Rosser property for the system D of λ-calculus with intersection types. As a consequence we can get the confluence property for developments directly, without making use of the strong normalization property for developments, by using only the typability in D and a suitable embedding of developments in this system. As an application we get a proof of the Church-Rosser theorem for the untyped λ-calculus.

Keywords: λ-calculus, Church-Rosser property, developments, intersection types, reducibility.

1 INTRODUCTION

The Church-Rosser property (or confluence property) is a central property of λ-calculus. It has known many different proofs since it was first proved by A. Church and J.B. Rosser in 1936 [3]. Some of the classical proofs are contained in [2]. The property was also proved for the simply typed λ-calculus $\lambda_{\rightarrow}$ by G. Koletsos [10] and R. Statman [15] using the reducibility method and logical relations, respectively.

*This author is supported by "ΠΥΘΑΓΟΡΑΣ" grant, co-funded by the European Social Fund (75%) and the Hellenic Ministry of Education (25%) under Operational Programme on Education and Initial Vocational Training (ΕΠΕΑΕΚ II).

In this paper we prove, using a reducibility argument, the Church-Rosser property for the system D of λ-calculus with ***intersection types***. The intersection types assignment systems were introduced by M. Coppo, M. Dezani-Ciancaglini, and B. Venneri for characterizing via typability fundamental properties of the untyped λ-calculus such as solvability and strong normalization [4, 5, 6]. The system D is treated extensively by J.-L. Krivine in [13] where characterizations of normalization properties are given via the reducibility method by interpreting the types with suitable sets of λ-terms. A detailed study of this method for proving general properties of λ-calculus can be found in [7, 8], [11], and [9]. In our proof of the Church-Rosser property for system D we adapt the reducibility method of [11] to this system.

As a consequence of the Church-Rosser property for system D we get the confluence of a special kind of reduction called development. A development is a restricted reduction in which we select some initial redexes and keep reducing only them and their residuals throughout the reduction. In this way all developments are finite and have unique normal form [2, Chapter 11]. This property of developments was originally used by A. Church to prove the Church-Rosser property for the untyped λ-calculus. In [2] the confluence of developments is proved by using the well-known Newman's lemma, i.e. strong normalization and the weak Church-Rosser property imply the Church-Rosser property [14], so the strong normalization of developments is used as a prerequisite. In our proof, the confluence of developments comes directly, ***without using the strong normalization property***, from the Church-Rosser property for system D and by embedding the untyped λ-calculus into the system D. Note though that strong normalization has equal strength as typability in D (see [13, page 65] and [1, Theorem 7.4.11]).

As an application we can easily get a proof of the Church-Rosser theorem for the full untyped λ-calculus.

In section 2 of the paper we introduce the basic notions and prove via reducibility the Church-Rosser property for system D. In section 3 we define precisely an operator that establishes the embedding of the untyped λ-calculus into D and prove the confluence of developments. Finally, in section 4 we use the previous result to prove the Church-Rosser theorem for the untyped λ-calculus.

2 THE CHURCH-ROSSER PROPERTY FOR SYSTEM D

We start this section by presenting briefly some well-known definitions from λ-calculus and system D. The notation, terminology and the syntactic conventions are adopted mainly from [13].

PRELIMINARY DEFINITIONS

The *types* of D are the "propositional sentences" built inductively from the variables $X, Y, \ldots$ (the type variables) and the connectives $\cap$ and $\rightarrow$. We use capital letters $X, Y, \ldots$ for the type variables and small letters $x, y, \ldots$ for the individual variables by which we construct the λ-terms.

The untyped λ-*terms* are built inductively starting from the variables $x, y, \ldots$ and using the following rule: if t and u are terms then $(t)u$ (*application*) and $\lambda x.t$ (λ-*abstraction*) are terms. For simplicity we write $(u)t_1 t_2 \ldots t_n$ or even $ut_1 t_2 \ldots t_n$ for $(\ldots((u)t_1)t_2 \ldots)t_n$. The λx of a λ-abstraction term acts here as a variable binder and so we must distinguish between the *bound* and *free* occurrences of a variable in a term. We denote by $FV(u)$ the set of *free variables* in the term u and we write $u[t_1/x_1, \ldots, t_n/x_n]$ for the "simultaneous" *substitution* of the free occurrences of $x_1, \ldots, x_n$ in u by $t_1, \ldots, t_n$, respectively. When necessary we also adopt Barendregt's *variable convention* so that all bound variables are chosen to be different from the free variables.

A *context* Γ is a finite set of declarations $x : A$ where x is an individual variable, A is a type, and no x appears twice. $x : A$ means "variable x has type A". We write $\Gamma, x : A$ for the context $\Gamma \cup \{x : A\}$ where we always assume that x does not appear in Γ.

We define inductively the notion "in context Γ, term t has type A" written $\Gamma \vdash_D t : A$ (or more simply $\Gamma \vdash t : A$) :

Rule 1. $\Gamma, x : A \vdash x : A$ (hypothesis)

Rule 2. $\dfrac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \rightarrow B}$ ($\rightarrow$-introduction)

Rule 3. $\dfrac{\Gamma \vdash t : A \quad \Gamma \vdash u : A \rightarrow B}{\Gamma \vdash (u)t : B}$ ($\rightarrow$-elimination)

Rule 4. $\dfrac{\Gamma \vdash t : A \cap B}{\Gamma \vdash t : A}$ ($\cap_1$-elimination) $\dfrac{\Gamma \vdash t : A \cap B}{\Gamma \vdash t : B}$ ($\cap_2$-elimination)

Rule 5. $\dfrac{\Gamma \vdash t : A \quad \Gamma \vdash t : B}{\Gamma \vdash t : A \cap B}$ ($\cap$-introduction)

Note that $\cap$ is a special conjunction which behaves rather as a set-theoretic *intersection*.

We call $\Gamma \vdash t : A$ a *typing* of t. If a term gets a typing by the above rules then it is a *typed* or *typable* term. It is easy to check that $FV(t) \subseteq \{x_1, \ldots, x_k\}$ whenever $x_1 : A_1, \ldots, x_k : A_k \vdash t : A$.

Let Λ denote the set of all (untyped) λ-terms. If $\mathcal{X}$ and $\mathcal{Y}$ are subsets of Λ, we define $\mathcal{X} \rightarrow \mathcal{Y}$ by:

$$u \in (\mathcal{X} \rightarrow \mathcal{Y}) \overset{def}{\Longleftrightarrow} \forall t \in \mathcal{X},\ (u)t \in \mathcal{Y}$$

The only reduction rule considered is β-*reduction* (notation $\xrightarrow{*}_\beta$) defined as the contextual, reflexive and transitive closure of the relation

$$(\lambda x.u)v \longrightarrow_\beta u[v/x]$$

between a *redex* $(\lambda x.u)v$ and its *contractum* $u[v/x]$. We write $t \longrightarrow_\beta t'$ when t' is obtained from t by contracting *one* redex in t, and $t \xrightarrow{*}_\beta t'$ when t' is obtained by a finite sequence (possibly empty) of contractions from t.

We say that a term t has the *Church-Rosser property* (t has CR) wrt[1] $\xrightarrow{*}_\beta$, if there exists a term t_3 such that $t_1 \xrightarrow{*}_\beta t_3$ and $t_2 \xrightarrow{*}_\beta t_3$ whenever $t \xrightarrow{*}_\beta t_1$ and $t \xrightarrow{*}_\beta t_2$. The β-reduction relation (or any other reduction relation defined on λ-terms) has the Church-Rosser property or is *confluent* if every term has the Church-Rosser property wrt $\xrightarrow{*}_\beta$ (wrt that relation, respectively).

We define next the formal machinery that will be needed in our work.

DEFINITION 1 A *direct reduct* of an application term $uv_1 \dots v_n$ ($n \geqslant 1$) is a term $u'v'_1 \dots v'_n$ such that $u \xrightarrow{*}_\beta u'$, $v_1 \xrightarrow{*}_\beta v'_1$, $\dots$, $v_n \xrightarrow{*}_\beta v'_n$ (therefore $uv_1 \dots v_n \xrightarrow{*}_\beta u'v'_1 \dots v'_n$).

We must note that a direct reduct of $uv_1 \dots v_n$ is defined wrt a fixed number n of operands, i.e. wrt a specific presentation of $uv_1 \dots v_n$ considered as a term constructed from u by the n consecutive applications $(u)v_1$, $(uv_1)v_2, \dots$, $(uv_1 \dots v_{n-1})v_n$. So any direct reduct of $uv_1 \dots v_n$ invariantly has the same form of presentation and a direct reduct of a direct reduct of $uv_1 \dots v_n$ is always a direct reduct of $uv_1 \dots v_n$.

LEMMA 2 *If* $uv_1 \dots v_n \xrightarrow{*}_\beta w$ *and* w *is not a direct reduct of* $uv_1 \dots v_n$ *then there exists a direct reduct* $u'v'_1 \dots v'_n$ *of* $uv_1 \dots v_n$ *where* $u' = \lambda x.u''$ *for some term* u'', *and* $u''[v'_1/x]v'_2 \dots v'_n \xrightarrow{*}_\beta w$, *i.e.*

$$uv_1 \dots v_n \xrightarrow{*}_\beta (\lambda x.u'')v'_1 \dots v'_n \longrightarrow_\beta u''[v'_1/x]v'_2 \dots v'_n \xrightarrow{*}_\beta w$$

Proof: Let w' be the first non direct reduct of $uv_1 \dots v_n$ in the reduction $uv_1 \dots v_n \xrightarrow{*}_\beta w$. Then

$$uv_1 \dots v_n \xrightarrow{*}_\beta u'v'_1 \dots v'_n \longrightarrow_\beta w' \xrightarrow{*}_\beta w$$

where $u'v'_1 \dots v'_n$ is a direct reduct of $uv_1 \dots v_n$. So w' cannot be a direct reduct of $u'v'_1 \dots v'_n$ because any direct reduct of $u'v'_1 \dots v'_n$ is obviously a direct reduct of $uv_1 \dots v_n$. The only way to get a non direct reduct of $u'v'_1 \dots v'_n$ is by contracting a redex which is not inside in any of $u', v'_1, \dots, v'_n$ and this is possible only if u' is a λ-abstraction $\lambda x.u''$ and the redex contracted is $(\lambda x.u'')v'_1$. Then $w' = u''[v'_1/x]v'_2 \dots v'_n$. □

LEMMA 3 *If* $t \xrightarrow{*}_\beta t'$ *and* $u \xrightarrow{*}_\beta u'$ *then* $u[t/x] \xrightarrow{*}_\beta u'[t'/x]$.

Proof: The proof can be found in any textbook on λ-calculus, for example [2, page 55]. □

[1] with respect to

DEFINITION 4 We define $\mathcal{CR}$ to be the set of λ-terms that have the Church-Rosser property, i.e. $\mathcal{CR} \stackrel{def}{=} \{t \in \Lambda : t \text{ has CR}\}$, and $\mathcal{CR}_0$ the set of λ-terms of the form $xv_1 \dots v_n$ ($n \geqslant 0$) where x is a variable and $v_1, \dots, v_n \in \mathcal{CR}$.

DEFINITION 5 $\mathcal{X} \subseteq \Lambda$ is said to be *saturated* when for all terms $u, t, t_1, \dots, t_n$ ($n \geqslant 0$) and for every variable x we have:

$$u[t/x]t_1 \dots t_n \in \mathcal{X} \Rightarrow (\lambda x.u)tt_1 \dots t_n \in \mathcal{X}$$

LEMMA 6 *(1) If $\mathcal{X}, \mathcal{Y} \subseteq \Lambda$ are saturated then $\mathcal{X} \cap \mathcal{Y}$ is saturated.*
(2) If $\mathcal{Y} \subseteq \Lambda$ is saturated and $\mathcal{X} \subseteq \Lambda$ then $\mathcal{X} \to \mathcal{Y}$ is saturated.

Proof: (1) If $u[t/x]t_1 \dots t_n \in \mathcal{X} \cap \mathcal{Y}$ then $(\lambda x.u)tt_1 \dots t_n \in \mathcal{X}, \mathcal{Y}$.
(2) If $u[t/x]t_1 \dots t_n \in \mathcal{X} \to \mathcal{Y}$ then $u[t/x]t_1 \dots t_n t_0 \in \mathcal{Y}$ for every $t_0 \in \mathcal{X}$, and because $\mathcal{Y}$ is saturated $(\lambda x.u)tt_1 \dots t_n t_0 \in \mathcal{Y}$. So $(\lambda x.u)tt_1 \dots t_n \in \mathcal{X} \to \mathcal{Y}$. □

PROPOSITION 7 *$\mathcal{CR}$ is saturated.*

Proof: Suppose that $u[t/x]t_1 t_2 \dots t_n \in \mathcal{CR}$. To prove that $(\lambda x.u)tt_1 \dots t_n \in \mathcal{CR}$ we suppose that $(\lambda x.u)tt_1 \dots t_n \xrightarrow{*}_\beta v$ and $(\lambda x.u)tt_1 \dots t_n \xrightarrow{*}_\beta w$ [Figure 1].

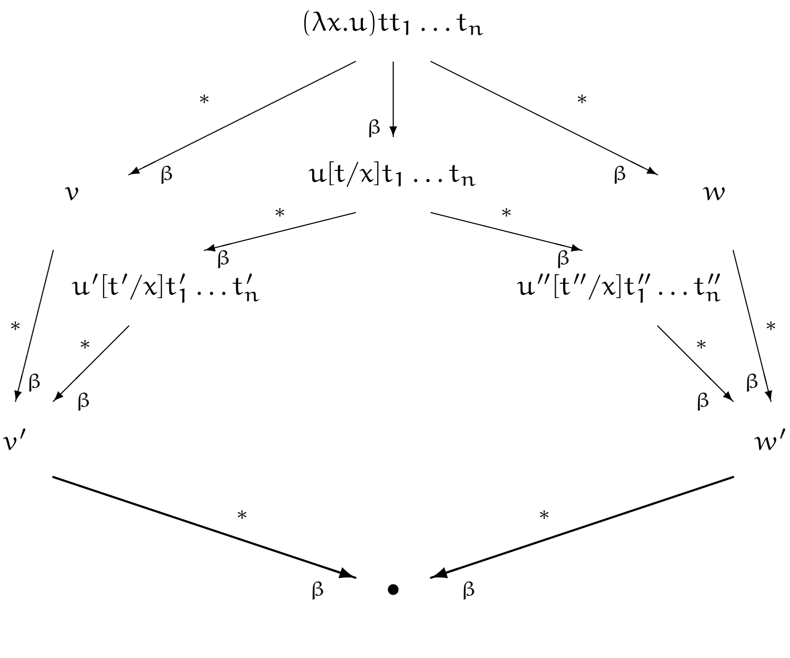

Figure 1: *Diagram of reductions for the proof of Proposition 7*

Consider the left reduction $(\lambda x.u)tt_1 \dots t_n \xrightarrow{*}_\beta v$. Then either v is a direct reduct of $(\lambda x.u)tt_1 \dots t_n$ or it is not.

In the *first case*, there exist terms $u', t', t'_1, \dots, t'_n$ such that

$$u \xrightarrow{*}_\beta u', \;\; t \xrightarrow{*}_\beta t', \;\; t_1 \xrightarrow{*}_\beta t'_1, \dots, \;\; t_n \xrightarrow{*}_\beta t'_n,$$

and $(\lambda x.u)tt_1 \dots t_n \xrightarrow{*}_\beta (\lambda x.u')t't'_1 \dots t'_n = v$. Then $v \longrightarrow_\beta u'[t'/x]t'_1 \dots t'_n$ (by one contraction). Let $v' = u'[t'/x]t'_1 \dots t'_n$. By Lemma 3, $u[t/x]t_1t_2 \dots t_n \xrightarrow{*}_\beta v'$ and so v and $u[t/x]t_1t_2 \dots t_n$ both reduce to v'.

In the *second case*, by Lemma 2 there exists a direct reduct $(\lambda x.u')t't'_1 \dots t'_n$ of $(\lambda x.u)tt_1 \dots t_n$ such that

$$u \xrightarrow{*}_\beta u', \ \ t \xrightarrow{*}_\beta t', \ \ t_1 \xrightarrow{*}_\beta t'_1, \dots, \ \ t_n \xrightarrow{*}_\beta t'_n,$$

and

$$(\lambda x.u)tt_1 \dots t_n \xrightarrow{*}_\beta (\lambda x.u')t't'_1 \dots t'_n \longrightarrow_\beta u'[t'/x]t'_1 \dots t'_n \xrightarrow{*}_\beta v$$

Let $v' = v$. By Lemma 3, $u[t/x]t_1t_2 \dots t_n \xrightarrow{*}_\beta u'[t'/x]t'_1 \dots t'_n$ and so $u[t/x]t_1t_2 \dots t_n \xrightarrow{*}_\beta v'$.

In *both cases* there exists a term v' such that $v \xrightarrow{*}_\beta v'$ (by one or zero contractions) and

$$u[t/x]t_1 \dots t_n \xrightarrow{*}_\beta v' \tag{1}$$

As the same argument holds also for the right reduction $(\lambda x.u)tt_1 \dots t_n \xrightarrow{*}_\beta w$, we can also obtain a term w' such that $w \xrightarrow{*}_\beta w'$ (by one or zero contractions) and

$$u[t/x]t_1 \dots t_n \xrightarrow{*}_\beta w' \tag{2}$$

The result follows from (1), (2), and the assumption that $u[t/x]t_1t_2 \dots t_n$ has CR. □

DEFINITION 8 An *interpretation* $\mathcal{I}$ is a mapping from type variables X to saturated subsets of Λ, denoted by $[\![X]\!]_\mathcal{I}$. We inductively extend $[\![\_]\!]_\mathcal{I}$ in to a mapping from types in the system D to subsets of Λ, in the following way:

(1) if A is a type variable then $[\![A]\!]_\mathcal{I}$ is already defined;

(2) if $A = B \cap C$ then $[\![A]\!]_\mathcal{I} \stackrel{def}{=} [\![B]\!]_\mathcal{I} \cap [\![C]\!]_\mathcal{I}$;

(3) if $A = B \to C$ then $[\![A]\!]_\mathcal{I} \stackrel{def}{=} ([\![B]\!]_\mathcal{I} \to [\![C]\!]_\mathcal{I}) \bigcap \mathcal{CR}$.

REMARK 9 We interpret the types of D by suitable sets of λ-terms that will guarantee the desired Church-Rosser property, i.e. $[\![A]\!]_\mathcal{I} \subseteq \mathcal{CR}$ [Lemma 14]. But in our variant of the reducibility method, we selected the interpretation of $B \to C$ to reside within $\mathcal{CR}$ in contrast to the usual reducibility interpretations where $B \to C$ is interpreted by $[\![B]\!]_\mathcal{I} \to [\![C]\!]_\mathcal{I}$. In the latter case we would be stuck with the proof of $\mathcal{CR} \to \mathcal{CR} \subseteq \mathcal{CR}$ as explained by the following reasoning.

Suppose that $t \in \mathcal{CR} \to \mathcal{CR}$ and let $t \xrightarrow{*}_\beta t_1$, $t \xrightarrow{*}_\beta t_2$. Then for any variable x not free in t, $(t)x \xrightarrow{*}_\beta (t_1)x$, $(t)x \xrightarrow{*}_\beta (t_2)x$ and because $x \in \mathcal{CR}$ we have that $(t)x \in \mathcal{CR}$. So we can find a term u such that $(t_1)x \xrightarrow{*}_\beta u$

and $(t_2)x \xrightarrow{*}_\beta u$. If u is a direct reduct of $(t_1)x, (t_2)x$ then $u = (t_3)x$ and $t_1 \xrightarrow{*}_\beta t_3$, $t_2 \xrightarrow{*}_\beta t_3$ therefore $t \in \mathcal{CR}$, i.e. the confluence from t simulates the confluence from $(t)x$ to u. Otherwise, $t_1 \xrightarrow{*}_\beta u_1$ and $t_2 \xrightarrow{*}_\beta u_2$ where $\lambda x.u \xrightarrow{*}_\eta u_1$ and $\lambda x.u \xrightarrow{*}_\eta u_2$, so u_1, u_2 are η-*equivalent*[2] and the confluence from $(t)x$ to u cannot be simulated by t [16].

PROPOSITION 10 *For every interpretation* $\mathcal{I}$ *and every type* A, $[\![A]\!]_{\mathcal{I}}$ *is saturated.*

Proof: By induction on the construction of type A. We consider only the case $A = B \to C$.

By the IH[3], $[\![C]\!]_{\mathcal{I}}$ is saturated. So $[\![B]\!]_{\mathcal{I}} \to [\![C]\!]_{\mathcal{I}}$ is saturated [Lemma 6]. By Proposition 7, $\mathcal{CR}$ is saturated and therefore $([\![B]\!]_{\mathcal{I}} \to [\![C]\!]_{\mathcal{I}}) \bigcap \mathcal{CR}$ is saturated [Lemma 6]. □

THEOREM 11 (soundness, adequacy) *Let* $\mathcal{I}$ *be an interpretation such that* $\mathcal{CR}_0 \subseteq [\![B]\!]_{\mathcal{I}} \subseteq \mathcal{CR}$ *for every type* B. *If* $x_1 : A_1, \ldots, x_k : A_k \vdash u : A$ *is a typing of* u, *then for all terms* $t_1 \in [\![A_1]\!]_{\mathcal{I}}, \ldots, t_k \in [\![A_k]\!]_{\mathcal{I}}$ *we have* $u[t_1/x_1, \ldots, t_k/x_k] \in [\![A]\!]_{\mathcal{I}}$.

Proof: We use induction on the typing in D of the term u. Consider the last rule used:

(1) For Rule 1, u is one variable between $x_1, \ldots, x_k$, say x_i, and $A = A_i$. Then $u[t_1/x_1, \ldots, t_k/x_k] = t_i$ where $t_i \in [\![A_i]\!]_{\mathcal{I}}$ by hypothesis.

(2) For Rule 2, $u = \lambda x.v$, $A = B \to C$ and we have:

$$\frac{x : B, x_1 : A_1, \ldots, x_k : A_k \vdash v : C}{x_1 : A_1, \ldots, x_k : A_k \vdash \lambda x.v : B \to C}$$

Because x is a bound variable in u, by the variable convention we can choose x such that $x \notin FV(t_1 t_2 \ldots t_k) \cup \{x_1, \ldots, x_k\}$. We want to prove that

$$(\lambda x.v)[t_1/x_1, \ldots, t_k/x_k] \in ([\![B]\!]_{\mathcal{I}} \to [\![C]\!]_{\mathcal{I}}) \bigcap \mathcal{CR}$$

By IH we have that (for all $t_i \in [\![A_i]\!]_{\mathcal{I}}$)

$$\forall t \in [\![B]\!]_{\mathcal{I}} \,,\ v[t/x, t_1/x_1, \ldots, t_k/x_k] \in [\![C]\!]_{\mathcal{I}}$$

(i) Because of the choice of x, the term $v[t/x, t_1/x_1, \ldots, t_k/x_k]$ is identical to the term $(v[t_1/x_1, \ldots, t_k/x_k])\,[t/x]$ modulo renaming of bound variables. Thus

$$(\lambda x.v[t_1/x_1, \ldots, t_k/x_k])\, t \in [\![C]\!]_{\mathcal{I}}$$

[2] The η-reduction relation (notation $\xrightarrow{*}_\eta$) is defined as the contextual, reflexive and transitive closure of the the relation $\lambda x.(v)x \longrightarrow_\eta v$ where $x \notin FV(v)$. The equivalence relation induced by $\xrightarrow{*}_\eta$ is called η-equivalence.

[3] induction hypothesis

because $[\![C]\!]_{\mathcal{J}}$ is saturated by Proposition 10. So

$$\lambda x.v[t_1/x_1, \ldots, t_k/x_k] \in [\![B]\!]_{\mathcal{J}} \to [\![C]\!]_{\mathcal{J}}$$

By the choice of x,

$$\lambda x.v[t_1/x_1, \ldots, t_k/x_k] = (\lambda x.v)[t_1/x_1, \ldots, t_k/x_k]$$

so

$$(\lambda x.v)[t_1/x_1, \ldots, t_k/x_k] \in [\![B]\!]_{\mathcal{J}} \to [\![C]\!]_{\mathcal{J}}$$

(ii) By hypothesis $\mathcal{CR}_0 \subseteq [\![B]\!]_{\mathcal{J}}$, $[\![C]\!]_{\mathcal{J}} \subseteq \mathcal{CR}$ and because $x \in \mathcal{CR}_0$ we have that

$$v[x/x, t_1/x_1, \ldots, t_k/x_k] \in \mathcal{CR}$$

Thus

$$v[t_1/x_1, \ldots, t_k/x_k] \in \mathcal{CR}$$

Since abstraction on the outside of a term does not add redexes $\lambda x.v[t_1/x_1, \ldots, t_k/x_k]$ has CR and because of the choice of x,

$$\lambda x.v[t_1/x_1, \ldots, t_k/x_k] = (\lambda x.v)[t_1/x_1, \ldots, t_k/x_k]$$

so

$$(\lambda x.v)[t_1/x_1, \ldots, t_k/x_k] \in \mathcal{CR}$$

(3) For Rule 3, $u = wv$ and for some type B we have:

$$\frac{x_1 : A_1, \ldots, x_k : A_k \vdash v : B \quad x_1 : A_1, \ldots, x_k : A_k \vdash w : B \to A}{x_1 : A_1, \ldots, x_k : A_k \vdash wv : A}$$

By IH, $v[t_1/x_1, \ldots, t_k/x_k] \in [\![B]\!]_{\mathcal{J}}$ and

$$w[t_1/x_1, \ldots, t_k/x_k] \in ([\![B]\!]_{\mathcal{J}} \to [\![A]\!]_{\mathcal{J}}) \bigcap \mathcal{CR}$$

so

$$(w[t_1/x_1, \ldots, t_k/x_k])\, v[t_1/x_1, \ldots, t_k/x_k] \in [\![A]\!]_{\mathcal{J}}$$

i.e. $(wv)[t_1/x_1, \ldots, t_k/x_k] \in [\![A]\!]_{\mathcal{J}}$.

(4) For Rule 4, we have for some type B:

$$\frac{x_1 : A_1, \ldots, x_k : A_k \vdash u : A \cap B}{x_1 : A_1, \ldots, x_k : A_k \vdash u : A}$$

By IH, $u[t_1/x_1, \ldots, t_k/x_k] \in [\![A]\!]_{\mathcal{J}} \cap [\![B]\!]_{\mathcal{J}}$, so $u[t_1/x_1, \ldots, t_k/x_k] \in [\![A]\!]_{\mathcal{J}}$ and the same holds for $\cap_2$-elimination.

(5) For Rule 5, $A = B \cap C$ and we have:

$$\frac{x_1 : A_1, \ldots, x_k : A_k \vdash u : B \quad x_1 : A_1, \ldots, x_k : A_k \vdash u : C}{x_1 : A_1, \ldots, x_k : A_k \vdash u : B \cap C}$$

By IH, $u[t_1/x_1, \ldots, t_k/x_k] \in [\![B]\!]_{\mathcal{I}}$ and $u[t_1/x_1, \ldots, t_k/x_k] \in [\![C]\!]_{\mathcal{I}}$, so $u[t_1/x_1, \ldots, t_k/x_k] \in [\![B \cap C]\!]_{\mathcal{I}}$. □

LEMMA 12 $\mathcal{CR}_0 \subseteq \mathcal{CR}$

Proof: Suppose that $xv_1 \ldots v_n \xrightarrow{*}_\beta u$ and $xv_1 \ldots v_n \xrightarrow{*}_\beta w$ where $v_1, \ldots, v_n$ have CR. Then u and w must necessarily be direct reducts of $xv_1 \ldots v_n$ [Lemma 2] of the form $u = xv'_1 \ldots v'_n$, $w = xv''_1 \ldots v''_n$ and $v_i \xrightarrow{*}_\beta v'_i$, $v_i \xrightarrow{*}_\beta v''_i$ for all $i \in \{1, \ldots, n\}$. But then there exist terms v'''_i $(1 \leqslant i \leqslant n)$ such that $v'_i \xrightarrow{*}_\beta v'''_i$ and $v''_i \xrightarrow{*}_\beta v'''_i$. By using the properties of β-reduction we can conclude that $u \xrightarrow{*}_\beta xv'''_1 \ldots v'''_n$ and $w \xrightarrow{*}_\beta xv'''_1 \ldots v'''_n$. □

LEMMA 13 $\mathcal{CR}_0 \subseteq (\mathcal{CR} \to \mathcal{CR}_0)$

Proof: Suppose that $xv_1 \ldots v_n \in \mathcal{CR}_0$ and $v \in \mathcal{CR}$. Then by definition of $\mathcal{CR}_0$, $v_i \in \mathcal{CR}$ $(1 \leqslant i \leqslant n)$ and therefore $xv_1 \ldots v_n v \in \mathcal{CR}_0$. □

LEMMA 14 *If* $\mathcal{I}$ *is an interpretation such that* $\mathcal{CR}_0 \subseteq [\![X]\!]_{\mathcal{I}} \subseteq \mathcal{CR}$ *for every type variable* X*, then* $\mathcal{CR}_0 \subseteq [\![A]\!]_{\mathcal{I}} \subseteq \mathcal{CR}$ *for every type* A.

Proof: We use induction on the construction of type A.

(1) If A is a type variable X then the result follows from the assumption.

(2) If $A = B \cap C$ then by IH, $\mathcal{CR}_0 \subseteq [\![B]\!]_{\mathcal{I}} \subseteq \mathcal{CR}$ and $\mathcal{CR}_0 \subseteq [\![C]\!]_{\mathcal{I}} \subseteq \mathcal{CR}$. So $\mathcal{CR}_0 \subseteq ([\![B]\!]_{\mathcal{I}} \cap [\![C]\!]_{\mathcal{I}}) \subseteq \mathcal{CR}$.

(3) If $A = B \to C$ then evidently $([\![B]\!]_{\mathcal{I}} \to [\![C]\!]_{\mathcal{I}}) \bigcap \mathcal{CR} \subseteq \mathcal{CR}$. By IH, $\mathcal{CR}_0 \subseteq [\![C]\!]_{\mathcal{I}}$, $[\![B]\!]_{\mathcal{I}} \subseteq \mathcal{CR}$, so $(\mathcal{CR} \to \mathcal{CR}_0) \subseteq ([\![B]\!]_{\mathcal{I}} \to [\![C]\!]_{\mathcal{I}})$ and by Lemma 13, $\mathcal{CR}_0 \subseteq ([\![B]\!]_{\mathcal{I}} \to [\![C]\!]_{\mathcal{I}})$. By Lemma 12, $\mathcal{CR}_0 \subseteq ([\![B]\!]_{\mathcal{I}} \to [\![C]\!]_{\mathcal{I}}) \bigcap \mathcal{CR}$. □

THEOREM 15 (church-rosser for typed terms) *If* t *is typed in the system D then* t *has CR.*

Proof: Suppose that $x_1 : A_1, \ldots, x_k : A_k \vdash t : A$ is a typing of t. Let $\mathcal{I}$ be an interpretation such that $[\![X]\!]_{\mathcal{I}} = \mathcal{CR}$ for every type variable X. Then by Lemma 14, $\mathcal{CR}_0 \subseteq [\![A_i]\!]_{\mathcal{I}}$ for all $i \in \{1, \ldots, k\}$ and because all x_i's belong to $\mathcal{CR}_0$ we have $x_i \in [\![A_i]\!]_{\mathcal{I}}$ for all i's. By the soundness theorem $t[x_1/x_1, \ldots, x_k/x_k] \in [\![A]\!]_{\mathcal{I}}$, i.e. $t \in [\![A]\!]_{\mathcal{I}}$ and again by Lemma 14, $t \in \mathcal{CR}$. □

3 CONFLUENCE OF DEVELOPMENTS

We have proved that every term t typed in the system D has the Church-Rosser property. Therefore the β-reduction relation for *typed* terms is confluent.

We proceed to prove the confluence of a "restricted" kind of reduction on the *untyped* terms by defining an embedding of the untyped terms into the typed terms. This "restricted" reduction defines the notion of a *development* [2, 13]. In our proof we are motivated by the proof of the *theorem of finite developments* as presented in [13, pages 45–49]. The rest of this section up to Lemma 33 makes explicit the machinery used there.

First we need to define an operator $\Psi(\_,\_)$ such that for any pair $(t, \mathcal{F})$ with $t \in \Lambda$ and $\mathcal{F}$ a set of occurrences of redexes in t, $\Psi(t, \mathcal{F})$ will be produced from the term t where all the redexes $(\lambda x.u)v$ in t *not* belonging to $\mathcal{F}$ are "frozen" by replacing them with $((c)\lambda x.u)v$, where c is a *new* distinguished variable for λ-terms that is never substituted. By doing this we leave as redexes in t only the ones in $\mathcal{F}$. In addition, we will block the possibility of creating new redexes from β-reductions in t out of the contraction of the redexes in $\mathcal{F}$. For example, if t contains a *sub*term $(y)v$, then after β-reducing a redex in t, some subterm of the form $\lambda x.u$ may substitute y and create a new redex. In order to avoid this situation we will also put in front of every subterm of the form $(w)v$, with w not a λ-abstraction, the distinguished variable c, i.e. we replace $(w)v$ with $((c)w)v$. Thus we also "freeze" the applications in t so that they cannot be transformed into redexes.

REMARK 16 In what follows, $\mathcal{F}$ is a *set* of *occurrences* of redexes in t, i.e. of redexes accompanied with a pointer showing their location in term t. For example, the same redex $(\lambda x.x)x$ occurs in two different locations in $t = ((\lambda x.x)x)(\lambda x.x)x$ and thus may appear twice in $\mathcal{F}$ but with a different pointer in each case.

However, for brevity reasons, we will refer to $\mathcal{F}$ as a set of redexes in t and will not specify the accompanying pointer of the redexes.

DEFINITION 17 Let $t \in \Lambda$ and $\mathcal{F}$ a set of redexes in t. We define formally the operator $\Psi(\_,\_)$ by induction on t:

(1) if t is a variable x then $\mathcal{F} = \emptyset$ and

$$\Psi(x, \emptyset) \stackrel{def}{=} x$$

(2) if t is a λ-abstraction $\lambda x.u$ then $\mathcal{F}$ is a set of redexes in u and

$$\Psi(\lambda x.u, \mathcal{F}) \stackrel{def}{=} \lambda x.\Psi(u, \mathcal{F})$$

(3) if t is an application uv and $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) is the set of redexes of u (resp. v) in $\mathcal{F}$ then $\mathcal{F} \setminus \{t\} = \mathcal{F}_1 \cup \mathcal{F}_2$ and

$$\Psi(uv, \mathcal{F}) \stackrel{def}{=} \begin{cases} (c)\,\Psi(u, \mathcal{F}_1)\,\Psi(v, \mathcal{F}_2) & \text{if } t \notin \mathcal{F} \\ \Psi(u, \mathcal{F}_1)\,\Psi(v, \mathcal{F}_2) & \text{otherwise} \end{cases}$$

We call $\Psi(t, \mathcal{F})$ the *freezing* of $(t, \mathcal{F})$.

EXAMPLE 18

1. If $t = (\lambda x.x)(\lambda x.x)y$ and $\mathcal{F} = \{(\lambda x.x)y\}$, then $\Psi(t, \mathcal{F}) = ((c)\lambda x.x)(\lambda x.x)y$.
2. If $t = (\lambda x.(x)x)\lambda x.(x)x$ and $\mathcal{F} = \{(\lambda x.(x)x)\lambda x.(x)x\}$, then $\Psi(t, \mathcal{F}) = (\lambda x.(c)xx)\lambda x.(c)xx$.

Let c be the new variable introduced above.

DEFINITION 19 We define inductively a subset of the λ-terms with c, denoted Λ_c, in the following way:

(1) if x is a variable distinct from c, then $x \in \Lambda_c$ (variable)

(2) if x is a variable distinct from c and $T \in \Lambda_c$, then $\lambda x.T \in \Lambda_c$ (λ-abstraction)

(3) if $T, U \in \Lambda_c$, then $(c)TU \in \Lambda_c$ (*non-redex application*)

(4) if $T, U \in \Lambda_c$ and T is a λ-abstraction, then $TU \in \Lambda_c$ (*redex application*)

Note that there are terms of Λ not in Λ_c, for example c, $(\lambda x.x)yz$, $((c)\lambda x.x)yz \notin \Lambda_c$ but $((c)(\lambda x.x)y)z$, $((c)((c)\lambda x.x)y)z \in \Lambda_c$.

LEMMA 20 *(1) If* $T, U \in \Lambda_c$ *and* $x \neq c$*, then* $T[U/x] \in \Lambda_c$*. (2)* Λ_c *is closed under* β*-reduction, i.e. if* $T \in \Lambda_c$ *and* $T \xrightarrow{*}_\beta T'$ *then* $T' \in \Lambda_c$.

Proof: (1) By induction on T. (2) By induction on T using (1). □

LEMMA 21 *Every term of* Λ_c *is typed in the system D.*

Proof: We can actually prove that for every term $T \in \Lambda_c$ *and every context* Γ *for all the free variables of* T*, except* c, there exist types C, A such that $\Gamma, c : C \vdash_D T : A$. The proof can be found in [13, pages 46–47]. The use of intersection types is crucial in this proof but we will see later [Proposition 36] that with a slight modification of Λ_c the proof can also be adapted for the simply typed λ-calculus. □

LEMMA 22 *The range of the freezing operator* $\Psi(\_, \_)$ *is a subset of* Λ_c.

Proof: We prove by an easy induction on t that if $t \in \Lambda$ and $\mathcal{F}$ is a set of redexes in t, then $\Psi(t, \mathcal{F}) \in \Lambda_c$. □

DEFINITION 23 We define a surjective mapping from Λ_c onto Λ called *erasure* and denoted $|\_|$, by induction on $T \in \Lambda_c$:

(1) if T is a variable distinct from c, then $|T| \stackrel{def}{=} T$;

(2) if $T = \lambda x.U$ and $U \in \Lambda_c$, then $|T| \stackrel{def}{=} \lambda x.|U|$;

(3) if $T = (c)UV$ and $U, V \in \Lambda_c$, then $|T| \stackrel{def}{=} (|U|)|V|$;

(4) if $T = (\lambda x.U)V$ and $U, V \in \Lambda_c$, then $|T| \stackrel{def}{=} (\lambda x.|U|)\,|V|$.

Thus $|T|$ is obtained by leaving out the variable c in T. It is noticeable that erasure does not preserve types.

We will now show, in the following four lemmas, that $\Psi(\_\,,\_)$ defines a one-to-one correspondence between the pairs $(t, \mathcal{F})$ and the terms of Λ_c, i.e. an embedding of the untyped terms into the typed terms.

LEMMA 24 *If* $t \in \Lambda$ *and* $\mathcal{F}$ *is a set of redexes in* t*, then* $|\Psi(t, \mathcal{F})| = t$.

Proof: By an easy induction on $t \in \Lambda$ using Lemma 22. □

LEMMA 25 *If* $t \in \Lambda$ *and* $\mathcal{F}$ *is a set of redexes in* t*, then* $\mathcal{F} = \{|R| : R$ *is a redex in* $\Psi(t, \mathcal{F})\}$.

Proof: By induction on $t \in \Lambda$ using Lemma 24. □

LEMMA 26 *If* $T \in \Lambda_c$*,* $t = |T|$ *and* $\mathcal{F} = \{|R| : R$ *redex in* $T\}$ *then* $\mathcal{F}$ *is a set (possibly empty) of redexes in* t *and* $\Psi(t, \mathcal{F}) = T$*, i.e.* $\Psi(\_\,,\_)$ *is surjective.*

Proof: By induction on $T \in \Lambda_c$. □

LEMMA 27 *For every* $T \in \Lambda_c$ *there exists one and only one pair* $(t, \mathcal{F})$ *with* $t \in \Lambda$ *and* $\mathcal{F}$ *a set of redexes in* t*, such that* $\Psi(t, \mathcal{F}) = T$*. Therefore* $\Psi(\_\,,\_)$ *is a one-to-one mapping onto* Λ_c*.*

Proof: Due to the previous lemma it suffices to prove the "only one" part. This is easily proved using Lemma 24 and Lemma 25. □

DEFINITION 28 Let $t \in \Lambda$, $\mathcal{F}$ a set of redexes in t and $t \longrightarrow_\beta t_1$ by contraction of a redex r in t. If $T = \Psi(t, \mathcal{F} \cup \{r\})$, R the redex in T with $|R| = r$ [Lemma 25] and T_1 the term obtained by contraction of R in T, then by Lemma 27 there exists $\mathcal{F}_1$ such that $\Psi(t_1, \mathcal{F}_1) = T_1$ (in fact $t_1 = |T_1|$ and $\mathcal{F}_1 = \{|R| : R$ redex in $T_1\}$). We call $\mathcal{F}_1$ the set of *residuals of* $\mathcal{F}$ *in* t_1 *relative to* r.

EXAMPLE 29 Let $t = (\lambda x.(x)x)\lambda x.(x)x \longrightarrow_\beta (\lambda x.(x)x)\lambda x.(x)x = t_1$ and $\mathcal{F} = \{(\lambda x.(x)x)\lambda x.(x)x\}$. Then

$$\begin{aligned} T &= \Psi(t, \mathcal{F}) = (\lambda x.(c)xx)\lambda x.(c)xx \\ T_1 &= ((c)\lambda x.(c)xx)\lambda x.(c)xx \\ \mathcal{F}_1 &= \emptyset \end{aligned}$$

So the set of residuals of $\mathcal{F}$ in t_1 relative to redex $(\lambda x.(x)x)\lambda x.(x)x$ is $\emptyset$.

EXAMPLE 30 Let $t = (\lambda x.(x)x)(\lambda x.x)x \longrightarrow_\beta ((\lambda x.x)x)(\lambda x.x)x = t_1$ and $\mathcal{F} = \{(\lambda x.x)x\}$. Then

$$\begin{aligned} T &= \Psi(t, \mathcal{F} \cup \{(\lambda x.(x)x)(\lambda x.x)x\}) = (\lambda x.(c)xx)(\lambda x.x)x \\ T_1 &= ((c)(\lambda x.x)x)(\lambda x.x)x \\ \mathcal{F}_1 &= \{(\lambda x.x)x, (\lambda x.x)x\} \end{aligned}$$

So the set of residuals of $\mathcal{F}$ in t_1 relative to redex $(\lambda x.(x)x)(\lambda x.x)x$ is $\{(\lambda x.x)x, (\lambda x.x)x\}$, i.e. two distinct *occurrences* of the same redex $(\lambda x.x)x$.

DEFINITION 31 Let $t \in \Lambda$, $\mathcal{F}$ a set of redexes in t and the β-reduction

$$t \longrightarrow_\beta t_1 \longrightarrow_\beta \ldots t_{n-1} \longrightarrow_\beta t_n$$

obtained by contracting consecutively the redexes r in t, r_1 in t_1, ..., r_{n-1} in t_{n-1}. We define, by induction on n, the set $\mathcal{F}_n$ of *residuals of* $\mathcal{F}$ *in* t_n *relative to* $(r, r_1, \ldots, r_{n-1})$: if $n = 1$ then $\mathcal{F}_1$ is defined above; if $n \geqslant 2$ then $\mathcal{F}_n$ is the set of residuals of $\mathcal{F}_{n-1}$ in t_n relative to r_{n-1} where $\mathcal{F}_{n-1}$ is the set of residuals of $\mathcal{F}$ in t_{n-1} relative to $(r, r_1, \ldots, r_{n-2})$.

Intuitively, given a β-reduction of a term t we select a set $\mathcal{F}$ of redexes in the term, we "mark" those redexes (in Λ_c we "freeze" all the other redexes by blocking them with the variable c) and we follow their evolution throughout the reduction.

DEFINITION 32 Let $t \in \Lambda$ and $\mathcal{F}$ a set of redexes in t. A *development* of $(t, \mathcal{F})$ is a β-reduction $t \longrightarrow_\beta t_1 \longrightarrow_\beta \ldots t_{n-1} \longrightarrow_\beta t_n$ $(n \geqslant 0)$ obtained by contracting consecutively the redexes $r, r_1, \ldots, r_{n-1}$ where $r \in \mathcal{F}$ and r_i is a residual of $\mathcal{F}$ in t_i relative to $(r, r_1, \ldots, r_{i-1})$, for all i. If $\mathcal{F}_n$ is the set of residuals of $\mathcal{F}$ in t_n relative to $(r, r_1, \ldots, r_{n-1})$ then we denote the development by $(t, \mathcal{F}) \xrightarrow{*}_d (t_n, \mathcal{F}_n)$ or $t \xrightarrow{\mathcal{F}}_d t_n$. As in the case of β-reduction we write $(t, \mathcal{F}) \longrightarrow_d (t_1, \mathcal{F}_1)$ for the one-step development where $t \longrightarrow_\beta t_1$.

In a development of $(t, \mathcal{F})$ we always contract redexes that are residuals of the initial set $\mathcal{F}$ of redexes. This is achieved within Λ_c by "freezing" the applications in t, blocking them with the variable c, so that they will not become redexes themselves.

LEMMA 33 *Let* $t \in \Lambda$ *and* $\mathcal{F}$ *a set of redexes in* t. *There exists a one-to-one correspondence between the developments of* $(t, \mathcal{F})$ *and the* β*-reductions of* $\Psi(t, \mathcal{F})$.

Proof: It suffices to show that

$$(t, \mathcal{F}) \longrightarrow_d (t', \mathcal{F}') \Leftrightarrow \Psi(t, \mathcal{F}) \longrightarrow_\beta \Psi(t', \mathcal{F}')$$

But this is immediate from Lemma 27 and the way of defining residuals. □

THEOREM 34 (confluence of developments) *If* $t \xrightarrow{\mathcal{F}_1}_d t_1$ *and* $t \xrightarrow{\mathcal{F}_2}_d t_2$ *then there exist sets* $\mathcal{F}^1$, *resp.* $\mathcal{F}^2$, *of redexes in* t_1, *resp.* t_2, *and a term* $t_3 \in \Lambda$ *such that* $t_1 \xrightarrow{\mathcal{F}^1}_d t_3$ *and* $t_2 \xrightarrow{\mathcal{F}^2}_d t_3$.

Proof: The proof is sketched in Figure 2. Let $t \xrightarrow{\mathcal{F}_1}_d t_1$ and $t \xrightarrow{\mathcal{F}_2}_d t_2$. Then there exist $\mathcal{F}_1^1, \mathcal{F}_2^2$ such that $(t, \mathcal{F}_1) \xrightarrow{*}_d (t_1, \mathcal{F}_1^1)$ and $(t, \mathcal{F}_2) \xrightarrow{*}_d (t_2, \mathcal{F}_2^2)$. By extending the initial sets of redexes $\mathcal{F}_1, \mathcal{F}_2$ to $\mathcal{F}_1 \cup \mathcal{F}_2$ and contracting the same redexes, we get the developments $(t, \mathcal{F}_1 \cup \mathcal{F}_2) \xrightarrow{*}_d (t_1, \mathcal{F}_1^1 \cup \mathcal{F}_2^1)$ and $(t, \mathcal{F}_1 \cup \mathcal{F}_2) \xrightarrow{*}_d (t_2, \mathcal{F}_1^2 \cup \mathcal{F}_2^2)$ for some $\mathcal{F}_2^1$ (resp. $\mathcal{F}_1^2$) which are the residuals of $\mathcal{F}_2$

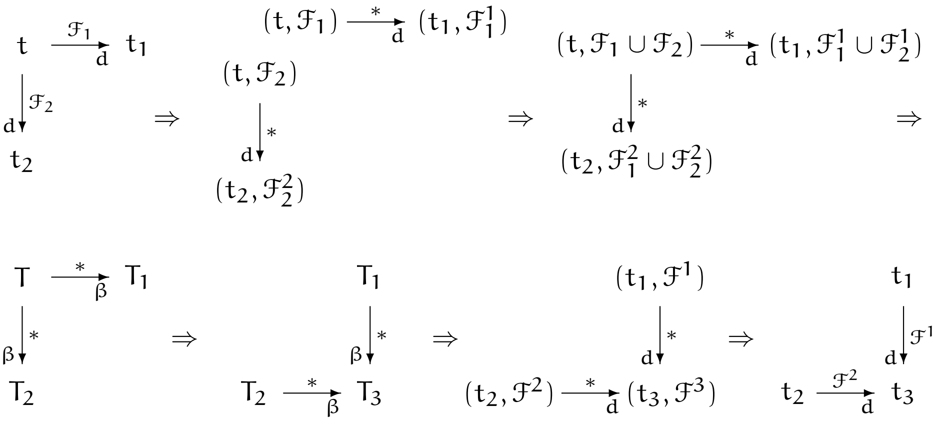

Figure 2: *Sketching the proof of Theorem 34*

(resp. $\mathcal{F}_1$) for the corresponding reductions from t.[4] Let $\mathcal{F}^1 = \mathcal{F}_1^1 \cup \mathcal{F}_2^1$ and $\mathcal{F}^2 = \mathcal{F}_1^2 \cup \mathcal{F}_2^2$. By Lemma 33 there exist $T, T_1, T_2 \in \Lambda_c$ such that

$$T = \Psi(t, \mathcal{F}_1 \cup \mathcal{F}_2), \quad T_1 = \Psi(t_1, \mathcal{F}^1), \quad T_2 = \Psi(t_2, \mathcal{F}^2)$$

and $T \xrightarrow{*}_\beta T_1$, $T \xrightarrow{*}_\beta T_2$. But T is typed in the system *D* [Lemma 21] therefore T has CR [Theorem 15]. So there exist $T_3 \in \Lambda_c$ such that $T_1 \xrightarrow{*}_\beta T_3$, $T_2 \xrightarrow{*}_\beta T_3$. By Lemma 33 there exist $t_3, \mathcal{F}^3$ such that $T_3 = \Psi(t_3, \mathcal{F}^3)$ and

$$(t_1, \mathcal{F}^1) \xrightarrow{*}_d (t_3, \mathcal{F}^3), \quad (t_2, \mathcal{F}^2) \xrightarrow{*}_d (t_3, \mathcal{F}^3)$$

i.e. $t_1 \xrightarrow{\mathcal{F}^1}_d t_3$ and $t_2 \xrightarrow{\mathcal{F}^2}_d t_3$. □

3.1 DIGRESSION: A PROOF WITHOUT INTERSECTION TYPES

In the above proof we made use of the fact that typability in D implies the Church-Rosser property [Theorem 15] and for this reason we employed the "freezing" mechanism of Λ_c to simulate the process of a development. We can however prove Theorem 34 *without introducing system* D [12]. All we need is the *simply typed* λ*-calculus* $\lambda_\rightarrow$ for which the analogous Church-Rosser theorem, i.e. typability in $\lambda_\rightarrow$ implies the Church-Rosser property, can be found in [10, 15].

The simply typed λ-calculus can be defined as a restriction of system D by ommitting the intersection types and the corresponding rules ($\cap_1$-elimination), ($\cap_2$-elimination), and ($\cap$-introduction). The typing relation will be denoted by $\vdash_{\lambda_\rightarrow}$.

The "freezing" mechanism of Λ_c must now be adapted to the new situation. We consider a denumerable set $\mathcal{C} = \{c_0, c_1, \ldots\}$ of *new* distinguished variables.

[4] The residuals of a set of redexes are determined by the residuals of the individual redexes.

DEFINITION 35 We define inductively a subset of the λ-terms with the variables $c_0, c_1, \ldots$, denoted $\Lambda_{\overline{\mathbf{c}}}$, as before:

(1) if x is a variable distinct from $c_0, c_1, \ldots$, then $x \in \Lambda_{\overline{\mathbf{c}}}$ (variable)

(2) if x is a variable distinct from $c_0, c_1, \ldots$ and $T \in \Lambda_{\overline{\mathbf{c}}}$, then $\lambda x.T \in \Lambda_{\overline{\mathbf{c}}}$ (λ-abstraction)

(3) if $T, U \in \Lambda_{\overline{\mathbf{c}}}$, then $(c_i)TU \in \Lambda_{\overline{\mathbf{c}}}$ for any i (*non-redex application*)

(4) if $T, U \in \Lambda_{\overline{\mathbf{c}}}$ and T is a λ-abstraction, then $TU \in \Lambda_{\overline{\mathbf{c}}}$ (*redex application*)

Any term of $\Lambda_{\overline{\mathbf{c}}}$ can be transformed to a term of Λ_c by just replacing the variables $c_0, c_1, \ldots$ with c. We can easily prove as before that the set $\Lambda_{\overline{\mathbf{c}}}$ is closed under β-reduction and that every term of $\Lambda_{\overline{\mathbf{c}}}$ is typed in the system D.

PROPOSITION 36 *Let* $T \in \Lambda_{\overline{\mathbf{c}}}$ *be a term where each of the variables* $c_0, c_1, \ldots$ *has at most one occurrence, and* Γ *be any context for the free variables of* T*, except* $c_0, c_1, \ldots$. *Then there exist types* $A, C_0, C_1, \ldots, C_n$ *of the simply-typed* λ*-calculus such that* $\Gamma, c_0 : C_0, \ldots, c_n : C_n \vdash_{\lambda_\to} T : A$.

Proof: We use induction on T. We sketch the proof for the last two cases (3) and (4).

(3) Let $T = c_0UV$. From hypothesis we can suppose that $\mathrm{FV}(U) \cap \mathcal{C} = \{c_1, \ldots, c_k\}$ and $\mathrm{FV}(V) \cap \mathcal{C} = \{c_{k+1}, \ldots, c_n\}$. By IH, there exist types B, D, $C_1, \ldots, C_k$, $C_{k+1}, \ldots C_n$ such that

$$\Gamma, c_1 : C_1, \ldots, c_k : C_k \vdash_{\lambda_\to} U : B$$

and

$$\Gamma, c_{k+1} : C_{k+1}, \ldots, c_n : C_n \vdash_{\lambda_\to} V : D$$

Then we can deduce that

$$\Gamma, c_1 : C_1, \ldots, c_k : C_k, c_{k+1} : C_{k+1}, \ldots, c_n : C_n, c_0 : B \to D \to A \vdash_{\lambda_\to} c_0UV : A$$

(4) Let $T = (\lambda x.V)U$. From hypothesis we can suppose that $\mathrm{FV}(U) \cap \mathcal{C} = \{c_0, \ldots, c_k\}$ and $\mathrm{FV}(V) \cap \mathcal{C} = \{c_{k+1}, \ldots, c_n\}$. By IH, there exist types $B, C_0, \ldots, C_k$ such that

$$\Gamma, c_0 : C_0, \ldots, c_k : C_k \vdash_{\lambda_\to} U : B$$

By IH again, for the context $\Gamma, x : B$ there exist types $A, C_{k+1}, \ldots, C_n$ such that

$$\Gamma, x : B, c_{k+1} : C_{k+1}, \ldots, c_n : C_n \vdash_{\lambda_\to} V : A$$

Then we can easily deduce

$$\Gamma, c_0 : C_0, \ldots, c_k : C_k, c_{k+1} : C_{k+1}, \ldots, c_n : C_n \vdash_{\lambda_\to} (\lambda x.V)U : A$$

The cases (1) and (2) are immediate. □

So the terms of $\Lambda_{\bar{c}}$ with at most one occurrence of $c_0, c_1, \ldots$ are typable in the simply-typed λ-calculus and as we stated above they have the Church-Rosser property. Consider now the terms T, T_1, T_2 in the proof of Theorem 34. If we replace each occurrence of the variable c in T by a *new* variable in $\mathcal{C}$ we get a term $T' \in \Lambda_{\bar{c}}$ with at most one occurrence of $c_0, c_1, \ldots$ and with the same redexes as T. By simulating the reductions $T \xrightarrow{*}_\beta T_1$ and $T \xrightarrow{*}_\beta T_2$ we get in $\Lambda_{\bar{c}}$ the reductions $T' \xrightarrow{*}_\beta T_1'$ and $T' \xrightarrow{*}_\beta T_2'$ where T_1', T_2' have the same redexes as T_1, T_2, respectively. Because T' has CR, there exists a term $T_3' \in \Lambda_{\bar{c}}$ such that $T_1' \xrightarrow{*}_\beta T_3'$ and $T_2' \xrightarrow{*}_\beta T_3'$. It remains to replace the variables $c_0, c_1, \ldots$ by c and recover the term $T_3 \in \Lambda_c$ and the reductions $T_1 \xrightarrow{*}_\beta T_3$, $T_2 \xrightarrow{*}_\beta T_3$. Then we can proceed with the rest of the proof of Theorem 34.

4 APPLICATION: THE CHURCH-ROSSER THEOREM

DEFINITION 37 We define a reduction relation on Λ called *one-reduction* (notation $\xrightarrow{*}_1$) by: $t \xrightarrow{*}_1 t' \overset{def}{\iff} \exists \mathcal{F}, \mathcal{F}'$ such that $(t, \mathcal{F}) \xrightarrow{*}_d (t', \mathcal{F}')$.[5]

LEMMA 38 $\xrightarrow{*}_\beta$ *is the transitive closure of* $\xrightarrow{*}_1$.

Proof: Let $t \xrightarrow{*}_\beta t'$. We use induction on the length n of the reduction. If $n = 0$, i.e. $t' = t$, then for some $\mathcal{F}$, $(t, \mathcal{F}) \xrightarrow{*}_d (t, \mathcal{F})$ trivially. If $n \geqslant 1$ then $t \xrightarrow{*}_\beta t'' \longrightarrow_\beta t'$ for some $t'' \in \Lambda$. By IH there exist terms $t_1, t_2, \ldots, t_k \in \Lambda$ such that

$$t \xrightarrow{*}_1 t_1 \xrightarrow{*}_1 t_2 \ldots \xrightarrow{*}_1 t_k \xrightarrow{*}_1 t''$$

If r is the reduced redex in $t'' \longrightarrow_\beta t'$ then $(t'', \{r\}) \longrightarrow_d (t', \emptyset)$, so $t'' \xrightarrow{*}_1 t'$, i.e. finally

$$t \xrightarrow{*}_1 t_1 \xrightarrow{*}_1 t_2 \ldots \xrightarrow{*}_1 t_k \xrightarrow{*}_1 t'' \xrightarrow{*}_1 t'$$

□

THEOREM 39 (church-rosser) *If* $t \in \Lambda$ *then* t *has CR.*

Proof: We have to show that if $t \xrightarrow{*}_\beta t_1$ and $t \xrightarrow{*}_\beta t_2$ then there exists $t_3 \in \Lambda$ such that $t_1 \xrightarrow{*}_\beta t_3$ and $t_2 \xrightarrow{*}_\beta t_3$. This is immediate from Lemma 38 and Theorem 34 by a simple diagram chasing of Figure 3. □

ACKNOWLEDGEMENT

The authors would like to thank the two anonymous referees for their very helpful comments.

[5]This relation is almost the same as the one defined in [2, Definition 11.2.27] with the difference that there $\mathcal{F}' = \emptyset$. The reason is that when $\mathcal{F}' = \emptyset$ all residuals are "consumed" and the development ends with a *unique* term [2, Theorem 11.2.25].

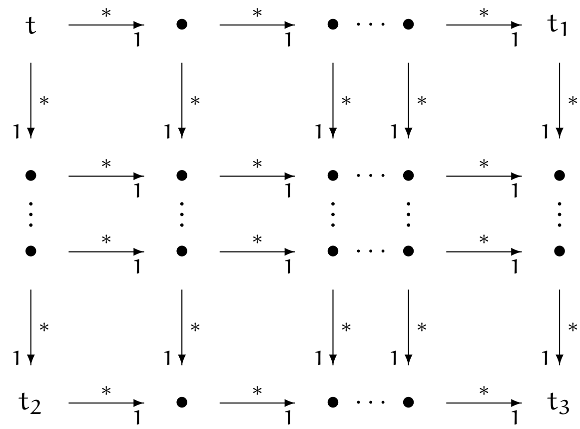

Figure 3: *Diagram of reductions for the proof of Theorem 39*

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