An atomic theory with no prime models

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Abstract: We construct an atomic uncountable theory with no prime models. This contrasts with the countable case.

We follow the standard textbook [1]. We start with the basic definitions to set up the notation. Let $\mathfrak{L}$ be a first order language, and T be a complete theory in $\mathfrak{L}$. A formula $\phi(x_0, \ldots x_{n-1})$ is complete in T if for all $\psi(x_0, \ldots x_{n-1})$, we have

$$T \models \phi \to \psi \text{ or } T \models \phi \to \neg\psi.$$

A formula $\theta(x_0, \ldots x_{n-1})$ is completable in T iff there is a complete formula $\phi(x_0, \ldots x_{n-1})$ with $T \models \phi \to \theta$. A theory T is atomic if every $\mathfrak{L}$ formula which is consistent with T is completable in T. An $\mathfrak{L}$-structure $\mathfrak{M}$ is said to be atomic, if every n-tuple $a_0, \ldots a_{n-1} \in M$ satisfies a complete formula in $\mathrm{Th}(\mathfrak{M})$. A model $\mathfrak{M}$ is a prime model if it embeds elementarily in any model of $\mathrm{Th}(\mathfrak{M})$. A Theorem of Vaught [1] 2.3.2 says that If T is a complete *countable* theory, then T has a countable atomic - equivalently a prime model - if and only if it is atomic. The proof in the countable case appeals to the Henkin-Orey omitting types theorem which we recall. Let Σ be a set of formulas. T locally realizes Σ if there is a formula $\phi(x_0, \ldots x_{n-1})$ in $\mathfrak{L}$ such that

(i) ϕ is consistent with T

(ii) For all $\sigma \in \Sigma, T \models \phi \to \sigma$.

T locally omits Σ iff T does not locally realize Σ. Thus T locally omits Σ if if and only if for any formula $\phi(x_0 \ldots x_{n-1})$ which is consistent with T, there exists $\sigma \in \Sigma$ such that $\phi \wedge \neg\sigma$ is consistent with T. $\mathfrak{M}$ omits Σ if no tuple in $\mathfrak{M}$ realizes all formulas in Σ. The Omitting Types Theorem [1] 2.2.15. states that

THEOREM 1 *(Henkin-Orey) Let* T *be a consistent theory in a countable language* $\mathfrak{L}$*, and for all* $i < \omega$*, let* $\Sigma_i(x_1 \dots x_n)$ *be a set of formulas in* n_i *variables. If* T *locally omits each* Σ_i*, then* T *has a model which omits each* Σ_i*.*

Now the idea of Vaught's Theorem is follows. Let T be an atomic theory. For each $n < \omega$, let Σ_n be the negation of complete formulas $\psi(x_0 \dots x_{n-1})$ in T. Then T locally omits each set Σ_n. A model omitting all Σ_n will be the desired model [1] 2.3.5. We show that the non-trivial implication (T is atomic implies T has a prime model) does not hold for uncountable languages of any greater power. The question as to wether Vaught's Theorem extends to the uncountable case, is natural, however to the best of our knowledge it was not dealt with in the literature. (It is proved though that atomic theories of size $\leqslant \aleph_1$ do have atomic models [4] and [3].) For a language $\mathfrak{L}$, $Md_{\mathfrak{L}}$ denote the class of $\mathfrak{L}$ structures and $\mathfrak{F}_L$ denotes the set of $\mathfrak{L}$ formulas.

Proof: Let $\mathfrak{T}'^0 = (T^0, <^0)$ be a normal Aronszajn tree.i.e,

(1) $\mathfrak{T}'^0$ is a poset, and

(2) for every $a \in T^0, T^0|a = \{x \in T^0 : x <^0 a\}$ is well ordered by $<^0$. If for any ordinal β

$$T^0_\beta = \{a \in T^0 : \text{ the order type of } (T^0|a, T^0|a \restriction <^0) = \beta\}.$$

Then the following hold:

(3) $|T^0_0| = 1$.

(4) There are no ordered subsets of $\mathfrak{T}'^0$ of order type ω_1.

(5) For every $\beta < \omega_1, |T^0_\beta| \leqslant \aleph_0$.

(6) If $\beta < \omega_1$ is a limit ordinal , $a, b \in T^0_\beta$ and $T^0|a = T^0|b$, then $a = b$.

(7) Each $a \in T^0$ has $\aleph_0$ immediate successors.

(8) If $\beta < \mu < \omega_1$ and $x \in T^0_\beta$ then there is a $y \in T^0_\mu$ such that $x <^0 y$. (It is well known that in ZFC normal Aronszajn trees exist.)

Let $\mathfrak{L}$ be a first order language with one binary relation symbol $<$ and $\aleph_1$ unary relation symbols $T_\beta(\beta < \omega_1)$. Let $\mathfrak{T}^0 = (\mathfrak{T}'^0, T^0_\beta)_{\beta<\omega_1} \in Md_{\mathfrak{L}}$. Let $\Sigma = Th(\mathfrak{T}'^0)$, the first order theory of $\mathfrak{T}^0$. Then Σ is complete. Define $\mathfrak{T}^1$ as follows

$$T^1 = \{f : f \text{ is a function from } \beta < \omega_1 \text{ into } \omega$$

$$\text{with } |\{\beta : f(\beta) > 0\}| < \aleph_0\}.$$

Let $\mathfrak{T}'^1 = (T^1, <_1)$ where

$$f <^1 g \text{ iff } f \subseteq g.$$

For $\beta < \omega_1$, define T^1_β as follows :

$$T^1_\beta = \{f \in T^1 : Dom(f) = \beta\}.$$

Now let

$$\mathfrak{T}^1 = (\mathfrak{T}'^1, T^1_\beta)_{\beta<\omega_1}.$$

We now prove

(9) $\mathfrak{T}^1 \models \Sigma$ so that $\mathfrak{T}^0 \equiv \mathfrak{T}^1$.

(10) Σ is atomic.

(11) Every elementary submodel of $\mathfrak{T}^0$ satisfies (4) but no elementary submodel of $\mathfrak{T}^1$ satisfy (4). This will prove our result (because a prime model would be elementary embeddable in $\mathfrak{T}^0$ and $\mathfrak{T}^1$.)

By (2), (3) and (6), $\mathfrak{T}'^0$ is a meet semilattice. It is also easy to see that $\mathfrak{T}'^1$ is a meet semilattice. We shall work in a definitional expansion of $(\mathfrak{L}, \Sigma)$ with a binary relation . with definition

$$(\forall v_0 \forall v_1 \forall v_2)(v_2 = v_0.v_1 \longleftrightarrow v_2 \leqslant v_0 \wedge v_2 \leqslant v_1 \wedge$$

$$\forall v_3)(v_3 \leqslant v_0 \wedge v_3 \leqslant v_1 \longrightarrow v_3 \leqslant v_2)).$$

(12) $\mathfrak{T}^1$ satisfies $(1)-(3)$ and $(5)-(8)$.

For $k, m \in \omega$ and $\beta \in \omega_1$, let $\phi^{k,m,\beta} \in \mathfrak{F}_\mathfrak{L}$ be defined as $T_\beta(v_k.v_m)$. For $n \in \omega$, $i < 2$ and $q \in {}^nT^i$ define

$$\Gamma_q = \{\phi^{k,m,\beta} : k, m < n, \text{ and } \mathfrak{T}^i \models \phi^{k,m,\beta}[q]\}.$$

For any $q \in {}^nT^i$, let $\phi_q = \bigwedge \Gamma_q$. For $i, j < 2$, $n \in \omega$, $q \in {}^nT^i$ and $r \in {}^nT^j$ define

$$q I_n r \text{ iff } \models \phi_q \longleftrightarrow \phi_r.$$

By (7) (8) and (12) we have

(13) if $q \in {}^nT^i$ and $r \in {}^nT^j$, qI_nr and $a \in T^i$, then there is a $b \in T^j$ such that

$$(q_0, \ldots q_{n-1}, a)I_{n+1}(r_0 \ldots r_{n-1}, b)$$

Using (13) for $i \neq j$ (9) follows. By (13) again, for $i = j = 0$, for every $q \in {}^nT^0$, ϕ_q is an atomic formula. This proves (10). Now we prove (11). Trivially every elementary submodel of $\mathfrak{T}^0$ satisfies (1)-(8). Let

$$\mathfrak{T}^* = (T^*, <^*, T^*_\beta)_{\beta<\omega_1}$$

be an elementary submodel of $\mathfrak{T}^1$. For any $\beta < \omega_1$, choose $f_\beta \in T^*_\beta$. Define $H : \omega_1 \to \omega_1$ by

$$H(\beta) = \text{ the greatest } \mu \text{ such that } f_\beta(\mu) \neq 0, \text{ if such } \mu \text{ exists}$$

$$= 0 \text{ otherwise }.$$

H is well defined since $\{\mu : f_\beta(\mu) \neq 0\}$ is finite. H is a regressive function so by Fodor's theorem [2][8.7], there exists a $\mu < \omega_1$ such that $|H^{-1}\{\mu\}| = \aleph_1$. T^*_μ is countable so by (8) there is an $h \in T^*_\mu$ such that for each δ, if $\mu < \delta < \omega_1$, then $h \cup \{(k, 0) : \mu < k \leqslant \delta\} \in T^*$. It follows thus that

$$\{h \cup \{(k, 0) : \mu < k \leqslant \delta\} : \mu < \delta < \omega_1\}$$

is an ordered subset of $(T^*, <^*)$ of order type ω_1. □

REFERENCES

[1] CC. Chang and H.J. Keisler *Model Theory.* Studies in Logic and Foundations of Mathematics. North Holland.

[2] Jech T., *Set Theory*. The third Millenuim Edition Revised and Expanded. Springer.

[3] J. Knight *Prime and Atomic models* Journal of Symbolic Logic vol 43, no 3, 1978 p. 385-393

[4] M. Laskowski, S. Shelah *On the existence of atomic models* Journal of Symbolic Logic, vol 58, no 4, 1993, p. 1189-1194

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