Reduction in first-order logic compared with reduction in implicational logic

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Abstract: In this paper we discuss strong normalization for natural deduction in the $\to\forall$ - fragment of first-order logic. The method of collapsing types is used to transfer the result (concerning strong normalization) from implicational logic to first-order logic. The result is improved by a complement, which states that the length of any reduction sequence of derivation term r in first-order logic is equal to the length of the corresponding reduction sequence of its collapse term r^c in implicational logic.

Our basic logic calculus is the $\to\forall$ - fragment of minimal natural deduction for first-order logic over simply typed lambda-terms. This restriction regarding the minimal fragment does not mean a loss in general, since the full classical first-order logic can be embedded in this system by adding stability axiom. The method of collapsing types developed in [2] is used to get some results concerning the strong normalization of derivation terms in first-order logic.

1 PRELIMINARIES

Let us fix our language. Assume that we have a countable infinite set of function symbols $f, g, h \ldots$, and predicate symbols $P, Q, R \ldots$, each of arity $\geqslant 0$. Terms (object terms) $d, e, \ldots$, are defined inductively from object variables $x, y, z \ldots$, by the following rules:

1. object variable x is a term,
2. if $\vec{d}$ is a list of terms and the arity of function symbol f is the length of the list $\vec{d}$, then $f\vec{d}$ is a term,

3. terms are defined only by rules 1 and 2.

The set $FV(d)$ of free object variables of an object term d is defined as usual.

Atomic formulas are $\perp$ (falsity) and $P\vec{d}$, where $\vec{d}$ is a list of terms, P is a predicate symbol and the arity of P is the length of $\vec{d}$.

Formulas are built from atomic formulas by implication $\varphi \to \psi$ and universal quantification $\forall x \varphi$.

Derivation terms $r^{\varphi}, t^{\varphi}, s^{\varphi}, q^{\varphi} \ldots$ (and also its set $FA(r^{\varphi})$ of free assumption variables) are built from *assumption variables* $u^{\varphi}, v^{\varphi}, w^{\varphi} \ldots$ by the introduction and elimination rules for $\to$ and $\forall$:

φ — u^{φ} is a derivation term with $FA(u^{\varphi}) = \{u^{\varphi}\}$;

$\to^{+}$ — implication introduction — if r^{ψ} is a derivation term, then $(\lambda u^{\varphi} r^{\psi})^{\varphi\to\psi}$ is a derivation term with $FA((\lambda u^{\varphi} r^{\psi})) = FA(r^{\psi}) \setminus \{u^{\varphi}\}$;

$\to^{-}$ — implication elimination — if $t^{\varphi\to\psi}$ and s^{φ} are derivation terms, then $(t^{\varphi\to\psi} s^{\varphi})^{\psi}$ is a derivation term with $FA(t^{\varphi\to\psi} s^{\varphi}) = FA(t^{\varphi\to\psi}) \cup FA(s^{\varphi})$;

$\forall^{+}$ — universal quantification introduction — if r^{φ} is a derivation term and x is an object variable which satisfies the condition $x \notin \cup\{FV(\psi) | u^{\psi} \in FA(r^{\varphi})\}$, then $(\lambda x r^{\varphi})^{\forall x \varphi}$ is a derivation term with $FA(\lambda\, x r^{\varphi}) = FA(r^{\varphi})$;

$\forall^{-}$ — universal quantification elimination — if $t^{\forall x \varphi}$ is a derivation term and d is an object term, then $(t^{\forall x \varphi} d)^{\varphi_x[d]}$ is a derivation term with $FA(t^{\forall x \varphi} d) = FA(t^{\forall x \varphi})$.

We write $r^{\varphi}\left[u_1^{\psi_1}, \ldots, u_m^{\psi_m}\right]$ to indicate that the assumption variables free in r^{φ} are in the list $u_1^{\psi_1}, \ldots, u_m^{\psi_m}$. We also use the notation $r : \varphi$ instead of r^{φ}.

DEFINITION 1 A formula φ is called *derivable* from assumptions $\psi_1, \ldots, \psi_m$, if there is a derivation term $r^{\varphi}\left[u_1^{\psi_1}, \ldots, u_m^{\psi_m}\right]$ with different assumption variables $u_1^{\psi_1}, \ldots, u_m^{\psi_m}$.

In the case of *classical logic*: for any predicate symbol P the term $\mathtt{stab}_P : \forall\vec{x}.\neg\neg P\vec{x} \to P\vec{x}$ is a derivation term.

From now on we will use the word *term* for derivation terms (until there is no confusion with the notion of object terms) and *type* for formulas.

As we have mentioned the $\to \forall$ - fragment of minimal logic contains full classical first-order logic. As in [1] (Subsections 1.1 and 2.1) this can be seen as follows:

1. Associate with any formula φ in the language of classical first-order logic a finite list $\vec{\varphi}$ of formulas in our $\to\forall$ - fragment, by induction on φ:

$$\begin{array}{rcl} P\vec{d} & \mapsto & P\vec{d} \\ \neg\varphi & \mapsto & \vec{\varphi} \to\perp \\ \varphi \to \psi & \mapsto & \vec{\varphi} \to \psi_1, \dots, \vec{\varphi} \to \psi_n \\ \varphi \wedge \psi & \mapsto & \vec{\varphi}, \vec{\psi} \\ \varphi \vee \psi & \mapsto & (\vec{\varphi} \to\perp), (\vec{\psi} \to\perp) \to\perp \\ \forall x\varphi & \mapsto & \forall x\varphi_1, \dots, \forall x\varphi_m \\ \exists x\varphi & \mapsto & \forall x(\vec{\varphi} \to\perp) \to\perp \end{array}$$

where we write $\vec{\varphi} \to \psi$ for $(\varphi_1 \to (\varphi_2 \to \cdots (\varphi_m \to \psi) \cdots))$.

2. In any model $\mathcal{M}$, where $\perp$ is interpreted by falsity, we clearly have that a formula φ in the language of full first-order logic holds under an assignment α iff all formulas in the assigned sequence $\vec{\varphi}$ hold under α (in our $\to\forall$ - fragment of minimal logic).

3. Our derivation calculus for the $\to\forall$ - fragment is complete in the following sense:

   a formula φ is derivable from stability assumptions $\forall\vec{x}.\neg\neg P\vec{x} \to P\vec{x}$ for all predicate symbols P in φ iff φ is valid in any model under any assignment.

2 STRONG NORMALIZATION

It was shown in [1] that for pure implicational logic any term can be reduced to a normal form (w.r.t. $\to_1$ conversion, the one step reduction using β-conversion rule) and this form is uniquely determined. Moreover, it was shown that any reduction sequence terminates, i.e. any term is strongly normalizable. A derivation term is said to be in normal form if it is impossible to perform a reduction. Here we use the method of collapsing types (ref [2]) to transfer the result (concerning strong normalization, obtained in [1]) from implicational logic to first-order logic.

It must be mentioned that the general β-conversion rule is extended to first-order logic. In particular, we have

$$(\lambda u^{\varphi} t^{\psi}) s^{\varphi} \text{ converts into } (\to_1) t_u\,[]$$

where t, s are derivation terms, u is an assumption variable; and

$$(\lambda x r^{\varphi}) d \text{ converts into } (\to_1) r^{\varphi_x[d]}$$

where x is an object variable, d is an object term and r is a derivation term.

For any formula φ of first-order logic we define its *collapse* φ^c by

$$
\begin{array}{rcll}
(P\vec{d})^c & \equiv & P & \text{(c1)} \\
(\varphi \to \psi)^c & \equiv & \varphi^c \to \psi^c & \text{(c2)} \\
(\forall x \varphi)^c & \equiv & \top \to \psi^c & \text{(c3)}
\end{array}
$$

where $\top :\equiv \bot \to \bot$ (i.e. $\top$ means tautology). Though, $\bot$ is an atomic formula it behaves like predicate symbols, i.e. $(\bot)^c \equiv \bot$, therefore $(\top)^c \equiv \top$.

For any derivation term r^ψ in first-order logic we can now define its *collapse* $(r^\psi)^c$. It is obvious from this definition that for any derivation term r^ψ in first-order logic with free assumption variables $u_1^{\varphi_1}, \dots, u_m^{\varphi_m}$ the *collapse* $(r^\psi)^c$ is a derivation term $(r^c)^{\varphi^c}$ in implicational logic with free assumption variables $u_1^{\varphi_1^c}, \dots, u_m^{\varphi_m^c}$.

$$
\begin{array}{rcll}
(u^\varphi)^c & \equiv & u^{\varphi^c} & \text{(c4)} \\
(\lambda u^\varphi r)^c & \equiv & \lambda u^{\varphi^c} r^c & \text{(c5)} \\
(t^{\varphi \to \psi} s^\varphi)^c & \equiv & t^c s^c & \text{(c6)} \\
(\lambda x r)^c & \equiv & \lambda u^\top r^c & \text{(c7)} \\
(t^{\forall x \varphi} d)^c & \equiv & t^c (\lambda z^\bot z^\bot)^\top & \text{(c8)}
\end{array}
$$

Note that for any derivation term r^ψ, assumption variable u^φ and derivation term s^φ we have that $r^c\,[s^c]$ is a derivation term in implicational logic (where the substitution of s^c is done for the assumption variable u^{φ^c}), which is the collapse of $r\,[s]$. Also for any derivation term r^ψ, object variable x and object term d we have that $r_x\,[d]$ is a derivation term of $\psi_x\,[d]$ with collapse $(r_x\,[d])^c \equiv r^c$.

LEMMA 1 *If* $r \to_1 r'$ *in first-order logic, then* $r^c \to_1 (r')^c$ *in implicational logic ([1] - Subsection 2.2).*

From Lemma 1 and the theorem, which states that any term in implicational logic is strongly normalizable, the following main result was obtained in [1]:

THEOREM 2 *Any derivation term* r *in first-order logic is strongly normalizable.*

Indeed, since the collapse r^c of the term r is a term in implicational logic and any term in implicational logic is strongly normalizable, i.e. any reduction sequence starting from r^c terminates, then from Lemma 1 we conclude that any reduction sequence starting from r also terminates, hence r is strongly normalizable. But it is still conceivable that r terminates (in terms of reduction sequence) before r^c, i.e. the reduction sequence of r^c as defined is longer than the reduction sequence of r that we chose. Our aim is to show that it is impossible, and both of the terms do the same number of one-step reductions.

First of all it is necessary to emphasize that it is not so obvious, since there is no bijective correspondence between a derivation term in first-order logic and its *collapse*.

NOTE. Although, to any derivation term in first-order logic we identically associate collapse, it is not necessary for the converse to be true. The following instances illustrate this fact.

EXAMPLE 1. Assume the collapse is $t^c = \lambda u^\top r^c$. Then there are two possible forms of derivation term t (ambiguity):

1. on the one hand, since $\top^c \equiv \top$, then $t^c = \lambda u^\top r^c = \lambda u^{\top^c} r^c = (\lambda u^\top r)^c$ according to (c5); so, $t = \lambda u^\top r$;

2. on the other hand, $t^c = \lambda u^\top r^c = (\lambda x_{obj} r)^c$ according to (c7); so, $t = \lambda x_{obj} r$.

We write x_{obj} instead of x to indicate the fact that x is an object variable. This notion is extended on object terms too, e.g. d_{obj} instead of d. For the convenience, sometimes the obj pattern will be omitted, but implied.

EXAMPLE 2. Assume $\varphi^c = \top \to P$, where P is any predicate symbol. Then there are two possible forms of a formula φ (ambiguity):

1. on the one hand, since $(\bot\to\bot)^c = \top^c \equiv \top = (\bot\to\bot)$, then $\varphi^c = \top \to P = (\bot\to\bot) \to P = (\bot\to\bot)^c \to P^c = ((\bot\to\bot) \to P)^c$ according to (c1) and (c2); so, $\varphi = (\bot\to\bot) \to P = \top \to P$;

2. on the other hand, $\varphi^c = \top \to P = \top \to P^c = (\forall x_{obj} P)^c$ according to (c3); so, $\varphi = \forall x_{obj} P$.

Now we reformulate Theorem 2:

THEOREM 3 *Any derivation term r in first-order logic is strongly normalizable. Moreover, for any reduction sequence $r = r_1 \to_1 \cdots \to_1 r_n = r'$ of a derivation term r with normal form r', the length is identical to the length of the reduction sequence $r^c = r_1^c \to_1 \cdots \to_1 r_n^c = (r')^c$ in implicational logic and $(r')^c$ is the normal form of r^c.*

Proof: The first part of the theorem is plain due to Theorem 2. From Lemma 1 it simply follows that if r^c terminates, so does r. It remains to prove that if r terminates, so does r^c, i.e. r^c terminates as soon as r. Assume that $r \to^* r'$ and r' is the normal form of r; that is r terminates and the last term of normalization reduction sequence is r'. Here $\to^*$ denotes transitive and reflexive closure of $\to_1$. From Lemma 1 we obtain that $r^c \to^* (r')^c$ as well. Now it should be proved that $(r')^c$ cannot be normalized further, i.e. it terminates.

Let us suppose the opposite and come to contradiction. It means that there exists a term r''_c such that $(r')^c \to_1 r''_c$. So we have the next structure-view:

$$\begin{array}{lllll} r & \to^* & r' & & \text{- terminates} \\ \Downarrow_c & & \Downarrow_c & & \\ r^c & \to^* & (r')^c & \to_1 & r''_c \end{array}$$

Therefore, we conclude that $(r')^c$ has a form

$$(r')^c = t_c^L((\lambda u_c t_c)s_c)t_c^R$$

hence

$$r''_c = t_c^L(t_{c_u}[s_c])t_c^R$$

where t_c^L and t_c^R are arbitrary terms and may be empty.

Let us denote by t_c^M the middle part of $(r')^c$

$$t_c^M \equiv (\lambda u_c t_c)s_c.$$

More exactly $(r')^c$ has one of the two following forms:

(A) $[t_c^L((\lambda u_c t_c)s_c)]t_c^R = (t_c^L\ t_c^M)t_c^R$,

(B) $t_c^L[((\lambda u_c t_c)s_c)t_c^R] = t_c^L(t_c^M t_c^R)$.

REMARK. By $\tau(s)$ we denote the type of derivation term s, e.g. $\tau(s^{\varphi\to\psi}) = \varphi \to \psi$. Let us consider the term $t^c((\lambda z^\perp z^\perp)^\top)^c$ in case when $\tau(t^c) = \top \to \varphi^c = \top^c \to \varphi^c$. It is obvious that $((\lambda z^\perp z^\perp)^\top)^c = (\lambda z^\perp z^\perp)^\top$. According to (c6) and (c8) there are two possible forms of term r which collapse is $r^c = t^c((\lambda z^\perp z^\perp)^\top)^c$:

1. on the one hand $r = t^{\top\to\varphi}(\lambda z^\perp z^\perp)^\top$;

2. on the other hand $r = t^{\forall x_{obj}\varphi} d_{obj}$.

Inter alia, this remark can be viewed as one more example, which shows the accuracy of the note about inverse problem mentioned above.

We now consider the two forms of $(r')^c$:

FOR THE FORM (A):

$$(r')^c = (t_c^L\ t_c^M)t_c^R$$

We consider two cases depending on the form of t_c^R.

CASE (A-I). $t_c^R = (\lambda z^\perp z^\perp)^\top = ((\lambda z^\perp z^\perp)^\top)^c$.

Let us denote: $q^c \equiv t_c^L((\lambda u_c t_c)s_c)$, hence $(r')^c = q^c((\lambda z^\perp z^\perp)^\top)^c$.

From the remark mentioned above we obtain that either

$$r' = q^{\top\to\varphi}(\lambda z^\perp z^\perp)^\top$$

or

$$r' = q^{\forall x\varphi} d.$$

CASE (A-1-1). $r' = q^{\top\to\varphi}(\lambda z^{\perp}z^{\perp})^{\top}$ and $q^c = t_c^L((\lambda u_c t_c)s_c) = t_c^L\ t_c^M$ and $\tau(q^c) = \top \to \varphi^c$.

Since $t_c^M = (\lambda u_c t_c)s_c \neq (\lambda z^{\perp}z^{\perp})^{\top}$ then according to (c4)-(c8) we conclude that there is only one possible form for q^c, that is-(c6). It follows that $\exists\ t_L, t_M$ terms, which satisfy these equations: $t_c^L = (t_L)^c$ and $t_c^M = (t_M)^c$, hence $q^c = (t_L)^c(t_M)^c$. Let us denote: $(t_{ML})^c \equiv \lambda u_c t_c$, so we have $(t_M)^c = (\lambda u_c t_c)s_c = (t_{ML})^c s_c$. Depending on the form s_c ($= (\lambda z^{\perp}z^{\perp})^{\top}$ or not) we get either $t_M = t_{ML}^{\forall x\psi}s$, where $s = e_{obj}$, or $t_M = t_{ML}^{\psi\to\omega}s$, where $s^c = s_c$ (s_c is a derivation term). As we have $(t_{ML})^c \equiv \lambda u_c t_c$, then according to (c5) and (c7) there are two possible forms of term t_{ML} which collapse is $\lambda u_c t_c$: $t_{ML} = \lambda x_{obj}t$, if $\tau(u_c) = \top$ or $t_{ML} = \lambda ut$, if $\tau(u_c) \neq \top$, where $u^c = u_c$ and $t^c = t_c$. Therefore, $t_c^M = (t_M)^c = [(\lambda x_{obj}t)e_{obj}]^c$ or $t_c^M = (t_M)^c = [(\lambda ut)s]^c$, which means that in both cases the term r' contains subterm $(\lambda x_{obj}t)e_{obj}$ or $(\lambda ut)s$ respectively, i.e. we could have performed one more $\to_1$ reduction for r', which contradicts our theorem condition that r' terminates.

CASE (A-1-2). $r' = q^{\forall x\varphi}d$ and $q^c = t_c^L((\lambda u_c t_c)s_c) = t_c^L\ t_c^M$ and $\tau(q^c) = \top \to \varphi^c$. This case is similar to the case (a-1-1).

CASE (A-2). $t_c^R \neq (\lambda z^{\perp}z^{\perp})^{\top}$.

According to (c4)-(c8) we conclude that there is only one possible form for $(r')^c$, that is (c6): it follows that $r' = q^{\varphi\to\psi}t_R$, where $(t_R)^c = t_c^R$ and $q^c = t_c^L\ t_c^M$, hence we come to the case (a-1-1) when $\tau(q^c) = \varphi^c \to \psi^c$.

FOR THE FORM (B):

$$(r')^c = t_c^L(t_c^M t_c^R)$$

Since $t_c^M t_c^R$ does not have the form $(\lambda z^{\perp}z^{\perp})^{\top}$, it follows that according to (c4)-(c8) there is only one possible form for $(r')^c$, that is (c6). Hence, $r' = t_L\ q$, where $(t_L)^c = t_c^L$ and $q^c = t_c^M t_c^R = ((\lambda u_c t_c)s_c)t_c^R$. According to (c4)-(c8) q^c may have one of the two following forms: (c6) or (c8). Depending on the form of t_c^R($= (\lambda z^{\perp}z^{\perp})^{\top}$ or not) we get either $q = t_M d_{obj}$, where $(t_M)^c = t_c^M$ or $q = t_M t_R$, where $(t_M)^c = t_c^M$ and $(t_R)^c = t_c^R$ respectively. In both cases we have t_M which satisfies the equation $(t_M)^c = (\lambda u_c t_c)s_c$. The rest is similar to the case (a-1-1).

All the cases have been considered, hence the theorem is proved by the methods of contradiction. □

Using the last result (Theorem 3) the upper bound for the length of arbitrary reduction sequences obtained in [3] (obtained only for implicational logic) can be extended to include first-order logic. So we obtain, that in first-order logic any reduction sequence for a term r is bounded by

$$2_{g(r^c)}(l(r^c)),$$

where r^c is the collapse of the term r, $l(r^c)$ and $g(r^c)$ denote the length and degree of the term r^c respectively. Here $2_m(n)$ is recursively defined by $2_0(n) = n$ and $2_{m+1}(n) = 2^{2_m(n)}$.

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