

# *The McKinsey–Lemmon logic is barely canonical*

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*Abstract:* We study a canonical modal logic introduced by Lemmon, and axiomatised by an infinite sequence of axioms generalising McKinsey’s formula. We prove that the class of all frames for this logic is not closed under elementary equivalence, and so is non-elementary. We also show that any axiomatisation of the logic involves infinitely many non-canonical formulas.

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## I INTRODUCTION

Our story starts with *McKinsey’s formula*,<sup>1</sup>

$$M : \quad \Box\Diamond p \rightarrow \Diamond\Box p. \quad (I)$$

M has long been studied by modal logicians. On the one hand, the normal modal logic K4M (also known as K4.1) axiomatised by M together with the transitivity axiom  $\Box p \rightarrow \Box\Box p$  is a well-behaved logic. It is canonical (i.e., valid in its own canonical frame), and hence Kripke complete. The class of all frames

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<sup>1</sup>McKinsey actually studied (in [18]) the system S4 augmented with  $\Box\Diamond p \wedge \Box\Diamond q \rightarrow \Diamond(p \wedge q)$ , but Sobociński showed in [19] that this is the same system as S4 + M. For this and further discussion, see [9].

validating K4M is elementary: it is the class of transitive frames such that colloquially, every world sees a world that can see at most itself (see, e.g., [3, proposition 3.46] or [2, example 3.57]).

On the other hand, M itself is rather wild. The logic KM that M axiomatises alone is determined by its finite frames [4], and so it is Kripke complete. However, the class of all frames validating KM is not elementary [6], and not even closed under elementary equivalence [20].<sup>2</sup> KM is not the logic of *any* elementary class of frames [6], and is not canonical [7]. M is often called the simplest formula not equivalent to a Sahlqvist formula (see [2, §3.6] or [3, §10.3] for details of Sahlqvist formulas).

KM was cited by Lemmon in [17] as a logic that had not yielded to the ‘canonical model’ completeness method expounded in that work. Lemmon then generalised M to an infinite sequence of formulas

$$M_k : \quad \diamond((\diamond p_1 \rightarrow \Box p_1) \wedge \dots \wedge (\diamond p_k \rightarrow \Box p_k)), \quad \text{for } k \geq 1. \quad (2)$$

M is equivalent (in the basic normal modal logic K) to  $M_1$ . It may help to observe that since  $\diamond p \rightarrow \Box p$  is equivalent to  $\Box \neg p \vee \Box p$ , we can rewrite  $M_k$  equivalently as

$$M_0 = \top, \quad M_k = \diamond \bigwedge_{i < k} (\Box p_i \vee \Box \neg p_i) \quad \text{for } k \geq 1, \quad (3)$$

where for later convenience we use the propositional variables  $p_0, \dots, p_{k-1}$ . We will use this form of the  $M_k$  throughout the paper. Now we can see that the validity of  $M_1$  in a Kripke frame  $\mathcal{F}$  says that for any partition of the worlds of  $\mathcal{F}$  into at most two sets (corresponding to the interpretations of  $p$  and  $\neg p$  in a Kripke model over  $\mathcal{F}$ ), any world sees a world whose successors all lie in a single partition set.  $M_k$  says the same as  $M_1$  but for a partition into at most  $2^k$  sets. Clearly,  $M_{k+1} \vdash M_k$  for all  $k \geq 1$ . Lemmon showed by a short proof-theoretic argument that assuming transitivity, all the  $M_k$  are equivalent to  $M_1$ .

Lemmon defined  $KM^\infty$  to be the modal logic axiomatised by the axioms in (2). This logic, standing between KM and K4M, is the subject of our paper. Lemmon proved that it is the logic of the class of Kripke frames satisfying

$$m^\infty : \quad \forall x \exists y (R(x, y) \wedge \forall z z' (R(y, z) \wedge R(y, z') \rightarrow z = z')). \quad (4)$$

This condition says that every world sees a world with at most one successor. By considering partitions as above, it is easily seen that  $KM^\infty$  is valid in all frames with this property. Lemmon proved completeness by a compactness argument that showed that the canonical frame for  $KM^\infty$  satisfies  $m^\infty$ . This means that  $KM^\infty$  is canonical. The logic obtained from  $KM^\infty$  by adding the transitivity axiom is K4M, so since the transitivity axiom is also canonical, this gives another proof of the canonicity of K4M.

<sup>2</sup>Actually, the class of frames validating any modal logic is elementary iff it is closed under elementary equivalence [21].

Since  $KM^\infty$  is the logic of an elementary class of frames — those satisfying  $m^\infty$  — its canonicity also follows from *Fine's Theorem* that the modal logic of an elementary class of frames is canonical [5]. However, the proof by compactness is different, and the method applies in some cases where Fine's result does not [11, 10]. A similar compactness argument was used by Hughes in [14]. It was generalised by Balbiani et al. in [1, §3], where it is shown that if  $\sigma(p_0, \dots, p_{n-1})$  is a Sahlqvist formula with first-order correspondent  $\phi(x)$ , then  $\{\diamond \bigwedge_{i < k} \sigma(p_0^i, \dots, p_{n-1}^i) : k \geq 1\}$ , where the  $p_i^j$  are distinct propositional variables, axiomatises the modal logic of the class of frames satisfying  $\forall x \exists y (R(x, y) \wedge \phi(y))$ . By Fine's Theorem, this logic is canonical.  $KM^\infty$  is covered by taking  $\sigma = \diamond p \rightarrow \Box p$ , and the logic in [14] is covered by taking  $\sigma = \Box p \rightarrow p$ . [12] derives axioms for the modal logic of an arbitrary elementary class of frames; as an example, (3) is obtained effectively from a formulation of (4) in hybrid logic.

In [15, §6], Jónsson showed using new algebraic proofs that the  $M_k$  are theorems of K4M, that  $KM^\infty$  is canonical, and hence that K4M is canonical.

In [1, §5], it is shown that  $KM^\infty$  (and also the logic axiomatised by  $M_k$ , for each finite  $k$ ) has the finite model property and is decidable. It is stated that  $KM^\infty$  is PSPACE-complete and that the proof will appear in a sequel.

OUR PAPER: Here, we add to the impression that  $KM^\infty$  lies somewhat nearer to KM than to K4M. First, we show that, just as for KM, the class of all frames for  $KM^\infty$  is non-elementary, and not even closed under elementary equivalence (Theorem 2.2 below). The proof is similar to that of [20] for KM, and the result was also proved independently by the same argument in [1]. In remark 3.9, we show that the class of frames for  $KM^\infty$  is not closed under ultraproducts.

We also study the canonicity of  $KM^\infty$ . We have seen that it shares canonicity with K4M. But we will show that it is *only barely canonical*. A formula is said to be *canonical* if the logic that it axiomatises is canonical. We prove:

- (Theorem 4.3) For no  $k \geq 1$  is  $M_k$  canonical. This generalises the result of [7] that  $M$  is not canonical.
- $KM^\infty$  cannot be axiomatised by canonical formulas. Hence, it is not axiomatisable by Sahlqvist formulas.
- (Theorem 4.4) Indeed, any axiomatisation of  $KM^\infty$  has infinitely many non-canonical axioms.

It follows by Fine's Theorem that  $KM^\infty$  is *only barely the logic of an elementary class of frames*. No  $M_k$  (for any  $k \geq 1$ ) axiomatises the logic of any elementary class of frames; and any axiomatisation of  $KM^\infty$  contains infinitely many axioms that, taken individually, fail to axiomatise the logic of any elementary class of frames. But  $KM^\infty$  itself is the logic of a finitely axiomatisable elementary class of frames.

Thus, the canonicity of  $KM^\infty$ , and its being the logic of an elementary class of frames, do not arise from properties of any finite number of axioms. They only emerge in the limit when all the axioms are taken together. This striking phenomenon has been seen before. In an algebraic setting, [13] showed that the variety RRA of representable relation algebras, and also the variety of modal algebras of ‘infinite chromatic number’, cannot be axiomatised by finitely many non-canonical axioms plus arbitrarily many canonical ones. Analogous results on elementary frame classes then follow from Fine’s Theorem as above. We use the same proof methods here. For each  $k, l$  with  $2 \leq l \leq k < \omega$ , we construct an inverse system of finite Kripke frames validating  $M_k$ , whose inverse limit is a frame that validates  $M_l$  but not  $M_{l+1}$ . The frames are based on those used in the proof in [7] of non-canonicity of  $KM$ . We can then deduce the third result above by first-order compactness.

**ORGANISATION OF PAPER:** In section 2 we prove that the class of frames for  $KM^\infty$  is non-elementary (Theorem 2.2). In section 3, we introduce some particular frames, and determine which  $M_k$  they validate. They will be put to use in section 4, where we show that no  $M_k$  is canonical (Theorem 4.3), and that any axiomatisation of  $KM^\infty$  involves infinitely many non-canonical axioms (Theorem 4.4).

**NOTATION** Let  $f : X \rightarrow Y$  be a map. We write  $\text{dom } f$  and  $\text{rng } f$  for the domain and range of  $f$ , respectively. If  $S \subseteq X$ , we write  $f|_S$  denote the restriction  $f$  to  $S$ . If  $S \subseteq Y$ , we write  $f^{-1}[S]$  for  $\{x \in X : f(x) \in S\}$ . If  $y \in Y$ , we write  $f^{-1}[y]$  for  $f^{-1}[\{y\}]$ . For sets  $X_i$  ( $i \in I$ ), we write  $\prod_{i \in I} X_i$  for the set of maps  $\eta : I \rightarrow \bigcup_{i \in I} X_i$  such that  $\eta(i) \in X_i$  for each  $i$ . We often write  $\eta(i)$  as  $\eta_i$  in this case.

Natural numbers will be regarded as ordinals. So for a natural number  $n < \omega$ , we identify  $n$  with  $\{0, 1, \dots, n - 1\}$ . For an ordinal  $\alpha$ , we write  ${}^\alpha 2$  for the set of maps  $f : \alpha \rightarrow 2$ , and  ${}^{<\omega} 2$  for  $\bigcup_{n < \omega} {}^n 2$ . Given a set  $X$  and a cardinal  $\kappa$ , the expression  $[X]^{\geq \kappa}$  denotes  $\{Y \subseteq X : |Y| \geq \kappa\}$ .

**KRIPKE SEMANTICS:** We set up our notation for this. A (*Kripke*) *frame*  $\mathcal{F} = (W, R)$  consists of a non-empty set  $W$  of ‘worlds’, together with a binary ‘accessibility’ relation  $R$  on  $W$ . We will write  $\text{dom } \mathcal{F}$  for  $W$ , and write  $R(x, y)$  to indicate that  $(x, y) \in R$ . An *R-successor* (respectively, *R-predecessor*) of  $w \in W$  is a world  $x \in W$  satisfying  $R(w, x)$  (respectively,  $R(x, w)$ ). We may indicate informally that  $R(w, x)$  by saying that  $w$  sees  $x$ , or that  $x$  is accessible from  $w$ . We will write  $R^w$  for the set of all *R*-successors of  $w$ .

We fix a countably infinite set  $V = \{p_0, p_1, \dots\}$  of propositional variables. An *assignment* into  $\mathcal{F}$  is a map  $h : V \rightarrow \wp(W)$ , the power set of  $W$ . The pair  $(\mathcal{F}, h)$  is called a (*Kripke*) *model*. We evaluate modal formulas at worlds of Kripke models in the usual way: for  $p \in V$ ,  $(\mathcal{F}, h), w \models p$  iff  $w \in h(p)$ ; booleans as usual; and  $(\mathcal{F}, h), w \models \diamond\phi$  (respectively,  $(\mathcal{F}, h), w \models \square\phi$ ) iff  $(\mathcal{F}, h), x \models \phi$  for some

(respectively, all)  $x \in R^w$ . A modal formula  $\phi$  is *valid* at a world  $w$  of a frame  $\mathcal{F}$  if  $(\mathcal{F}, h), w \models \phi$  for every assignment  $h$  into  $\mathcal{F}$ .  $\phi$  is *valid* in a frame  $\mathcal{F}$ , written  $\mathcal{F} \models \phi$ , if it is valid at every world of  $\mathcal{F}$ .

A frame  $(W, R)$  is a *generated subframe* of another,  $(W', R')$ , if  $W \subseteq W'$  and  $R = R' \cap (W \times W')$ . In this case, it is known (e.g., from [2, proposition 2.6] or [3, Theorem 2.7]) that for any  $h' : V \rightarrow \wp(W')$ , if  $h : V \rightarrow \wp(W)$  is given by  $h(p) = h'(p) \cap W$  for  $p \in V$ , then

$$(W, R, h), w \models \phi \iff (W', R', h'), w \models \phi$$

for every  $w \in W$  and every modal formula  $\phi$ . Hence, validity is preserved under generated subframes.

We will assume familiarity with basic notions of modal logic, such as canonical models. See [2, 3] for guidance if required.

## 2 THE FRAMES FOR $KM^\infty$ ARE NON-ELEMENTARY

Using a result of the first author and an argument along the lines of van Benthem's proof for  $KM$  in [20], we can establish our first result. We first quote Theorem 1 of [7] (reproduced as [8, Theorem 10.1]).

**THEOREM 2.1** *Let  $\mathcal{F} = (W, R)$  be a frame. Suppose that  $W$  contains a point  $r$  with the property that  $|R^m| \geq |R^r| + \omega$  for every  $m \in R^r$ . Then no  $M_k$  ( $k \geq 1$ ) is valid in  $\mathcal{F}$ .*

*Proof.* Let  $|R^r| = \kappa$ , and put  $R^r = \{m_i : i < \kappa\}$ . Define distinct points  $x_i, y_i \in W$  for  $i < \kappa$  by induction as follows. If  $i < \kappa$  and  $x_j, y_j$  have been defined for all  $j < i$ , we define  $x_i, y_i$  to be any distinct points of  $R^{m_i} \setminus \{x_j, y_j : j < i\}$ . This is possible because  $|R^{m_i}| \geq \kappa + \omega$ , so  $R^{m_i} \setminus \{x_j, y_j : j < i\}$  is infinite. Now define a Kripke model  $\mathcal{M}$  over  $\mathcal{F}$  by making  $p$  true at precisely  $\{y_i : i < \kappa\}$ . If  $\mathcal{M}, r \models \Diamond(\Box p \vee \Box \neg p)$ , then there is  $i < \kappa$  such that  $\mathcal{M}, m_i \models \Box p \vee \Box \neg p$ . But  $x_i, y_i \in R^{m_i}$ ,  $\mathcal{M}, x_i \models \neg p$ , and  $\mathcal{M}, y_i \models p$ , so this is impossible. Hence,  $\mathcal{M}, r \not\models M_1$ . Since  $M_{k+1} \vdash M_k$  for  $k \geq 1$ , no  $M_k$  for any  $k \geq 1$  is valid in  $\mathcal{F}$ .  $\square$

**THEOREM 2.2** *For each  $k \geq 1$ , the class of frames that validate  $M_k$  is not closed under elementary substructures, and hence is not elementary. The same holds for the class of frames validating  $KM^\infty$ .*

*Proof.* Let  $\mathcal{F} = (W, R)$  be the Kripke frame with  $W = \{r\} \cup [\omega]^{\geq \omega} \cup \omega$ , where  $r \notin [\omega]^{\geq \omega} \cup \omega$  is arbitrary, and  $R$  is given by:

$$\begin{aligned} R^r &= [\omega]^{\geq \omega}, \\ R^S &= S \quad \text{for each } S \in [\omega]^{\geq \omega}, \\ R^n &= \{n\} \quad \text{for each } n \in \omega. \end{aligned}$$

(So for  $n \in \omega$  and  $S \in [\omega]^{\geq \omega}$ ,  $R(S, n)$  holds iff  $n \in S$ .) Then  $\mathcal{F}$  is a frame for  $KM^\infty$ . For, given any  $k \geq 1$  and any assignment  $h : \{p_0, \dots, p_{k-1}\} \rightarrow \wp(W)$ ,

there is  $S \in [\omega]^{\geq \omega}$  such that for all  $x, y \in S$  and  $i < k$ , we have  $(\mathcal{F}, h), x \models p_i$  iff  $(\mathcal{F}, h), y \models p_i$ . Then  $(\mathcal{F}, h), S \models \bigwedge_{i < k} (\Box p_i \vee \Box \neg p_i)$ , so  $(\mathcal{F}, h), r \models M_k$ . Validity of  $M_k$  at all other points in  $\mathcal{F}$  is clear, as they have a successor (an element of  $\omega$ ) related only to itself. So  $\mathcal{F} \models M_k$  for each  $k$ , and  $\mathcal{F}$  validates  $KM^\infty$ .

But in any countable elementary substructure  $\mathcal{F}_0 = (W_0, R_0)$  of  $\mathcal{F}$ , it is easy to check that  $r \in W_0$ , and  $|R_0^r| \leq \omega = |R_0^S|$  for each  $S \in R_0^r$ . (For example, this follows from the preservation under elementary substructures of the formulas  $\exists x \forall y \neg R(y, x)$  and  $\forall x (R(r, x) \rightarrow \exists_{\geq n} y R(x, y))$  for each finite  $n$ .) By Theorem 2.1,  $\mathcal{F}_0 \not\models M_k$  for every  $k \geq 1$ .

Hence, the class of frames validating  $KM^\infty$  is not closed under elementary substructures and so cannot be elementary. Also, for each  $k \geq 1$ , the class of all frames validating  $M_k$  is not closed under elementary substructures (since  $\mathcal{F}$  validates  $M_k$  but  $\mathcal{F}_0$  does not).  $\square$

The proof shows that if  $L$  is any modal logic such that  $KM \subseteq L \subseteq KM^\infty$ , then the class of frames validating  $L$  is not elementary. This was proved independently in [1, Theorem 21] by the same argument.

### 3 FRAMES VALIDATING SOME $M_k$ AND NOT OTHERS

In this section, we study the canonicity properties of  $KM^\infty$  and its axiomatisations, using special frames based on those in [7].

#### 3.1 SQUAT FRAMES

**DEFINITION 3.1** Let  $\mathcal{F} = (W, R)$  be a frame. A world of  $W$  is called a *root* of  $\mathcal{F}$  if it has no  $R$ -predecessors, a *leaf* of  $\mathcal{F}$  if it has no  $R$ -successors other than itself, and a *midpoint*, otherwise. A world is *reflexive* if it is  $R$ -related to itself, and *irreflexive* otherwise.

$\mathcal{F}$  is said to be *squat* ([7] uses the term ‘trellis-like’) if it has a unique root, say  $r$ ;  $r$  is not a leaf; all ( $R$ -)successors of  $r$  are midpoints; and all successors of midpoints are reflexive leaves.

For example, the frame in Theorem 2.2 is squat.

**REMARK 3.2** We will often use the obvious fact that each  $M_k$  is valid at every world of a squat frame except perhaps the root. This is because each non-root has a reflexive leaf among its successors, and a reflexive leaf must clearly validate  $\bigwedge_{i < k} (\Box p_i \vee \Box \neg p_i)$  for any  $k \geq 1$ .

**DEFINITION 3.3** Let  $I \neq \emptyset$ , and for each  $i \in I$  let  $\mathcal{F}_i$  be a squat frame. We write  $\sum_{i \in I} \mathcal{F}_i$  for the squat frame consisting of a copy of each  $\mathcal{F}_i$  ( $i \in I$ ), the copies being disjoint except that their roots are identified.

Formally, if  $\mathcal{F}_i = (W_i, R_i)$  then  $\sum_{i \in I} \mathcal{F}_i = (W, R)$ , where  $W = (\bigcup_{i \in I} W_i \times \{i\})/\sim$ , the equivalence relation  $\sim$  is given by  $(w, i) \sim (w', i')$  iff  $(w, i) = (w', i')$

or  $w, w'$  are the roots of  $\mathcal{F}_i, \mathcal{F}_{i'}$  respectively, and

$$R = \{((w, i)/\sim, (w', i')/\sim) : i \in I, R_i(w, w')\},$$

where  $(w, i)/\sim$  denotes the  $\sim$ -class of  $(w, i)$ . If  $I = \{i_1, \dots, i_m\}$ , we write the sum as  $\mathcal{F}_{i_1} + \dots + \mathcal{F}_{i_m}$ .

We will usually identify each non-root world  $w$  of each  $\mathcal{F}_i$  with its ‘copy’  $(w, i)/\sim$  in  $\sum_{i \in I} \mathcal{F}_i$ .

**LEMMA 3.4** *Let  $\mathcal{F}_i$  ( $i \in I \neq \emptyset$ ) be squat frames. For any  $k < \omega$ , we have  $\sum_{i \in I} \mathcal{F}_i \models M_k$  iff  $\mathcal{F}_i \models M_k$  for some  $i \in I$ .*

*Proof.* Recall that  $M_0 = \top$  and  $M_k = \diamond \bigwedge_{j < k} (\Box p_j \vee \Box \neg p_j)$  for  $k \geq 1$ . The result is trivial for  $M_0$ . Let  $k \geq 1$ . Write  $r_i$  for the root of  $\mathcal{F}_i$  (each  $i$ ), and  $r$  for the root of  $\mathcal{F} = \sum_{i \in I} \mathcal{F}_i$ . Note that roots are by definition irreflexive. We will use remark 3.2 without explicit mention. We write  $\alpha = \bigwedge_{j < k} (\Box p_j \vee \Box \neg p_j)$ , so that  $M_k = \diamond \alpha$ .

$\Rightarrow$ : If  $\mathcal{F}_i \not\models M_k$  for each  $i \in I$ , then for each  $i$  there is an assignment  $h_i$  into  $\mathcal{F}_i$  such that  $(\mathcal{F}_i, h_i), r_i \not\models M_k$ . Let  $h$  be an assignment into  $\mathcal{F}$  such that for each  $i$ ,  $h$  agrees with  $h_i$  on the non-root worlds of  $\mathcal{F}_i$ . Assume for contradiction that  $(\mathcal{F}, h), r \models M_k$ . Pick a successor  $s$  of  $r$  with  $(\mathcal{F}, h), s \models \alpha$ . Since  $r$  is irreflexive,  $s \neq r$ . Suppose that  $s$  is in  $\mathcal{F}_i$ , say. Then  $s \in R_i^{r_i}$ , where  $R_i$  is the accessibility relation of  $\mathcal{F}_i$ . Now it is clear that the subframe of  $\mathcal{F}_i$  based on  $\{s\} \cup R_i^s$  is a generated subframe of both  $\mathcal{F}$  and  $\mathcal{F}_i$ . It follows that  $(\mathcal{F}_i, h_i), s \models \alpha$ , and hence  $(\mathcal{F}_i, h_i), r_i \models M_k$ , contradicting the choice of  $h_i$ . So  $(\mathcal{F}, h), r \not\models M_k$ , and  $M_k$  is not valid in  $\mathcal{F}$ .

$\Leftarrow$ : Suppose that  $i \in I$  and  $\mathcal{F}_i \models M_k$ . Let  $h$  be any assignment into  $\mathcal{F}$ . We show that  $(\mathcal{F}, h), r \models M_k$ . Let  $h_i$  be the ‘restriction’ of  $h$  to  $\mathcal{F}_i$ . By assumption,  $(\mathcal{F}_i, h_i), r_i \models M_k$ , so there is a successor  $s$  of  $r_i$  in  $\mathcal{F}_i$  with  $(\mathcal{F}_i, h_i), s \models \alpha$ . By definition of  $\mathcal{F}$ ,  $s$  is a successor of  $r$  in  $\mathcal{F}$ . As before,  $(\mathcal{F}, h), s \models \alpha$ . So  $(\mathcal{F}, h), r \models M_k$  as required.  $\square$

### 3.2 SPECIAL SQUAT FRAMES

The following squat frames are modifications of frames used in [7] to prove non-canonicity of  $M$ . We will use them to study the canonicity of  $KM^\infty$ .

**DEFINITION 3.5** For each  $k, n < \omega$ , we define  $\mathcal{G}_n^k$  to be the squat frame with a root  $r$ , a set  $L_n^k = {}^{k+n}2$  of leaves, and a set  $[L_n^k]^{\geq 2^n} = \{Y \subseteq L_n^k : |Y| \geq 2^n\}$  of midpoints. See Figure 1. The accessibility relation  $R$  on  $\mathcal{G}_n^k$  is given by:

- $R^r = [L_n^k]^{\geq 2^n}$ ,
- $R^s = s$  for each  $s \in [L_n^k]^{\geq 2^n}$ ,
- $R^x = \{x\}$  for each  $x \in L_n^k$ .

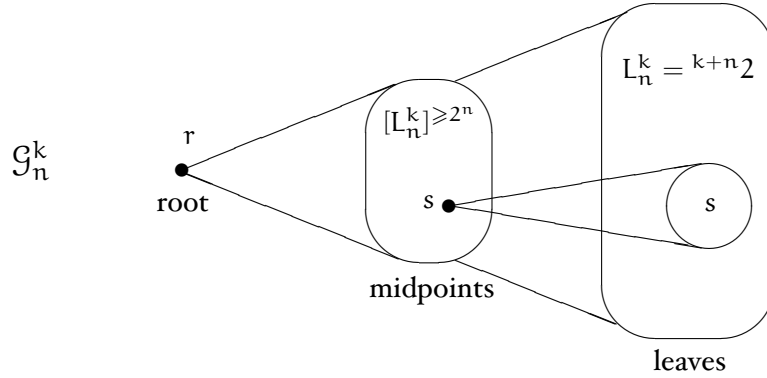


Figure 1: The squat frame  $\mathcal{G}_n^k$

Fix  $k < \omega$ . The following lemmas determine which  $M_l$  ( $l < \omega$ ) are valid in which  $\mathcal{G}_n^k$ .

LEMMA 3.6  $\mathcal{G}_0^k$  validates  $M_l$  for every  $l < \omega$ .

*Proof.*  $M_0 = \top$  is valid, so suppose  $l \geq 1$ . All singleton subsets of  $L_0^k$  are midpoints of  $\mathcal{G}_0^k$ . So each of these midpoints has a unique successor. But any point with at most one successor validates  $\bigwedge_{i < l} (\Box p_i \vee \Box \neg p_i)$ . Since the root sees all midpoints,  $M_l$  is valid at the root, and hence (remark 3.2) valid in  $\mathcal{G}_0^k$ .  $\square$

LEMMA 3.7 For each  $n < \omega$ ,  $M_k$  is valid in  $\mathcal{G}_n^k$ .

*Proof.* Certainly,  $M_0$  is valid. Assume that  $k \geq 1$ . By remark 3.2, we only need check that  $M_k$  is valid at the root. Let  $h : V \rightarrow \wp(\text{dom } \mathcal{G}_n^k)$  be an arbitrary assignment. Then  $h$  induces a partition of  $L_n^k$  into at most  $2^k$  sets, namely, the equivalence classes of the equivalence relation on  $L_n^k$  given by  $x \sim y$  iff  $x \in h(p_i) \iff y \in h(p_i)$  for each  $i < k$ . Since  $|L_n^k| = 2^{k+n}$ , at least one partition set  $s$  must have cardinality at least  $2^n$ , and so is in  $[L_n^k] \geq 2^n$ . Then  $(\mathcal{G}_n^k, h), s \models \Box p_i \vee \Box \neg p_i$  for each  $i < k$ . As  $s$  is accessible from the root, we see that  $(\mathcal{G}_n^k, h), r \models M_k$ . Since  $h$  was arbitrary, the proof is complete.  $\square$

LEMMA 3.8 If  $n \geq 1$  then  $M_{k+1}$  is not valid in  $\mathcal{G}_n^k$ .

*Proof.* As  $n \geq 1$ , we may assign truth values to the variables  $p_0, \dots, p_k$  at points in  $L_n^k$  by:  $p_i$  is true at  $\eta \in {}^{k+n}2$  iff  $\eta(i) = 1$ . Let  $s \subseteq L_n^k$  and suppose that for each  $i \leq k$ ,  $p_i$  has the same truth value on every element of  $s$ . Define  $\xi \in {}^{k+1}2$  by: for each  $i \leq k$ ,  $\xi(i) = 1$  iff  $p_i$  is true at every element of  $s$ . Then  $\xi = \chi \upharpoonright (k+1)$



for every  $x \in s$ . It follows that  $|s| \leq 2^{k+n-(k+1)} = 2^{n-1}$ . As all midpoints of  $\mathcal{G}_n^k$  have at least  $2^n$  elements,  $s$  cannot be in  $\mathcal{G}_n^k$ . So  $\bigwedge_{i \leq k} (\Box p_i \vee \Box \neg p_i)$  is false at every midpoint in  $\mathcal{G}_n^k$ , and therefore  $M_{k+1}$  is false at the root under this assignment.  $\square$

Note that the accessibility relation of  $\mathcal{G}_n^k$  is not transitive. This is essential. For as we mentioned in the introduction,  $M_{k+1} \vdash M_k$  for all  $k$ , and any transitive frame validating  $M_1$  actually validates all the  $M_k$ . It follows that any transitive frame validating  $M_k$  (for any  $k \geq 1$ ) must also validate  $M_{k+1}$ . So transitivity would violate the lemmas. They show that  $M_k \not\vdash M_{k+1}$ . So in the absence of transitivity, the  $M_k$  are strictly increasing in strength. It follows easily that  $KM^\infty$  is not finitely axiomatisable (Corollary 4.5 below).

**REMARK 3.9** These results will show that the class of frames that validate  $KM^\infty$  is not closed under ultraproducts, thereby reproving Theorem 2.2. For each  $n < \omega$ , let  $\mathcal{F}_n = \sum_{k < \omega} \mathcal{G}_n^k$ . By Lemma 3.7,  $\mathcal{G}_n^k$  validates  $M_k$  for each  $k$ . By Lemma 3.4,  $\mathcal{F}_n$  also validates  $M_k$  for each  $k$ . Now consider a non-principal ultraproduct  $\mathcal{F}$  of the  $\mathcal{F}_n$ . Every midpoint of  $\mathcal{F}_n$  has at least  $2^n$  successors. By standard saturation properties of ultraproducts, or by direct inspection, each midpoint of  $\mathcal{F}$  has  $2^\omega$  successors, and  $|\text{dom } \mathcal{F}| = 2^\omega$  as well. By Theorem 2.1,  $\mathcal{F}$  validates no  $M_k$  for any  $k \geq 1$ . Our result now follows from the well known fact that a class of structures is elementary iff it is closed under ultraproducts and ultraroots. The same argument shows that for any  $k \geq 1$ , the class of frames validating  $M_k$  is not closed under ultraproducts. Of course, these results follow from Theorem 2.2, since the class of frames that validate a modal logic is always closed under ultraroots.

### 3.3 DESCRIPTIVE FRAMES AND INVERSE LIMITS

We wish to apply a result of the first author on inverse limits of families of descriptive frames, so we will recall what these are.

**DEFINITION 3.10** A *general frame* is a triple  $(W, R, P)$ , where  $(W, R)$  is a Kripke frame, and  $P \subseteq \wp(W)$  is non-empty and closed under intersection, complement, and the map  $l_R : S \mapsto \{x \in W : \forall y (R(x, y) \rightarrow y \in S)\}$  (for  $S \subseteq W$ ).

A general frame  $(W, R, P)$  is said to be a *descriptive frame* if

1. If  $x, y \in W$  are distinct, then there is some  $S \in P$  with  $x \in S$  and  $y \notin S$ .
2. If  $x, y \in W$  and  $\neg R(x, y)$ , then there is some  $S \in P$  with  $x \in l_R(S)$  and  $y \notin S$ .
3.  $\bigcap \mu \neq \emptyset$  for every ‘ultrafilter’  $\mu \subseteq P$  — i.e., a subset of  $P$  satisfying, for all  $S, S' \in P$ , (i)  $S' \supseteq S \in \mu \Rightarrow S' \in \mu$ , (ii)  $S, S' \in \mu \Rightarrow S \cap S' \in \mu$ , and (iii)  $S \in \mu \iff (W \setminus S) \notin \mu$ .

For information about descriptive frames, see, e.g., [8, §§1.9–1.11], [3, §8.4], and [2, §5.5].

DEFINITION 3.11

1. If  $\mathcal{F} = (W, R)$  is a Kripke frame, we write  $\mathcal{F}^+$  for  $(W, R, \wp(W))$ . ([8, 1.3.5] uses this notation in a different way.) Clearly, if  $\mathcal{F}$  is finite (i.e.,  $W$  is finite), then  $\mathcal{F}^+$  is a descriptive frame.
2. If  $\mathcal{F} = (W, R, P)$  is a descriptive frame, we write  $\mathcal{F}_+$  for its underlying Kripke frame  $(W, R)$ . (We will not use this notation for non-descriptive general frames because it would clash with well known algebraic notation.)

DEFINITION 3.12 Let  $\mathcal{F} = (W, R, P)$  be a general frame and  $\phi$  a modal formula. We say that  $\phi$  is *valid in*  $\mathcal{F}$ , written  $\mathcal{F} \models \phi$ , if  $(W, R, h), w \models \phi$  for every assignment  $h : V \rightarrow P$  and every  $w \in W$ .

Clearly,  $\phi$  is valid in a Kripke frame  $\mathcal{F}$  iff it is valid in the general frame  $\mathcal{F}^+$ :

$$\mathcal{F} \models \phi \iff \mathcal{F}^+ \models \phi. \quad (5)$$

We will also need the notions of bounded morphism, frame homomorphism, and inverse family.

DEFINITION 3.13 Recall that given frames  $\mathcal{F} = (W, R)$  and  $\mathcal{F}' = (W', R')$ , a map  $f : W \rightarrow W'$  is said to be a *bounded morphism* from  $\mathcal{F}$  to  $\mathcal{F}'$  if for all  $w \in W$  and  $v' \in W'$ , we have  $R'(f(w), v')$  iff there is  $v \in R^w$  with  $f(v) = v'$ .

We remark that the generated subframes of a frame  $\mathcal{F}$  are precisely the ranges of bounded morphisms into  $\mathcal{F}$ .

DEFINITION 3.14 [8, DEFINITION 1.5.1] Let  $\mathcal{F} = (W, R, P)$  and  $\mathcal{F}' = (W', R', P')$  be general frames. We say that  $f : \mathcal{F} \rightarrow \mathcal{F}'$  is a *frame homomorphism* if  $f : (W, R) \rightarrow (W', R')$  is a bounded morphism and  $f^{-1}[S'] \in P$  for every  $S' \in P'$ .

Clearly, if  $f : \mathcal{F} \rightarrow \mathcal{F}'$  is a bounded morphism between Kripke frames, then  $f : \mathcal{F}^+ \rightarrow \mathcal{F}'^+$  is a frame homomorphism.

DEFINITION 3.15 [8, DEFINITION 1.11.1] An *inverse family* of descriptive frames is an object

$$\mathcal{J} = ((I, \leq), (\mathcal{F}_i : i \in I), (f_{ij} : i \geq j \text{ in } I)),$$

where  $(I, \leq)$  is an upwards-directed partial order ('upwards-directed' means that any finite subset of  $I$  has an upper bound in  $I$ ),  $\mathcal{F}_i = (W_i, R_i, P_i)$  is a descriptive frame for each  $i \in I$ , and for each  $i, j \in I$  with  $i \geq j$ ,  $f_{ij} : \mathcal{F}_i \rightarrow \mathcal{F}_j$

is a frame homomorphism such that (a)  $f_{ii}$  is the identity map on  $W_i$ , and (b)  $f_{jk} \circ f_{ij} = f_{ik}$  whenever  $k \leq j \leq i$  in  $I$ .

The *inverse limit*  $\lim_{\leftarrow} \mathcal{J}$  of  $\mathcal{J}$  is defined to be  $\mathcal{F} = (W, R, P)$ , where

$$\begin{aligned} W &= \{x \in \prod_{i \in I} W_i : f_{ij}(x_i) = x_j \text{ for each } i \geq j \text{ in } I\}, \\ R &= \{(x, y) \in W : R_i(x_i, y_i) \text{ for each } i \in I\}, \\ P &= \{f_i^{-1}[S] : i \in I, S \in P_i\}. \end{aligned}$$

In the third line, for each  $i \in I$ ,  $f_i : W \rightarrow W_i$  is the projection given by  $f_i(x) = x_i$ .

The main fact we need about inverse limits is:

**FACT 3.16** [8, I.II.2(8), I.II.4] *In the above notation, the inverse limit  $\mathcal{F}$  of  $\mathcal{J}$  is itself a descriptive frame. Moreover, for any modal formula  $\phi$ , if  $\phi$  is valid in  $\mathcal{F}_i$  for every  $i \in I$ , then  $\phi$  is valid in  $\mathcal{F}$ .*

### 3.4 INVERSE LIMITS OF THE SQUAT FRAMES

We will now apply this to our squat frames  $\mathcal{G}_n^k$ .

**DEFINITION 3.17** Let  $k < \omega$  and  $n \leq m < \omega$ . We define  $\pi_{mn}^k : \text{dom } \mathcal{G}_m^k \rightarrow \text{dom } \mathcal{G}_n^k$  as follows:

- It takes the root of  $\mathcal{G}_m^k$  to the root of  $\mathcal{G}_n^k$ .
- $\pi_{mn}^k(x) = x \upharpoonright (k+n)$  for each leaf  $x \in L_m^k = {}^{k+m}2$ .
- $\pi_{mn}^k$  maps a set  $s \in [L_m^k]^{\geq 2^m}$  to the set  $\{\pi_{mn}^k(x) : x \in s\}$ .  
(It is clear that  $|\pi_{mn}^k(s)| \geq |s|/2^{m-n}$ , so that indeed,  $\pi_{mn}^k(s) \in [L_n^k]^{\geq 2^n}$ .)

**LEMMA 3.18** *Let  $k < \omega$  and  $n \leq m \leq l < \omega$ . Then  $\pi_{mn}^k : \mathcal{G}_m^k \rightarrow \mathcal{G}_n^k$  is a surjective bounded morphism,  $\pi_{nn}^k$  is the identity on  $\text{dom } \mathcal{G}_n^k$ , and  $\pi_{ln}^k = \pi_{mn}^k \circ \pi_{lm}^k$ .*

*Proof.* Straightforward. □

We need a little notation: if  $\mathcal{F}, \mathcal{F}', \mathcal{G}, \mathcal{G}'$  are squat frames and  $f : \mathcal{F} \rightarrow \mathcal{F}'$ ,  $g : \mathcal{G} \rightarrow \mathcal{G}'$  are bounded morphisms taking roots to roots, then we define  $f+g : \mathcal{F}+\mathcal{G} \rightarrow \mathcal{F}'+\mathcal{G}'$  to be the map (clearly a well defined bounded morphism) taking the root of  $\mathcal{F}+\mathcal{G}$  to the root of  $\mathcal{F}'+\mathcal{G}'$ , and given on the remaining worlds  $x$  by

$$(f+g)(x) = \begin{cases} f(x), & \text{if } x \in \text{dom } \mathcal{F}, \\ g(x), & \text{otherwise.} \end{cases}$$

Until §4, fix  $k, l < \omega$ . We will define two inverse families of descriptive frames made from finite squat frames (for  $\pi_{mn}^k$  and  $-^+$  see definitions 3.17 and 3.11):

1.  $\mathcal{J}^k = ((\omega, <), ((\mathcal{G}_n^k)^+ : n < \omega), (\pi_{mn}^k : n \leq m < \omega)),$
2.  $\mathcal{J}^{k,l} = ((\omega, <), ((\mathcal{G}_n^k + \mathcal{G}_1^l)^+ : n < \omega), (\pi_{mn}^k + \iota : n \leq m < \omega)),$  where  $\iota$  is the identity map on  $\text{dom } \mathcal{G}_1^l$ .

The general frames here are descriptive frames because they are of the form  $\mathcal{F}^+$  for a finite Kripke frame  $\mathcal{F}$ . We are interested in the inverse limits of these families. For short, write

$$\begin{aligned} \mathcal{G}_\infty &= (\varprojlim (\mathcal{J}^k))_+ \\ \mathcal{F}_\infty &= (\varprojlim (\mathcal{J}^{k,l}))_+ \end{aligned} \tag{6}$$

LEMMA 3.19  $\mathcal{F}_\infty \cong \mathcal{G}_\infty + \mathcal{G}_1^l$ .

*Proof.* Let  $r$  be the root of  $\mathcal{F}_\infty$  and  $r'$  the root of  $\mathcal{G}_\infty + \mathcal{G}_1^l$ . By definition, for  $\eta \in \mathcal{F}_\infty$  we have  $\eta_n \in \text{dom}(\mathcal{G}_n^k + \mathcal{G}_1^l)$  for each  $n < \omega$ . Define

$$\eta' = \begin{cases} r', & \text{if } \eta = r, \\ \eta, & \text{if } \eta \neq r \text{ and } \eta_n \in \text{dom } \mathcal{G}_n^k \text{ for each } n < \omega, \\ \eta_0, & \text{if } \eta \neq r \text{ and } \eta_n \in \text{dom } \mathcal{G}_1^l \text{ for each } n < \omega. \end{cases}$$

It can be checked that  $(\eta \mapsto \eta') : \mathcal{F}_\infty \rightarrow \mathcal{G}_\infty + \mathcal{G}_1^l$  is well defined and is the required isomorphism.  $\square$

$\mathcal{F}_\infty$  is the underlying Kripke frame of the inverse limit of an inverse family of descriptive frames whose underlying Kripke frames all validate  $M_{\max(k,l)}$  (by Lemmas 3.7 and 3.4). Now  $k, l < \omega$  are arbitrary, and it could be that  $k \gg l$ . Nevertheless, and perhaps surprisingly,  $\mathcal{F}_\infty$  need not validate  $M_k$ . Indeed, we will show that  $\mathcal{F}_\infty \models M_l$  but  $\mathcal{F}_\infty \not\models M_{l+1}$ .

This will be proved by showing that  $\mathcal{G}_\infty \not\models M_n$  for any  $n \geq 1$ . The proof will need some technical lemmas. The first one is almost immediate from the definition of  $\mathcal{G}_\infty$ :

LEMMA 3.20  $\mathcal{G}_\infty$  is a squat frame with at most  $2^\omega$  worlds.

*Proof (sketch).* The maps  $\pi_{nm}^k$  take roots to roots, midpoints to midpoints, and leaves to leaves. So each element in  $\text{dom } \mathcal{G}_\infty$  is a sequence in  $\prod_{n < \omega} \text{dom } \mathcal{G}_n^k$  consisting entirely of roots, entirely of midpoints, or entirely of reflexive leaves. It is not so hard to see that such a sequence is a root, midpoint, or reflexive leaf of  $\mathcal{G}_\infty$ , respectively. (In particular, because the maps  $\pi_{mn}^k$  are bounded morphisms, we can inductively construct a sequence of leaves that is a successor in  $\mathcal{G}_\infty$  of any given sequence in  $\mathcal{G}_\infty$  consisting of midpoints. We will prove a stronger result in Corollary 3.25 below.) It follows easily that  $\mathcal{G}_\infty$  is squat. Since the  $\mathcal{G}_n^k$  are finite,  $|\text{dom } \mathcal{G}_\infty| \leq |\prod_{n < \omega} \text{dom } \mathcal{G}_n^k| = 2^\omega$ .  $\square$

The next fact we need — that each midpoint of  $\mathcal{G}_\infty$  has  $2^\omega$  successors — is a little harder to prove. Let  $\mathcal{G}_\infty = (W, R)$ , say. Fix an arbitrary midpoint  $s = (s_n : n < \omega)$  of  $\mathcal{G}_\infty$ . So (i) each  $s_n$  is a midpoint of  $\mathcal{G}_n^k$ , and (ii)  $\pi_{mn}^k(s_m) = s_n$  whenever  $n \leq m$ . By the definitions, this says:

(i)  $s_n \subseteq {}^{k+n}2$  and  $|s_n| \geq 2^n$  for each  $n < \omega$ ,

(ii)  $s_n = \{x \upharpoonright (k+n) : x \in s_m\}$  whenever  $n \leq m < \omega$ .

An element  $x = (x_n : n < \omega)$  of  $\mathcal{G}_\infty$  is a leaf of  $\mathcal{G}_\infty$  iff  $x_n \in {}^{k+n}2$  for each  $n$ . In this case,  $x_n = x_m \upharpoonright (k+n)$  for each  $n \leq m < \omega$ , and  $x \in R^s$  iff  $x_n \in s_n$  for all  $n$ .

**DEFINITION 3.21** Let  $n < \omega$  and  $x \in s_n$ .

1. For  $n \leq m < \omega$ , write  $s_m^x = \{y \in s_m : y \upharpoonright (k+n) = x\}$ .

2. For  $c < \omega$ , we say that  $x$  is *c-big* if  $|s_m^x| \geq 2^{m-n-c}$  for every  $m \geq n$ .

Since  $s_m^x \subseteq {}^{k+m}2$  and  $x \in {}^{k+n}2$ , we see that

$$|s_m^x| \leq 2^{m-n} \text{ for any } n \leq m < \omega \text{ and } x \in s_n. \quad (7)$$

Clearly,

$$s_l^x = \bigcup \{s_l^y : y \in s_m^x\}, \text{ whenever } n \leq m \leq l < \omega \text{ and } x \in s_n. \quad (8)$$

Also note that ‘c-big’ gets *weaker* as  $c$  grows: any c-big element is  $(c+1)$ -big.

**LEMMA 3.22** *If  $x \in s_n$  is not c-big, then for all large enough  $l \geq n$  we have  $|s_l^x| < 2^{l-n-c}$ .*

*Proof.* By assumption, there is  $m \geq n$  such that  $|s_m^x| < 2^{m-n-c}$ . Take any  $l \geq m$ . By (7),  $|s_l^y| \leq 2^{l-m}$  for each  $y \in s_m^x$ . So by (8),  $|s_l^x| \leq 2^{l-m} \cdot |s_m^x| < 2^{l-m} \cdot 2^{m-n-c} = 2^{l-n-c}$ .  $\square$

**COROLLARY 3.23** *There is some k-big  $x \in s_0$ .*

*Proof.* If not, then since  $s_0$  is finite, by the preceding lemma we may choose large enough  $n < \omega$  such that  $|s_n^x| < 2^{n-k}$  for every  $x \in s_0$ . Now  $s_0 \subseteq {}^k2$ , so  $|s_0| \leq 2^k$ . By (ii) above,  $s_n = \bigcup \{s_n^x : x \in s_0\}$ . Hence,  $|s_n| < 2^{n-k} \cdot |s_0| \leq 2^n$ , contradicting (i) above.  $\square$

**PROPOSITION 3.24** *For any  $n, c < \omega$  and any c-big  $x \in s_n$ , there is some  $m > n$  such that  $s_m^x$  contains at least two c-big elements.*

*Proof.* By induction on  $c$ . If  $c = 0$ , then  $|s_m^x| \geq 2^{m-n}$  for all  $m \geq n$ . But (cf. (7)) we have  $s_m^x \subseteq \{y \in {}^{k+m}2 : y \upharpoonright (k+n) = x\}$ , and the right-hand side has cardinality  $2^{m-n}$ . So in fact,

$$s_m^x = \{y \in {}^{k+m}2 : y \upharpoonright (k+n) = x\} \quad \text{for all } m \geq n. \quad (9)$$

Now, for  $l \geq m \geq n$  and any  $y \in s_m^x$ , we have

$$\begin{aligned} s_l^y &= \{z \in s_l^y : z \upharpoonright (k+m) = y\} && \text{by definition of } s_l^y, \\ &= \{z \in s_l^x : z \upharpoonright (k+m) = y\} && \text{since } y \upharpoonright (k+n) = x, \\ &= \{z \in {}^{k+l}2 : z \upharpoonright (k+m) = y\} && \text{by (9)}. \end{aligned}$$

So  $|s_l^y| = 2^{l-m}$ , and hence every element of every  $s_m^x$  ( $m \geq n$ ) is 0-big.

Suppose that  $1 \leq c < \omega$ , and inductively assume the proposition for smaller  $c$ . Let  $x \in s_n$  be  $c$ -big. There are two cases.

*Case 1: some element of  $\bigcup_{m \geq n} s_m^x$  is  $(c-1)$ -big.* Suppose that  $m \geq n$  and  $y \in s_m^x$  is  $(c-1)$ -big. By the inductive hypothesis, there is  $l > m$  such that  $s_l^y$  (and hence  $s_l^x$ ) contains at least two  $(c-1)$ -big (and hence  $c$ -big) elements, as required.

*Case 2: otherwise.* So  $x$  itself is not  $(c-1)$ -big, and by Lemma 3.22 we may take  $m \geq c+n$  such that  $|s_m^x| < 2^{m-n-(c-1)}$ . We show that there are at least two  $c$ -big elements of  $s_m^x$ .

Assume for contradiction that  $s_m^x$  has at most one  $c$ -big element. By (ii) above,  $s_m^x \neq \emptyset$ . So we may take  $y \in s_m^x$  such that  $s_m^x \setminus \{y\}$  contains no  $c$ -big elements. By the case assumption,  $y$  is not  $(c-1)$ -big. So using Lemma 3.22 repeatedly, there is large enough  $l > m$  such that

$$\begin{aligned} |s_l^y| &\leq 2^{l-m-c+1} - 1, \\ |s_l^z| &\leq 2^{l-m-c} - 1 \quad \text{for all } z \in s_m^x \setminus \{y\}. \end{aligned}$$

Now by (8),  $s_l^x = s_l^y \cup \bigcup \{s_l^z : z \in s_m^x \setminus \{y\}\}$ . We know that  $|s_m^x| < 2^{m-n-(c-1)}$ , so

$$|s_m^x \setminus \{y\}| \leq 2^{m-n-c+1} - 2.$$

Because  $x$  is  $c$ -big, we have  $|s_l^x| \geq 2^{l-n-c}$ . We conclude that

$$\begin{aligned} 2^{l-n-c} &\leq |s_l^x| \\ &\leq |s_l^y| + \sum_{z \in s_m^x \setminus \{y\}} |s_l^z| \\ &\leq 2^{l-m-c+1} - 1 + (2^{m-n-c+1} - 2)(2^{l-m-c} - 1) \\ &= 2^{l-m-c+1} - 1 + 2^{l-n-2c+1} - 2^{l-m-c+1} - 2^{m-n-c+1} + 2 \\ &= 2^{l-n-2c+1} - 2^{m-n-c+1} + 1 \\ &< 2^{l-n-2c+1} \quad (\text{since } m \geq c+n, \text{ so } m-n-c+1 \geq 1). \end{aligned}$$

Hence  $l-n-c < l-n-2c+1$ , and so  $c < 1$ , contradicting our assumption. So again,  $s_m^x$  has at least two  $c$ -big elements, as required.

This completes the induction and the proof.  $\square$

COROLLARY 3.25  $|R^s| \geq 2^\omega$ .

*Proof.* For each  $\sigma \in {}^n 2$  (each  $n < \omega$ ), we will choose  $k$ -big  $\widehat{\sigma} \in s_m$  for some  $m \geq n$  by induction on  $n$ , such that  $\widehat{\sigma^0}, \widehat{\sigma^1}$  are distinct elements of  $s_l^{\widehat{\sigma}}$  for some  $l > m$ . Here, we write  $\widehat{\sigma^i}$  for the map  $\tau \in {}^{n+1} 2$  given by  $\tau \upharpoonright n = \sigma$  and  $\tau(n) = i$  (for  $i = 0, 1$ ).

We have  ${}^0 2 = \{\emptyset\}$ . Let  $\widehat{\emptyset}$  be any  $k$ -big element of  $s_0$ ; by Corollary 3.23, such an element exists. Inductively, if  $k$ -big  $\widehat{\sigma} \in s_m$  has been chosen, by Proposition 3.24 we can choose  $l > m$  and distinct  $k$ -big  $\widehat{\sigma^0}, \widehat{\sigma^1} \in s_l^{\widehat{\sigma}}$ .

Now, for each  $\eta \in {}^\omega 2$ ,  $\{\widehat{\eta \upharpoonright n} : n < \omega\}$  generates a leaf  $\lambda(\eta) = ((\widehat{\eta \upharpoonright n}) \upharpoonright (k+n)) : n < \omega \in R^s$ , and the  $\lambda(\eta)$  for distinct  $\eta$  are pairwise distinct. So  $\lambda : {}^\omega 2 \rightarrow R^s$  is one-one, and hence  $|R^s| \geq 2^\omega$ .  $\square$

The corollary holds for any midpoint  $s$  of  $\mathcal{G}_\infty$ . This completes our analysis of the structure of  $\mathcal{G}_\infty$ . The underlying ‘combinatorial principle’ we used is that for any  $k < \omega$ , any subtree of the infinite binary tree whose  $n$ th level has at least  $2^{n-k}$  nodes, for each  $n$ , has  $2^\omega$  branches.

It now follows that:

PROPOSITION 3.26  $\mathcal{G}_\infty \not\models M_n$  for every  $n \geq 1$ .

*Proof.* Let  $r$  be the root of  $\mathcal{G}_\infty$ . Then  $R^r$  is the set of midpoints of  $\mathcal{G}_\infty$ . By Lemma 3.20 and Corollary 3.25, for any  $s \in R^r$  we have  $|R^r| + \omega \leq 2^\omega = |R^s|$ . The result follows by Theorem 2.1.  $\square$

We can now prove what we wanted.

COROLLARY 3.27  $\mathcal{F}_\infty \models M_1$  but  $\mathcal{F}_\infty \not\models M_{l+1}$ .

*Proof.* Recall from Lemma 3.19 that  $\mathcal{F}_\infty \cong \mathcal{G}_\infty + \mathcal{G}_1^l$ . By Lemma 3.7,  $\mathcal{G}_1^l \models M_l$ , so by Lemma 3.4,  $\mathcal{F}_\infty \models M_l$  as well. By Proposition 3.26,  $\mathcal{G}_\infty \not\models M_{l+1}$ ; and by Lemma 3.8,  $\mathcal{G}_1^l \not\models M_{l+1}$ . So by Lemma 3.4,  $\mathcal{F}_\infty \not\models M_{l+1}$ .  $\square$

We summarise our conclusions in the following

THEOREM 3.28 Let  $k, l < \omega$ .

1. There is an inverse family  $\mathcal{J}^{k,l}$  of finite descriptive frames validating  $M_{\max(k,l)}$ , such that if  $\mathcal{F}_\infty = (\lim_{\leftarrow} \mathcal{J}^{k,l})_+$  is the underlying Kripke frame of the inverse limit of  $\mathcal{J}^{k,l}$ , then  $\mathcal{F}_\infty \models M_l$  but  $\mathcal{F}_\infty \not\models M_{l+1}$ .
2. There is a descriptive frame  $\mathcal{D}^{k,l} = (W, R, P)$  with  $|P| = \omega$ , such that  $\mathcal{D}^{k,l} \models M_{\max(k,l)}$ ,  $\mathcal{D}_+^{k,l} \models M_l$ , and  $\mathcal{D}_+^{k,l} \not\models M_{l+1}$ .

*Proof.* The first part has already been established. For the second part, take  $\mathcal{D}^{k,l} = \lim_{\leftarrow} \mathcal{J}^{k,l} = (W, R, P)$ , say. It is clear from Definition 3.15 that  $P$  is countably infinite. By fact 3.16,  $\mathcal{D}^{k,l} \models M_{\max(k,l)}$ . The rest is as in the first part.  $\square$

#### 4 CANONICAL AXIOMS AND $KM^\infty$

A modal formula  $\phi$  is said to be *canonical* if it is valid in the canonical frame of the normal modal logic axiomatised by  $\phi$ . The following is more convenient here, and is well known to be equivalent to this:

**DEFINITION 4.1** A modal formula  $\phi$  is said to be *countably d-persistent* if whenever it is valid in a descriptive frame  $\mathcal{F} = (W, R, P)$  with  $P$  countable, it is also valid in its underlying Kripke frame  $\mathcal{F}_+$ .

**LEMMA 4.2** *Any canonical formula is countably d-persistent, and conversely.*

*Proof (sketch).* We only sketch the proof, because it is well known (see, e.g., [16, p. 221]). We assume familiarity with canonical models; see [2, 3] or any modal logic text for details. Write  $L$  for the set of all modal formulas written using only propositional variables from our countable set  $V$ . Let  $\mathcal{F} = (W, R, P)$  be a descriptive frame such that  $P$  is countable, and suppose that  $\phi \in L$  is canonical and valid in  $\mathcal{F}$ . Let  $\Lambda$  be the modal logic axiomatised by  $\phi$ , and let  $\mathcal{M} = (W^*, R^*, h^*)$  be its canonical model — so  $W^*$  is the set of all maximal  $\Lambda$ -consistent subsets of  $L$ .

Since  $P$  is countable, we may choose a surjective assignment  $h : V \rightarrow P$ . For  $w \in W$ , put  $\Gamma_w = \{\psi \in L : (\mathcal{F}, h), w \models \psi\}$ . Since  $\phi$  is valid in  $\mathcal{F}$ ,  $\Lambda$  is also valid in  $\mathcal{F}$ , and it follows that each  $\Gamma_w$  is maximal  $\Lambda$ -consistent (i.e., in  $W^*$ ). Using that  $\mathcal{F}$  is a descriptive frame and that  $h$  is surjective, it can be checked that the map  $f : W \rightarrow W^*$  given by  $f(w) = \Gamma_w$  is a one-one bounded morphism. Since  $\phi$  is assumed canonical, it is valid in  $(W^*, R^*)$ , and so also in its generated subframe based on  $\text{rng } f$ . But  $f$  is an isomorphism from  $(W, R)$  onto this. So  $\phi$  is valid in  $(W, R)$ , as required.

Conversely, if  $\phi$  is countably d-persistent then of course it is canonical, because the canonical model of the logic axiomatised by  $\phi$  can be viewed as a descriptive frame with countable ‘ $P$ ’-part (namely, the truth sets of formulas in  $L$ ), and  $\phi$  is valid in it.  $\square$

We can now prove our second main result. The case  $k = 1$  was proved in [7].

**THEOREM 4.3** *For no  $k \geq 1$  is  $M_k$  canonical.*

*Proof.* Let  $k \geq 1$ ; we prove that  $M_k$  is not countably d-persistent. For each  $n$ ,  $\mathcal{G}_n^k \models M_k$  by Lemma 3.7. By fact 3.16,  $\lim_{\leftarrow} \mathcal{J}^k \models M_k$  as well. By definition, the ‘ $P$ ’-part of  $\lim_{\leftarrow} \mathcal{J}^k$  is countable. But by Proposition 3.26,  $(\lim_{\leftarrow} \mathcal{J}^k)_+ = \mathcal{G}_\infty \not\models M_k$ .  $\square$



It follows that no  $M_k$  ( $k \geq 1$ ) is d-persistent (this stronger notion is defined as in Definition 4.1 but without the cardinality restriction).

To prove our third result, we want to use first-order compactness. To do this, we view a general frame  $(W, R, P)$  as a first-order structure whose domain is the disjoint union of  $W$  and  $P$ , with unary relations picking out  $W$  and  $P$ , and binary relations  $R \subseteq W \times W$  and  $\in \subseteq W \times P$  interpreted in the natural way. It is easy to write down a finite set  $\Delta$  of first-order sentences expressing that a structure  $(W, R, P)$  for this signature is a general frame.

As is well known (see, e.g., [2, definition 2.45]), every modal formula  $\phi$  has a *standard translation* to a formula  $ST_x(\phi)$  of first-order logic, with a free variable  $x$ . We modify this here by regarding propositional variables as first-order variables. For a propositional variable  $p$ , we define  $ST_x(p)$  to be  $x \in p$ . We put  $ST_x(\top) = \top$ , etc.,  $ST_x(\phi \wedge \psi) = ST_x(\phi) \wedge ST_x(\psi)$  and similarly for negation, and  $ST_x(\Box\phi) = \forall y(R(x, y) \rightarrow ST_y(\phi))$  and  $ST_x(\Diamond\phi) = \exists y(R(x, y) \wedge ST_y(\phi))$ . Here,  $y$  is a new variable. For a formula  $\phi(p_1, \dots, p_n)$ , we write  $ST(\phi)$  for the universal closure  $\forall x \in W \forall p_1 \dots p_n \in P ST_x(\phi)$ . For a set  $X$  of formulas, we write  $ST(X)$  for  $\{ST(\phi) : \phi \in X\}$ . Clearly, a modal formula  $\phi$  is valid in a general frame  $\mathcal{G}$  iff  $ST(\phi)$  is true in it in first-order semantics:

$$\mathcal{G} \models \phi \iff \mathcal{G} \models ST(\phi). \quad (10)$$

Hence (cf. (5)),  $\phi$  is valid in a Kripke frame  $\mathcal{F}$  iff  $ST(\phi)$  is true in  $\mathcal{F}^+$  in first-order semantics:

$$\mathcal{F} \models \phi \iff \mathcal{F}^+ \models ST(\phi). \quad (11)$$

With these preliminaries in hand, we can prove our third theorem.

**THEOREM 4.4** *Any axiomatisation of the logic  $KM^\infty$  has infinitely many non-canonical axioms.*

*Proof.* Suppose on the contrary that (without loss of generality)  $KM^\infty$  is axiomatised by a single axiom  $B$  together with a set  $\Sigma$  of canonical formulas. Since  $\Sigma \cup \{B\}$  and  $\{M_k : k < \omega\}$  axiomatise the same logic, the two first-order theories

$$\begin{aligned} \Delta \cup ST(\Sigma \cup \{B\}), \\ \Delta \cup \{ST(M_k) : k < \omega\} \end{aligned}$$

have the same models. (Here,  $\Delta$  is as above.) Therefore, bearing in mind that for  $m > n$ ,  $M_m \vdash M_n$  and hence  $\Delta \cup ST(M_m) \models ST(M_n)$ , first-order compactness yields:

- (a) there is  $l < \omega$  such that  $\Delta \cup ST(M_l) \models ST(B)$ ,
- (b) there is a finite subset  $X \subseteq \Sigma$  such that  $\Delta \cup ST(X \cup \{B\}) \models ST(M_{l+1})$ ,
- (c) there is finite  $k$  such that  $\Delta \cup ST(M_k) \models ST(X)$ . (Necessarily,  $k > l$ .)

Let  $\mathcal{D} = \mathcal{D}^{k,l}$  be the descriptive frame of Theorem 3.28(2). The ‘P’-part of  $\mathcal{D}$  is countable,  $\mathcal{D} \models M_{\max(k,l)}$ ,  $\mathcal{D}_+ \models M_l$ , and  $\mathcal{D}_+ \not\models M_{l+1}$ .

We have  $\mathcal{D} \models M_k$ . Plainly,  $\mathcal{D} \models \Delta$ . Now, by (c) and (10), we obtain  $\mathcal{D} \models X$ . The formulas in  $X$  are assumed canonical, so by Lemma 4.2,  $\mathcal{D}_+ \models X$  as well. By (11),  $(\mathcal{D}_+)^+ \models ST(X)$ .

As  $\mathcal{D}_+ \models M_l$ , (11) gives  $(\mathcal{D}_+)^+ \models ST(M_l)$ . Clearly,  $(\mathcal{D}_+)^+ \models \Delta$ . So by (a),  $(\mathcal{D}_+)^+ \models ST(B)$ .

Now we have  $(\mathcal{D}_+)^+ \models \Delta \cup ST(X \cup \{B\})$ , so by (b) and (11), we arrive at  $\mathcal{D}_+ \models M_{l+1}$ , a contradiction.  $\square$

The following corollary follows immediately.

COROLLARY 4.5  $KM^\infty$  is not finitely axiomatisable.

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