Natural Derivations for Priest, An Introduction to Non-Classical Logic

Tony Roy*
Department of Philosophy
California State University, San Bernardino
troy@csusb.edu

Received by Greg Restall
Published March 20, 2006
http://www.philosophy.unimelb.edu.au/ajl/2006

Abstract: This document collects natural derivation systems for logics described in Priest, *An Introduction to Non-Classical Logic* [4]. It provides an alternative or supplement to the semantic tableaux of his text. Except that some chapters are collapsed, there are sections for each chapter in Priest, with an additional, final section on quantified modal logic. In each case, (i) the language is briefly described and key semantic definitions stated, (ii) the derivation system is presented with a few examples given, and (iii) soundness and completeness are proved. There should be enough detail to make the parts accessible to students would work through parallel sections of Priest.

This document collects natural derivation systems for logics described in Priest, *An Introduction to Non-Classical Logic* [4]. It thus provides an alternative or supplement to the semantic tableaux of his text. Some of the derivation systems may also be of interest in their own right. They are all Fitch-style systems on the model of [1, 12], and many other places. Though a classical system is presented for chapter 1, prior acquaintance with some such system is assumed. Associated goal-directed derivation strategies are discussed extensively in [12, chapter 6].

Except that some chapters are collapsed, there are sections for each chapter in Priest, with an additional, final section on quantified modal logic. In each

*Thanks to Graham Priest for encouragement, and to referee(s) for this *Journal* who took the time to plow through the details, and make many helpful suggestions. Thanks also to Darcy Otto for getting me started on LaTeX, with which this document was produced.

case, (i) the language is briefly described and key semantic definitions stated, (ii) the derivation system is presented with a few examples given, and (iii) soundness and completeness are proved. Notation is common with Priest, though some nomenclature is improvised to keep things systematic. Though cases of some proofs are left to the reader, there should be enough detail to make the parts accessible to students who would work through parallel sections of Priest.

Demonstrations of completeness are all on the model of the standard argument for classical logic, and simplified considerably by the use of "subscripts" and "overlines" in derivations. For the most part, I take over approaches from tableaux in Priest. Thus, e.g., subscripts are like indexes from his tableaux. Overlines are like underlines in [13]. Advantages of the approach to completeness are particularly dramatic when quantifiers are introduced, as exhibited in the section on quantified modal logic.

TABLE OF CONTENTS

1 CLASSICAL LOGIC: *CL* (CH. 1)

1.1 LANGUAGE / SEMANTIC NOTIONS

LCL The LANGUAGE consists of propositional parameters $p_0, p_1 \ldots$ combined in the usual way with the operators, $\neg$, $\wedge$, $\vee$, $\supset$, and $\equiv$. So each propositional parameter is a FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$ and $(A \equiv B)$.

ICL An INTERPRETATION is a function v which assigns to each propositional parameter either 1 (true) or 0 (false).

TCL For complex expressions,

- (¬) $v(\neg A) = 1$ if $v(A) = 0$, and 0 otherwise.
- (∧) $v(A \wedge B) = 1$ if $v(A) = 1$ and $v(B) = 1$, and 0 otherwise.
- (∨) $v(A \vee B) = 1$ if $v(A) = 1$ or $v(B) = 1$, and 0 otherwise.
- (⊃) $v(A \supset B) = 1$ if $v(A) = 0$ or $v(B) = 1$, and 0 otherwise.
- (≡) $v(A \equiv B) = 1$ if $v(A) = v(B)$, and 0 otherwise.

For a set Γ of formulas, $v(\Gamma) = 1$ iff $v(A) = 1$ for each $A \in \Gamma$; then,

VCL $\Gamma \models_{CL} A$ iff there is no *CL* interpretation v such that $v(\Gamma) = 1$ and $v(A) = 0$.

1.2 NATURAL DERIVATIONS: *NCL*

NCL is just the sentential portion of the system *ND* from [12, chapter 6]. Refer to that source for examples and further discussion (compare, e.g., [1]). Every line of a derivation is a premise, an assumption, or justified from previous lines by a rule. The rules include *introduction* and *exploitation* rules for each operator, and *reiteration*. In the parenthetical "exit strategy" for assumptions, 'c' indicates a contradiction is to be sought, 'g' a goal at the bottom of the scope line.

R (*reiteration*)

```
a | P
  |
  | P          a R
```

¬I (*negation intro*)

```
a | | P        A (c, ¬I)
  | |---
  | |
  | | Q
b | | ¬Q
  | ¬P         a-b ¬I
```

¬E (*negation exploit*)

```
a | | ¬P       A (c, ¬E)
  | |---
  | |
  | | Q
b | | ¬Q
  | P          a-b ¬E
```

∧I (*conjunction intro*)

```
a | P
b | Q
  |
  | P ∧ Q      a,b ∧I
```

∧E (*conjunction exploit*)

```
a | P ∧ Q
  |
  | P          a ∧E
```

∧E (*conjunction exploit*)

```
a | P ∧ Q
  |
  | Q          a ∧E
```

∨I (*disjunction intro*)

```
a | P
  |
  | P ∨ Q      a ∨I
```

∨I (*disjunction intro*)

```
a | P
  |
  | Q ∨ P      a ∨I
```

∨E (*disjunction exploit*)

```
a | P ∨ Q
b | | P        A (g, a ∨E)
  | |---
  | |
c | | R
  |
d | | Q        A (g, a ∨E)
  | |---
  | |
e | | R
  | R          a,b-c,d-e ∨E
```

⊃I (*conditional intro*)

```
a | | P        A (g, ⊃I)
  | |---
  | |
b | | Q
  | P ⊃ Q      a-b ⊃I
```

⊃E (*conditional exploit*)

```
a | P ⊃ Q
b | P
  |
  | Q          a,b ⊃E
```

≡I (*biconditional intro*)

a	| P	A (g, ≡I)
b	| Q	
c	| Q	A (g, ≡I)
d	| P	
	P ≡ Q	*a-b,c-d* ≡I

≡E (*biconditional exploit*)

a	P ≡ Q	
b	P	
	Q	*a,b* ≡E

≡E (*biconditional exploit*)

a	P ≡ Q	
b	Q	
	P	*a,b* ≡E

NCL $\Gamma \vdash_{NCL} A$ iff there is an *NCL* derivation of A from the members of Γ.

As derived rules, we accept the following "ordinary" and "two-way" rules. The "two-way" rules are usually presented as *replacement* rules. Insofar as we will not have much call to use then that way, in order to streamline demonstrations of soundness, we treat them just as ordinary rules which work in either direction – where it is trivial that the rules are in fact derived in this sense from the rules of *NCL*.

Ordinary Derived Rules

modus tollens

MT	
	P ⊃ Q
	¬Q
	¬P

negated biconditional

NB		
	P ≡ Q	P ≡ Q
	¬P	¬Q
	¬Q	¬P

disjunctive syllogism

DS		
	P ∨ Q	P ∨ Q
	¬P	¬Q
	Q	P

Two-way Derived Rules

DN	P ◁▷ ¬¬P	*double negation*
Com	P ∧ Q ◁▷ Q ∧ P P ∨ Q ◁▷ Q ∨ P	*commutation*
Assoc	P ∧ (Q ∧ R) ◁▷ (P ∧ Q) ∧ R P ∨ (Q ∨ R) ◁▷ (P ∨ Q) ∨ R	*association*
Idem	P ◁▷ P ∧ P P ◁▷ P ∨ P	*idempotence*
Impl	P ⊃ Q ◁▷ ¬P ∨ Q ¬P ⊃ Q ◁▷ P ∨ Q	*implication*
Trans	P ⊃ Q ◁▷ ¬Q ⊃ ¬P	*transposition*
DeM	¬(P ∧ Q) ◁▷ ¬P ∨ ¬Q ¬(P ∨ Q) ◁▷ ¬P ∧ ¬Q	*De Morgan*
Exp	P ⊃ (Q ⊃ R) ◁▷ (P ∧ Q) ⊃ R	*exportation*

Equiv	$P \equiv Q$	$\lhd\rhd$ $(P \supset Q) \wedge (Q \supset P)$	*equivalence*
	$P \equiv Q$	$\lhd\rhd$ $(P \wedge Q) \vee (\neg P \wedge \neg Q)$	
Dist	$P \wedge (Q \vee R)$	$\lhd\rhd$ $(P \wedge Q) \vee (P \wedge R)$	*distribution*
	$P \vee (Q \wedge R)$	$\lhd\rhd$ $(P \vee Q) \wedge (P \vee R)$	

As examples, here are derivations to demonstrate the first form of Impl (among the relatively difficult of derivations for the derived rules).

$\neg P \vee Q \vdash_{NCL} P \supset Q$

```
 1 | ¬P ∨ Q              P
 2 | | ¬P                A (g, 1 ∨E)
 3 | | | P               A (g, ⊃I)
 4 | | | | ¬Q          A (c, ¬E)
 5 | | | | ¬P          2 R
 6 | | | | P           3 R
 7 | | | Q             4-6 ¬E
 8 | | P ⊃ Q           3-7 ⊃I
 9 | | Q               A (g, 1 ∨E)
10 | | | P             A (g, ⊃I)
11 | | | Q             9 R
12 | | P ⊃ Q           10-11 ⊃I
13 | P ⊃ Q             1,2-8,9-12 ∨E
```

$P \supset Q \vdash_{NCL} \neg P \vee Q$

```
 1 | P ⊃ Q               P
 2 | | ¬(¬P ∨ Q)        A (c, ¬E)
 3 | | | P              A (c, ¬I)
 4 | | | Q              1,3 ⊃E
 5 | | | ¬P ∨ Q         4 ∨I
 6 | | | ¬(¬P ∨ Q)      2 R
 7 | | ¬P               3-6 ¬I
 8 | | ¬P ∨ Q           7 ∨I
 9 | | ¬(¬P ∨ Q)        2 R
10 | ¬P ∨ Q             2-9 ¬E
```

1.3 SOUNDNESS AND COMPLETENESS

The following are standard arguments. Cases that are omitted are like ones worked, and so left to the reader.

THEOREM 1.1 *NCL is sound: If* $\Gamma \vdash_{NCL} A$ *then* $\Gamma \vDash_{CL} A$.

L1.1 If $\Gamma \subseteq \Gamma'$ and $\Gamma \vDash_{CL} P$, then $\Gamma' \vDash_{CL} P$.

Suppose $\Gamma \subseteq \Gamma'$ and $\Gamma \vDash_{CL} P$, but $\Gamma' \nvDash_{CL} P$. From the latter, by VCL, there is some v such that $v(\Gamma') = 1$ but $v(P) = 0$. But since $v(\Gamma') = 1$ and $\Gamma \subseteq \Gamma'$, $v(\Gamma) = 1$; so v is a *CL* interpretation such that $v(\Gamma) = 1$ but $v(P) = 0$; so by VCL, $\Gamma \nvDash_{CL} P$. This is impossible; reject the assumption: if $\Gamma \subseteq \Gamma'$ and $\Gamma \vDash_{CL} P$, then $\Gamma' \vDash_{CL} P$.

Main result: For each line in a derivation let A_i be the formula on line i and set Γ_i equal to the set of all premises and assumptions whose scope includes line i. Suppose $\Gamma \vdash_{NCL} A$. Then there is a derivation of A from premises in Γ where A appears under the scope of the premises alone. By induction on line number of this derivation, we show that for each line i of this derivation, $\Gamma_i \vDash_{CL} A_i$. The case when $A_i = A$ is the desired result.

Basis: A_1 is a premise or an assumption. Then $\Gamma_1 = \{A_1\}$; so $v(\Gamma_1) = 1$ iff $v(A_1) = 1$; so there is no v such that $v(\Gamma_1) = 1$ but $v(A_1) = 0$. So by VCL, $\Gamma_1 \vDash_{CL} A_1$.

Assp: For any $i, 1 \leqslant i < k, \Gamma_i \vDash_{CL} A_i$.

Show: $\Gamma_k \vDash_{CL} A_k$.

A_k is either a premise, an assumption, or arises from previous lines by R, $\supset$I, $\supset$E, $\wedge$I, $\wedge$E, $\neg$I, $\neg$E, $\vee$I, $\vee$E, $\equiv$I or $\equiv$E. If A_k is a premise or an assumption, then as in the basis, $\Gamma_k \vDash_{CL} A_k$. So suppose A_k arises by one of the rules.

(R)

($\supset$I) If A_k arises by $\supset$I, then the picture is like this,

$$
\begin{array}{r|l|l}
 & & P \\ \cline{3-3}
 & & \\
j & & Q \\
k & \multicolumn{2}{l}{P \supset Q}
\end{array}
$$

where $j < k$ and A_k is $P \supset Q$. By assumption, $\Gamma_j \vDash_{CL} Q$; and by the nature of access, $\Gamma_j \subseteq \Gamma_k \cup \{P\}$; so by L1.1, $\Gamma_k \cup \{P\} \vDash_{CL} Q$. Suppose $\Gamma_k \nvDash_{CL} P \supset Q$; then by VCL, there is some v such that $v(\Gamma_k) = 1$ but $v(P \supset Q) = 0$; from the latter, by TCL($\supset$), $v(P) = 1$ and $v(Q) = 0$; so $v(\Gamma_k) = 1$ and $v(P) = 1$; so $v(\Gamma_k \cup \{P\}) = 1$; so by VCL, $v(Q) = 1$. This is impossible; reject the assumption: $\Gamma_k \vDash_{CL} P \supset Q$, which is to say, $\Gamma_k \vDash_{CL} A_k$.

($\supset$E) If A_k arises by $\supset$E, then the picture is like this,

$$
\begin{array}{r|l}
i & P \supset Q \\
j & P \\
 & \\
k & Q
\end{array}
$$

where $i, j < k$ and A_k is Q. By assumption, $\Gamma_i \vDash_{CL} P \supset Q$ and $\Gamma_j \vDash_{CL} P$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L1.1, $\Gamma_k \vDash_{CL} P \supset Q$ and $\Gamma_k \vDash_{CL} P$. Suppose $\Gamma_k \nvDash_{CL} Q$; then by VCL, there is some v such that $v(\Gamma_k) = 1$ but $v(Q) = 0$; since $v(\Gamma_k) = 1$, by VCL, $v(P \supset Q) = 1$ and $v(P) = 1$; from the former, by TCL($\supset$), $v(P) = 0$ or $v(Q) = 1$; so $v(Q) = 1$. This is impossible; reject the assumption: $\Gamma_k \vDash_{CL} Q$, which is to say, $\Gamma_k \vDash_{CL} A_k$.

($\wedge$I)

($\wedge$E)

(¬I) If A_k arises by ¬I, then the picture is like this,

```
  | | P
  | |--
  | |
i | | Q
j | | ¬Q
k | ¬P
```

where $i, j < k$ and A_k is $\neg P$. By assumption, $\Gamma_i \models_{CL} Q$ and $\Gamma_j \models_{CL} \neg Q$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{P\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{P\}$; so by L1.1, $\Gamma_k \cup \{P\} \models_{CL} Q$ and $\Gamma_k \cup \{P\} \models_{CL} \neg Q$. Suppose $\Gamma_k \not\models_{CL} \neg P$; then by VCL, there is some v such that $v(\Gamma_k) = 1$ but $v(\neg P) = 0$; from the latter, by TCL(¬), $v(P) = 1$; so $v(\Gamma_k) = 1$ and $v(P) = 1$; so $v(\Gamma_k \cup \{P\}) = 1$; so by VCL, $v(Q) = 1$ and $v(\neg Q) = 1$; from the latter, by TCL(¬), $v(Q) = 0$. This is impossible; reject the assumption: $\Gamma_k \models_{CL} \neg P$, which is to say, $\Gamma_k \models_{CL} A_k$.

(¬E)

(∨I) If A_k arises by ∨I, then the picture is like this,

```
j | P                  j | P
  |          or          |
k | P ∨ Q              k | Q ∨ P
```

where $j < k$ and A_k is $P \vee Q$ or $Q \vee P$. Consider the first case. By assumption, $\Gamma_j \models_{CL} P$; but by the nature of access, $\Gamma_j \subseteq \Gamma_k$; so by L1.1, $\Gamma_k \models_{CL} P$. Suppose $\Gamma_k \not\models_{CL} P \vee Q$; then by VCL, there is some v such that $v(\Gamma_k) = 1$ but $v(P \vee Q) = 0$; since $v(\Gamma_k) = 1$, by VCL, $v(P) = 1$; but since $v(P \vee Q) = 0$, by TCL(∨), $v(P) = 0$ and $v(Q) = 0$. This is impossible; reject the assumption: $\Gamma_k \models_{CL} P \vee Q$, which is to say, $\Gamma_k \models_{CL} A_k$. And similarly when A_k is $Q \vee P$.

(∨E) If A_k arises by ∨E, then the picture is like this,

```
h | P ∨ Q
  | | P
  | |--
  | |
i | | R
  |
  | | Q
  | |--
  | |
j | | R
k | R
```

where $h, i, j < k$ and A_k is R. By assumption, $\Gamma_h \models_{CL} P \vee Q$, $\Gamma_i \models_{CL} R$ and $\Gamma_j \models_{CL} R$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k \cup \{P\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{Q\}$; so by L1.1, $\Gamma_k \models_{CL} P \vee Q$, $\Gamma_k \cup \{P\} \models_{CL} R$ and $\Gamma_k \cup \{Q\} \models_{CL} R$. Suppose $\Gamma_k \not\models_{CL} R$; then by VCL, there is some v such that $v(\Gamma_k) = 1$

but $v(R) = 0$. Since $v(\Gamma_k) = 1$, by VCL, $v(P \vee Q) = 1$; so by TCL($\vee$), $v(P) = 1$ or $v(Q) = 1$. Suppose, for the moment, that $v(P) = 1$; then $v(\Gamma_k) = 1$ and $v(P) = 1$; so $v(\Gamma_k \cup \{P\}) = 1$; so by VCL, $v(R) = 1$; this is impossible; reject the assumption: $v(P) \neq 1$; so $v(Q) = 1$; so $v(\Gamma_k) = 1$ and $v(Q) = 1$; so $v(\Gamma_k \cup \{Q\}) = 1$; so by VCL, $v(R) = 1$; this is impossible; reject the assumption: $\Gamma_k \vDash_{CL} R$, which is to say, $\Gamma_k \vDash_{CL} A_k$.

($\equiv$I)

($\equiv$E)

For any i, $\Gamma_i \vDash_{CL} A_i$.

THEOREM 1.2 *NCL is complete: if* $\Gamma \vDash_{CL} A$ *then* $\Gamma \vdash_{NCL} A$.

CON Γ is CONSISTENT iff there is no A such that $\Gamma \vdash_{NCL} A$ and $\Gamma \vdash_{NCL} \neg A$.

L1.2 If $\Gamma \nvdash_{NCL} \neg P$, then $\Gamma \cup \{P\}$ is consistent.

Suppose $\Gamma \nvdash_{NCL} \neg P$ but $\Gamma \cup \{P\}$ is inconsistent. Then there is some A such that $\Gamma \cup \{P\} \vdash_{NCL} A$ and $\Gamma \cup \{P\} \vdash_{NCL} \neg A$. But then we can argue,

1	Γ	
2	P	A (c, $\neg$I)
3	A	from $\Gamma \cup \{P\}$
4	$\neg A$	from $\Gamma \cup \{P\}$
5	$\neg P$	2-4 $\neg$I

So $\Gamma \vdash_{NCL} \neg P$. But this is impossible; reject the assumption: if $\Gamma \nvdash_{NCL} \neg P$, then $\Gamma \cup \{P\}$ is consistent.

L1.3 There is an enumeration of all the formulas, $A_1, A_2 \ldots$

Proof by construction in the usual way.[1]

MAX Γ is MAXIMAL iff for any A either $\Gamma \vdash_{NCL} A$ or $\Gamma \vdash_{NCL} \neg A$.

C(Γ') We construct a Γ' from Γ as follows. Set $\Omega_0 = \Gamma$. By L1.3, there is an enumeration, $A_1, A_2 \ldots$ of all the formulas; for any A_i in this series set,

$$\Omega_i = \Omega_{i-1} \quad \text{if } \Omega_{i-1} \vdash_{NCL} \neg A_i$$
$$\Omega_i = \Omega_{i-1} \cup \{A_i\} \quad \text{if } \Omega_{i-1} \nvdash_{NCL} \neg A_i$$

then

$$\Gamma' = \bigcup_{i \geqslant 0} \Omega_i$$

[1]For this, and extended discussion of the larger argument, see e.g. [12, §11.2].

L1.4 Γ' is maximal.

Suppose Γ' is not maximal. Then there is some A_i such that $\Gamma' \nvdash_{NCL} A_i$ and $\Gamma' \nvdash_{NCL} \neg A_i$. Whatever i may be, each member of Ω_{i-1} is in Γ'; so if $\Omega_{i-1} \vdash_{NCL} \neg A_i$ then $\Gamma' \vdash_{NCL} \neg A_i$; but $\Gamma' \nvdash_{NCL} \neg A_i$; so $\Omega_{i-1} \nvdash_{NCL} \neg A_i$; so by construction, $\Omega_i = \Omega_{i-1} \cup \{A_i\}$; so by construction, $A_i \in \Gamma'$; so $\Gamma' \vdash_{NCL} A_i$. This is impossible; reject the assumption: Γ' is maximal.

L1.5 If Γ is consistent, then each Ω_i is consistent.

Suppose Γ is consistent.

Basis: $\Omega_0 = \Gamma$ and Γ is consistent; so Ω_0 is consistent.

Assp: For any $i, 0 \leqslant i < k$, Ω_i is consistent.

Show: Ω_k is consistent.

Ω_k is either Ω_{k-1} or $\Omega_{k-1} \cup \{A_k\}$. Suppose the former; by assumption, Ω_{k-1} is consistent; so Ω_k is consistent. Suppose the latter; then by construction, $\Omega_{k-1} \nvdash_{NCL} \neg A_k$; so by L1.2, $\Omega_{k-1} \cup \{A_k\}$ is consistent; so Ω_k is consistent.

For any i, Ω_i is consistent.

L1.6 If Γ is consistent, then Γ' is consistent.

Suppose Γ is consistent, but Γ' is not; from the latter, there is some P such that $\Gamma' \vdash_{NCL} P$ and $\Gamma' \vdash_{NCL} \neg P$. Consider derivations D1 and D2 of these results and the premises $A_i \dots A_j$ of these derivations. Where A_j is the last of these premises in the enumeration of formulas, by the construction of Γ', each of $A_i \dots A_j$ must be a member of Ω_j; so D1 and D2 are derivations from Ω_j; so Ω_j is not consistent. But since Γ is consistent, by L1.5, Ω_j is consistent. This is impossible; reject the assumption: if Γ is consistent then Γ' is consistent.

C(v) We construct a *CL* interpretation v based on Γ' as follows. For any parameter p, set $v(p) = 1$ iff $\Gamma' \vdash_{NCL} p$.

L1.7 If Γ is consistent then for any A, $v(A) = 1$ iff $\Gamma' \vdash_{NCL} A$.

Suppose Γ is consistent. By L1.4, Γ' is maximal; by L1.6, Γ' is consistent. Now by induction on the number of operators in A,

Basis: If A has no operators, then it is a parameter p and by construction, $v(p) = 1$ iff $\Gamma' \vdash_{NCL} p$. So $v(A) = 1$ iff $\Gamma' \vdash_{NCL} A$.

Assp: For any i, $0 \leqslant i < k$, if A has i operators, then $v(A) = 1$ iff $\Gamma' \vdash_{NCL} A$.

Show: If A has k operators, then $v(A) = 1$ iff $\Gamma' \vdash_{NCL} A$.

If A has k operators, then it is of the form $\neg P$, $P \supset Q$, $P \wedge Q$, $P \vee Q$ or $P \equiv Q$ where P and Q have $< k$ operators.

(¬) A is $\neg P$. (i) Suppose $v(A) = 1$; then $v(\neg P) = 1$; so by TCL(¬), $v(P) = 0$; so by assumption, $\Gamma' \nvdash_{NCL} P$; so by maximality, $\Gamma' \vdash_{NCL} \neg P$, where this is to say, $\Gamma' \vdash_{NCL} A$. (ii) Suppose $\Gamma' \vdash_{NCL} A$; then $\Gamma' \vdash_{NCL} \neg P$; so by consistency, $\Gamma' \nvdash_{NCL} P$; so by assumption, $v(P) = 0$; so by TCL(¬), $v(\neg P) = 1$, where this is to say, $v(A) = 1$. So $v(A) = 1$ iff $\Gamma' \vdash_{NCL} A$.

(⊃) A is $P \supset Q$. (i) Suppose $v(A) = 1$ but $\Gamma' \nvdash_{NCL} A$; then $v(P \supset Q) = 1$ but $\Gamma' \nvdash_{NCL} P \supset Q$. From the latter, by maximality, $\Gamma' \vdash_{NCL} \neg(P \supset Q)$; from this it follows, by simple derivations, that $\Gamma' \vdash_{NCL} P$ and $\Gamma' \vdash_{NCL} \neg Q$; so by consistency, $\Gamma' \nvdash_{NCL} Q$; so by assumption, $v(P) = 1$ and $v(Q) = 0$; so by TCL(⊃), $v(P \supset Q) = 0$. This is impossible; reject the assumption: if $v(A) = 1$ then $\Gamma' \vdash_{NCL} A$.

(ii) Suppose $\Gamma' \vdash_{NCL} A$ but $v(A) = 0$; then $\Gamma' \vdash_{NCL} P \supset Q$ but $v(P \supset Q) = 0$. From the latter, by TCL(⊃), $v(P) = 1$ and $v(Q) = 0$; so by assumption, $\Gamma' \vdash_{NCL} P$ and $\Gamma' \nvdash_{NCL} Q$; but since $\Gamma' \vdash_{NCL} P \supset Q$ and $\Gamma' \vdash_{NCL} P$, by (⊃E), $\Gamma' \vdash_{NCL} Q$. This is impossible; reject the assumption: if $\Gamma' \vdash_{NCL} A$, then $v(A) = 1$. So $v(A) = 1$ iff $\Gamma' \vdash_{NCL} A$.

(∧)

(∨)

(≡)

For any A, $v(A) = 1$ iff $\Gamma' \vdash_{NCL} A$.

L1.8 If Γ is consistent, then $v(\Gamma) = 1$.

Suppose Γ is consistent and $A \in \Gamma$; then by construction, $A \in \Gamma'$; so $\Gamma' \vdash_{NCL} A$; so since Γ is consistent, by L1.7, $v(A) = 1$. And similarly for any $A \in \Gamma$. So $v(\Gamma) = 1$.

Main result: Suppose $\Gamma \models_{CL} A$ but $\Gamma \nvdash_{NCL} A$. By (DN), if $\Gamma \vdash_{NCL} \neg\neg A$, then $\Gamma \vdash_{NCL} A$; so $\Gamma \nvdash_{NCL} \neg\neg A$; so by L1.2, $\Gamma \cup \{\neg A\}$ is consistent; so by L1.8, there is a v constructed as above such that $v(\Gamma \cup \{\neg A\}) = 1$; so $v(\neg A) = 1$; so by TCL(¬), $v(A) = 0$; so $v(\Gamma) = 1$ and $v(A) = 0$; so by VCL, $\Gamma \not\models_{CL} A$. This is impossible; reject the assumption: if $\Gamma \models_{CL} A$, then $\Gamma \vdash_{NCL} A$.

2 NORMAL MODAL LOGICS: $K\alpha$ (CH. 2,3)

2.1 LANGUAGE / SEMANTIC NOTIONS

LKα The VOCABULARY consists of propositional parameters $p_0, p_1 \ldots$ with the operators, ¬, ∧, ∨, ⊃, ≡, □ and ◇. Each propositional parameter is a FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$, $(A \equiv B)$, $\Box A$ and $\Diamond A$.

IKα For any of these systems except Kυ, an INTERPRETATION is a triple $\langle W, R, v\rangle$ where W is a set of worlds, R is a subset of $W^2 = W \times W$, and v is a function such that for any $w \in W$ and p, $v_w(p) = 1$ or $v_w(p) = 0$. For $x, y, z \in W$, where α is empty or indicates some combination of the following constraints,

η	For any x, there is a y such that xRy	extendability
ρ	for all x, xRx	reflexivity
σ	for all x, y, if xRy then yRx	symmetry
τ	for all x, y, z, if xRy and yRz then xRz	transitivity

$\langle W, R, v\rangle$ is a Kα interpretation when R meets the constraints from α. For Kυ, a model is just a pair $\langle W, v\rangle$.

TK For complex expressions,

($\neg$) $v_w(\neg A) = 1$ if $v_w(A) = 0$, and 0 otherwise.

($\wedge$) $v_w(A \wedge B) = 1$ if $v_w(A) = 1$ and $v_w(B) = 1$, and 0 otherwise.

($\vee$) $v_w(A \vee B) = 1$ if $v_w(A) = 1$ or $v_w(B) = 1$, and 0 otherwise.

($\supset$) $v_w(A \supset B) = 1$ if $v_w(A) = 0$ or $v_w(B) = 1$, and 0 otherwise.

($\equiv$) $v_w(A \equiv B) = 1$ if $v_w(A) = v_w(B)$, and 0 otherwise.

($\Diamond$) $v_w(\Diamond A) = 1$ if some $x \in W$ such that wRx has $v_x(A) = 1$, and 0 otherwise.

($\Box$) $v_w(\Box A) = 1$ if all $x \in W$ such that wRx have $v_x(A) = 1$, and 0 otherwise.

For Kυ, substitute for ($\Diamond$) and ($\Box$),

$(\Diamond)_\upsilon$ $v_w(\Diamond A) = 1$ iff for some $x \in W$, $v_x(A) = 1$.

$(\Box)_\upsilon$ $v_w(\Box A) = 1$ iff for all $x \in W$, $v_x(A) = 1$.

For a set Γ of formulas, $v_w(\Gamma) = 1$ iff $v_w(A) = 1$ for each $A \in \Gamma$; then,

VKα $\Gamma \models_{K\alpha} A$ iff there is no Kα interpretation $\langle W, R, v\rangle$ ($\langle W, v\rangle$) and $w \in W$ such that $v_w(\Gamma) = 1$ and $v_w(A) = 0$.

2.2 NATURAL DERIVATIONS: *NK*α

Where s is any integer, let A_s be a SUBSCRIPTED FORMULA. For subscripts s and t allow also expressions of the sort, $s.t$. As in Priest, intuitively, subscripts indicate worlds, where A_s is true or false at world s, and $s.t$ just in case world s has access to world t. Derivation rules apply to these expressions. Rules for $\neg$, $\wedge$, $\vee$, $\supset$, and $\equiv$ are like ones from before, but with consistent subscripts. Rules for $\Box$ and $\Diamond$ are new.[2]

[2] There is no uniformity about how to do natural deduction in modal logic. Most avoid subscripts altogether. Another option uses subscripts of the sort $i.j \ldots k$ (cf. prefixes on tableaux in [2]); the result is elegant, but not so flexible as this account inspired by Priest, and we will need the flexibility, as we approach increasingly complex systems.

R
P_s
P_s

$\neg$I
| P_s
| Q_t
| $\neg Q_t$
$\neg P_s$

$\neg$E
| $\neg P_s$
| Q_t
| $\neg Q_t$
P_s

$\wedge$I
P_s
Q_s
$(P \wedge Q)_s$

$\wedge$E
$(P \wedge Q)_s$
P_s

$\wedge$E
$(P \wedge Q)_s$
Q_s

$\vee$I
P_s
$(P \vee Q)_s$

$\vee$I
P_s
$(Q \vee P)_s$

$\vee$E
$(P \vee Q)_s$
| P_s
| R_t
| Q_s
| R_t
R_t

$\supset$I
| P_s
| Q_s
$(P \supset Q)_s$

$\supset$E
$(P \supset Q)_s$
P_s
Q_s

$\equiv$I
| P_s
| Q_s
| Q_s
| P_s
$(P \equiv Q)_s$

$\equiv$E
$(P \equiv Q)_s$
P_s
Q_s

$\equiv$E
$(P \equiv Q)_s$
Q_s
P_s

$\Box$I
| $s.t$
| P_t
$\Box P_s$
where t does not appear in any undischarged premise or assumption

$\Box$E
$\Box P_s$
$s.t$
P_t

$\Diamond$I
P_t
$s.t$
$\Diamond P_s$

$\Diamond$E
$\Diamond P_s$
| $s.t$
| P_t
| Q_u
Q_u
where t does not appear in any undischarged premise or assumption and is not u

These are the rules of *NK*. Other systems *NK*α add from the following, for *access manipulation*, according to constraints in α.

AMη
| $s.t$
| P_u
P_u
where t does not appear in any undischarged premise or assumption and is not u

AMρ
$s.s$

AMσ
$s.t$
$t.s$

AMτ
$s.t$
$t.u$
$s.u$

AMρ has no premise. For *NK*υ, eliminate expressions of the sort s.t and rules for access manipulation. Let $\top$ be an arbitrary tautology (say, $p \supset p$). Then for $\Box$I, $\Box$E, $\Diamond$I and $\Diamond$E, substitute,

$\Box$Iυ	$\mid\ \top_t$ (assumption) … $\mid\ P_t$; $\Box P_s$	where t does not appear in any undischarged premise or assumption	
$\Box$Eυ	$\Box P_s$; P_t		
$\Diamond$Iυ	P_t ; $\Diamond P_s$		
$\Diamond$Eυ	$\Diamond P_s$; $\mid\ P_t$ (assumption) … $\mid\ Q_u$; Q_u	where t does not appear in any undischarged premise or assumption and is not u	

In these systems, every subscript is 0, appears in a premise, or appears in the t-place of an accessible assumption for $\Box$I, $\Diamond$E, ($\Box$Iυ, $\Diamond$Eυ) or AMη. Where Γ is a set of unsubscripted formulas, let Γ_0 be those same formulas, each with subscript 0. Then,

NKα $\Gamma \vdash_{NK\alpha} A$ iff there is an *NK*α derivation of A_0 from the members of Γ_0.

Derived rules carry over from *NCL* as one would expect, with subscripts constant throughout. Thus, e.g.,

MT: $(P \supset Q)_s$, $\neg Q_s$; $\neg P_s$

Impl: $(P \supset Q)_s \;\lhd\rhd\; (\neg P \vee Q)_s$
$(\neg P \supset Q)_s \;\lhd\rhd\; (P \vee Q)_s$

As examples, here are some derivations which exhibit left-hand forms of the following additional rule for *modal negation*,

MN: $\Box P_s \;\lhd\rhd\; \neg\Diamond\neg P_s$ $\quad \neg\Box P_s \;\lhd\rhd\; \Diamond\neg P_s$
$\Diamond P_s \;\lhd\rhd\; \neg\Box\neg P_s$ $\quad \neg\Diamond P_s \;\lhd\rhd\; \Box\neg P_s$

as derived in *NK* (and so any *NK*α, excluding *NK*υ, though this could be easily demonstrated as well).

$\neg\Diamond\neg P \vdash_{NK} \Box P$

1	$\neg\Diamond\neg P_0$	P
2	$\mid\ 0.1$	A (*g*, $\Box$I)
3	$\mid\ \mid\ \neg P_1$	A (*c*, $\neg$E)
4	$\mid\ \mid\ \Diamond\neg P_0$	2,3 $\Diamond$I
5	$\mid\ \mid\ \neg\Diamond\neg P_0$	1 R
6	$\mid\ P_1$	3-5 $\neg$E
7	$\Box P_0$	2-6 $\Box$I

$\Box P \vdash_{NK} \neg\Diamond\neg P$

1	$\Box P_0$	P
2	$\mid\ \Diamond\neg P_0$	A (*c*, $\neg$I)
3	$\mid\ \mid\ 0.1$	A (*g*, 2 $\Diamond$E)
4	$\mid\ \mid\ \neg P_1$	
5	$\mid\ \mid\ \mid\ \Diamond\neg P_0$	A (*c*, $\neg$I)
6	$\mid\ \mid\ \mid\ \neg P_1$	4 R
7	$\mid\ \mid\ \mid\ P_1$	1,3 $\Box$E
8	$\mid\ \mid\ \neg\Diamond\neg P_0$	5-7 $\neg$I
9	$\mid\ \neg\Diamond\neg P_0$	2,3-8 $\Diamond$E
10	$\mid\ \Diamond\neg P_0$	2 R
11	$\neg\Diamond\neg P_0$	2-10 $\neg$I

$\neg\Box\neg P \vdash_{NK} \Diamond P$

1	$\neg\Box\neg P_0$	P
2	$\neg\Diamond P_0$	A (c, ¬E)
3	0.1	A (g, □I)
4	P_1	A (c, ¬I)
5	$\Diamond P_0$	3,4 ◇I
6	$\neg\Diamond P_0$	2 R
7	$\neg P_1$	4-6 ¬I
8	$\Box\neg P_0$	3-7 □I
9	$\neg\Box\neg P_0$	1 R
10	$\Diamond P_0$	2-9 ¬E

$\Diamond P \vdash_{NK} \neg\Box\neg P$

1	$\Diamond P_0$	P
2	0.1	A (g, 1 ◇E)
3	P_1	
4	$\Box\neg P_0$	A (c, ¬I)
5	$\neg P_1$	2,4 □E
6	P_1	3 R
7	$\neg\Box\neg P_0$	4-6 ¬I
8	$\neg\Box\neg P_0$	1,2-7 ◇E

For examples in other systems, here are demonstrations of some characteristic principles:

$\vdash_{NK\eta} \Box P \supset \Diamond P$

1	$\Box P_0$	A (g, ⊃I)
2	0.1	A (g, AMη)
3	P_1	1,2 □E
4	$\Diamond P_0$	2,3 ◇I
5	$\Diamond P_0$	2-4 AMη
6	$(\Box P \supset \Diamond P)_0$	1-5 ⊃I

$\vdash_{NK\rho} \Box P \supset P$

1	$\Box P_0$	A (g, ⊃I)
2	0.0	AMρ
3	P_0	1,2 □E
4	$(\Box P \supset P)_0$	1-3 ⊃I

$\vdash_{NK\sigma} P \supset \Box\Diamond P$

1	P_0	A (g, ⊃I)
2	0.1	A (g, □I)
3	1.0	2 AMσ
4	$\Diamond P_1$	1,3 ◇I
5	$\Box\Diamond P_0$	2-4 □I
6	$(P \supset \Box\Diamond P)_0$	1-5 ⊃I

$\vdash_{NK\tau} \Box P \supset \Box\Box P$

1	$\Box P_0$	A (g, ⊃I)
2	0.1	A (g, □I)
3	1.2	A (g, □I)
4	0.2	2,3 AMτ
5	P_2	1,4 □E
6	$\Box P_1$	3-5 □I
7	$\Box\Box P_0$	2-6 □I
8	$(\Box P \supset \Box\Box P)_0$	1-7 ⊃I

$\vdash_{NK\sigma\tau} \Diamond P \supset \Box\Diamond P$

1	$\Diamond P_0$	A (g, $\supset$I)
2	0.1	A (g, 1 $\Diamond$E)
3	P_1	
4	0.2	A (g, $\Box$I)
5	2.0	4 AMσ
6	2.1	5,2 AMτ
7	$\Diamond P_2$	3,6 $\Diamond$I
8	$\Box\Diamond P_0$	4-7 $\Box$I
9	$\Box\Diamond P_0$	1,2-8 $\Diamond$E
10	$(\Diamond P \supset \Box\Diamond P)_0$	1-9 $\supset$I

$\vdash_{NK\upsilon} \Diamond P \supset \Box\Diamond P$

1	$\Diamond P_0$	A (g, $\supset$I)
2	P_1	A (g, 1 $\Diamond$E)
3	$\top_2$	A (g, $\Box$I)
4	$\Diamond P_2$	2 $\Diamond$I
5	$\Box\Diamond P_0$	3-4 $\Box$I
6	$\Box\Diamond P_0$	1,2-5 $\Diamond$E
7	$(\Diamond P \supset \Box\Diamond P)_0$	1-6 $\supset$I

2.3 SOUNDNESS AND COMPLETENESS

Preliminaries (excluding *NK*υ): Begin with generalized notions of validity. For a model $\langle W, R, v\rangle$, let m be a map from subscripts into W. Say $\langle W, R, v\rangle_m$ is $\langle W, R, v\rangle$ ***with*** map m. Then, where Γ is a set of expressions of our language for derivations, $v_m(\Gamma) = 1$ iff for each $A_s \in \Gamma$, $v_{m(s)}(A) = 1$, and for each $s.t \in \Gamma$, $\langle m(s), m(t)\rangle \in R$. Now expand notions of validity to include subscripted formulas, and alternate expressions as indicated in double brackets.

VKα* $\Gamma \models^*_{K\alpha} A_s$ $[\![s.t]\!]$ iff there is no $K\alpha$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma) = 1$ but $v_{m(s)}(A) = 0$ $[\![\langle m(s), m(t)\rangle \notin R]\!]$.

NKα* $\Gamma \vdash^*_{NK\alpha} A_s$ $[\![s.t]\!]$ iff there is an *NK*α derivation of A_s $[\![s.t]\!]$ from the members of Γ.

These notions reduce to the standard ones when all the members of Γ and A have subscript 0 (and so do not include expressions of the sort $s.t$). This is obvious for NKα*. In the other case, there is a $\langle W, R, v\rangle_m$ that makes all the members of Γ_0 true and A_0 false just in case there *is* a world in $\langle W, R, v\rangle$ that makes the unsubscripted members of Γ true and A false. For the following, cases omitted are like ones worked, and so left to the reader.

THEOREM 2.1 *NK*α *is sound: If* $\Gamma \vdash_{NK\alpha} A$ ***then*** $\Gamma \models_{K\alpha} A$.

L2.1 If $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{K\alpha} P_s$ $[\![s.t]\!]$, then $\Gamma' \models^*_{K\alpha} P_s$ $[\![s.t]\!]$.

Suppose $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{K\alpha} P_s$ $[\![s.t]\!]$, but $\Gamma' \not\models^*_{K\alpha} P_s$ $[\![s.t]\!]$. From the latter, by VKα*, there is some $K\alpha$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma') = 1$ but $v_{m(s)}(P) = 0$ $[\![\langle m(s), m(t)\rangle \notin R]\!]$. But since $v_m(\Gamma') = 1$ and $\Gamma \subseteq \Gamma'$, $v_m(\Gamma) = 1$; so $v_m(\Gamma) = 1$ but $v_{m(s)}(P) = 0$ $[\![\langle m(s), m(t)\rangle \notin R]\!]$; so by VK$\alpha$*, $\Gamma \not\models^*_{K\alpha} P_s$ $[\![s.t]\!]$. This is impossible; reject the assumption: if $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{K\alpha} P_s$ $[\![s.t]\!]$, then $\Gamma' \models^*_{K\alpha} P_s$ $[\![s.t]\!]$.

Main result: For each line in a derivation let $\mathcal{P}_i$ be the expression on line i and Γ_i be the set of all premises and assumptions whose scope includes line i. We set out to show "generalized" soundness: if $\Gamma \vdash^*_{NK\alpha} \mathcal{P}$ then $\Gamma \models^*_{K\alpha} \mathcal{P}$. As above, this reduces to the standard result when $\mathcal{P}$ and all the members of Γ are formulas with subscript 0. Suppose $\Gamma \vdash^*_{NK\alpha} \mathcal{P}$. Then there is a derivation of $\mathcal{P}$ from premises in Γ where $\mathcal{P}$ appears under the scope of the premises alone. By induction on line number of this derivation, we show that for each line i of this derivation, $\Gamma_i \models^*_{K\alpha} \mathcal{P}_i$. The case when $\mathcal{P}_i = \mathcal{P}$ is the desired result.

Basis: $\mathcal{P}_1$ is a premise or an assumption A_s $[\![s.t]\!]$. Then $\Gamma_1 = \{A_s\}$ $[\![\{s.t\}]\!]$; so for any $\langle W, R, v\rangle_m$, $v_m(\Gamma_1) = 1$ iff $v_{m(s)}(A) = 1$ $[\![\langle m(s), m(t)\rangle \in R]\!]$; so there is no $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_1) = 1$ but $v_{m(s)}(A) = 0$ $[\![\langle m(s), m(t)\rangle \notin R]\!]$. So by VK$\alpha$*, $\Gamma_1 \models^*_{K\alpha} A_s$ $[\![s.t]\!]$, where this is just to say, $\Gamma_1 \models^*_{K\alpha} \mathcal{P}_1$.

Assp: For any $i, 1 \leqslant i < k, \Gamma_i \models^*_{K\alpha} \mathcal{P}_i$.

Show: $\Gamma_k \models^*_{K\alpha} \mathcal{P}_k$.

$\mathcal{P}_k$ is either a premise, an assumption, or arises from previous lines by R, $\supset$I, $\supset$E, $\wedge$I, $\wedge$E, $\neg$I, $\neg$E, $\vee$I, $\vee$E, $\equiv$I, $\equiv$E, $\Box$I, $\Box$E, $\Diamond$I, $\Diamond$E or, depending on the system, AMη, AMρ, AMσ or AMτ. If $\mathcal{P}_k$ is a premise or an assumption, then as in the basis, $\Gamma_k \models^*_{K\alpha} \mathcal{P}_k$. So suppose $\mathcal{P}_k$ arises by one of the rules.

(R)

($\supset$I)

($\supset$E) If $\mathcal{P}_k$ arises by $\supset$E, then the picture is like this,

i	$(A \supset B)_s$
j	A_s
k	B_s

where $i, j < k$ and $\mathcal{P}_k$ is B_s. By assumption, $\Gamma_i \models^*_{K\alpha} (A \supset B)_s$ and $\Gamma_j \models^*_{K\alpha} A_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L2.1, $\Gamma_k \models^*_{K\alpha} (A \supset B)_s$ and $\Gamma_k \models^*_{K\alpha} A_s$. Suppose $\Gamma_k \not\models^*_{K\alpha} B_s$; then by VK$\alpha$*, there is some K$\alpha$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(B) = 0$; since $v_m(\Gamma_k) = 1$, by VKα*, $v_{m(s)}(A \supset B) = 1$ and $v_{m(s)}(A) = 1$; from the former, by TK($\supset$), $v_{m(s)}(A) = 0$ or $v_{m(s)}(B) = 1$; so $v_{m(s)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K\alpha} B_s$, which is to say, $\Gamma_k \models^*_{K\alpha} \mathcal{P}_k$.

($\wedge$I)

($\wedge$E)

(¬I) If $\mathcal{P}_k$ arises by ¬I, then the picture is like this,

$$\begin{array}{l|l} & \quad A_s \\ & \quad \vdots \\ i & \quad B_t \\ j & \quad \neg B_t \\ k & \neg A_s \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $\neg A_s$. By assumption, $\Gamma_i \models^*_{K\alpha} B_t$ and $\Gamma_j \models^*_{K\alpha} \neg B_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{A_s\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{A_s\}$; so by L2.1, $\Gamma_k \cup \{A_s\} \models^*_{K\alpha} B_t$ and $\Gamma_k \cup \{A_s\} \models^*_{K\alpha} \neg B_t$. Suppose $\Gamma_k \not\models^*_{K\alpha} \neg A_s$; then by VK$\alpha$*, there is a K$\alpha$ interpretation $\langle W, R, \nu\rangle_m$ such that $\nu_m(\Gamma_k) = 1$ but $\nu_{m(s)}(\neg A) = 0$; so by TK(¬), $\nu_{m(s)}(A) = 1$; so $\nu_m(\Gamma_k) = 1$ and $\nu_{m(s)}(A) = 1$; so $\nu_m(\Gamma_k \cup \{A_s\}) = 1$; so by VK$\alpha$*, $\nu_{m(t)}(B) = 1$ and $\nu_{m(t)}(\neg B) = 1$; from the latter, by TK(¬), $\nu_{m(t)}(B) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K\alpha} \neg A_s$, which is to say, $\Gamma_k \models^*_{K\alpha} \mathcal{P}_k$.

(¬E)

(∨I)

(∨E) If $\mathcal{P}_k$ arises by ∨E, then the picture is like this,

$$\begin{array}{l|l} h & (A \vee B)_s \\ & \quad A_s \\ & \quad \vdots \\ i & \quad C_t \\ & \quad B_s \\ & \quad \vdots \\ j & \quad C_t \\ k & C_t \end{array}$$

where $h, i, j < k$ and $\mathcal{P}_k$ is C_t. By assumption, $\Gamma_h \models^*_{K\alpha} (A \vee B)_s$, $\Gamma_i \models^*_{K\alpha} C_t$ and $\Gamma_j \models^*_{K\alpha} C_t$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k \cup \{A_s\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{B_s\}$; so by L2.1, $\Gamma_k \models^*_{K\alpha} (A \vee B)_s$, $\Gamma_k \cup \{A_s\} \models^*_{K\alpha} C_t$ and $\Gamma_k \cup \{B_s\} \models^*_{K\alpha} C_t$. Suppose $\Gamma_k \not\models^*_{K\alpha} C_t$; then by VK$\alpha$*, there is some K$\alpha$ interpretation $\langle W, R, \nu\rangle_m$ such that $\nu_m(\Gamma_k) = 1$ but $\nu_{m(t)}(C) = 0$. Since $\nu_m(\Gamma_k) = 1$, by VKα*, $\nu_{m(s)}(A \vee B) = 1$; so by TK(∨), $\nu_{m(s)}(A) = 1$ or $\nu_{m(s)}(B) = 1$. Suppose, for the moment, that $\nu_{m(s)}(A) = 1$; then $\nu_m(\Gamma_k) = 1$ and $\nu_{m(s)}(A) = 1$; so $\nu_m(\Gamma_k \cup \{A_s\}) = 1$; so by VK$\alpha$*, $\nu_{m(t)}(C) = 1$; this is impossible; reject the assumption: $\nu_{m(s)}(A) \neq 1$; so $\nu_{m(s)}(B) = 1$; so $\nu_m(\Gamma_k) = 1$ and $\nu_{m(s)}(B) = 1$; so $\nu_m(\Gamma_k \cup \{B_s\}) = 1$; so by VK$\alpha$*, $\nu_{m(t)}(C) = 1$; this is impossible; reject the assumption: $\Gamma_k \models^*_{K\alpha} C_t$, which is to say, $\Gamma_k \models^*_{K\alpha} \mathcal{P}_k$.

(≡I)

(≡E)

($\Box$I) If $\mathcal{P}_k$ arises by $\Box$I, then the picture is like this,

$$\begin{array}{r|l} & \begin{array}{|l} s.t \\ \hline \\ \\ A_t \end{array} \\ i & \\ k & \Box A_s \end{array}$$

where $i < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $\Box A_s$. By assumption, $\Gamma_i \models^*_{K\alpha} A_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{s.t\}$; so by L2.1, $\Gamma_k \cup \{s.t\} \models^*_{K\alpha} A_t$. Suppose $\Gamma_k \not\models^*_{K\alpha} \Box A_s$; then by VK$\alpha$*, there is a $K\alpha$ interpretation $\langle W, R, \nu\rangle_m$ such that $\nu_m(\Gamma_k) = 1$ but $\nu_{m(s)}(\Box A) = 0$; so by TK($\Box$), there is some $w \in W$ such that $m(s)Rw$ and $\nu_w(A) = 0$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, R, \nu\rangle_{m'}$; since t does not appear in Γ_k, it remains that $\nu_{m'}(\Gamma_k) = 1$; and since $m'(t) = w$ and $m'(s) = m(s)$, $\langle m'(s), m'(t)\rangle \in R$; so $\nu_{m'}(\Gamma_k \cup \{s.t\}) = 1$; so by VK$\alpha$*, $\nu_{m'(t)}(A) = 1$. But $m'(t) = w$; so $\nu_w(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K\alpha} \Box A_s$, which is to say, $\Gamma_k \models^*_{K\alpha} \mathcal{P}_k$.

($\Box$E) If $\mathcal{P}_k$ arises by $\Box$E, then the picture is like this,

$$\begin{array}{r|l} i & \Box A_s \\ j & s.t \\ & \\ k & A_t \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is A_t. By assumption, $\Gamma_i \models^*_{K\alpha} \Box A_s$ and $\Gamma_j \models^*_{K\alpha} s.t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L2.1, $\Gamma_k \models^*_{K\alpha} \Box A_s$ and $\Gamma_k \models^*_{K\alpha} s.t$. Suppose $\Gamma_k \not\models^*_{K\alpha} A_t$; then by VK$\alpha$*, there is some $K\alpha$ interpretation $\langle W, R, \nu\rangle_m$ such that $\nu_m(\Gamma_k) = 1$ but $\nu_{m(t)}(A) = 0$; since $\nu_m(\Gamma_k) = 1$, by VKα*, $\nu_{m(s)}(\Box A) = 1$ and $\langle m(s), m(t)\rangle \in R$; from the first of these, by TK($\Box$), any w such that $m(s)Rw$ has $\nu_w(A) = 1$; so $\nu_{m(t)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K\alpha} A_t$, which is to say, $\Gamma_k \models^*_{K\alpha} \mathcal{P}_k$.

($\Diamond$I)

($\Diamond$E) If $\mathcal{P}_k$ arises by $\Diamond$E, then the picture is like this,

$$\begin{array}{r|l} i & \Diamond A_s \\ & \begin{array}{|l} A_t \\ s.t \\ \hline \\ \\ B_u \end{array} \\ j & \\ k & B_u \end{array}$$

where $i, j < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption) and is not u, and $\mathcal{P}_k$ is B_u. By assumption, $\Gamma_i \models^*_{K\alpha} \Diamond A_s$ and $\Gamma_j \models^*_{K\alpha} B_u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k \cup \{A_t, s.t\}$; so by L2.1, $\Gamma_k \models^*_{K\alpha} \Diamond A_s$ and $\Gamma_k \cup \{A_t, s.t\} \models^*_{K\alpha} B_u$. Suppose $\Gamma_k \not\models^*_{K\alpha} B_u$; then by VK$\alpha$*, there is a $K\alpha$ interpretation $\langle W, R, \nu \rangle_m$ such that $\nu_m(\Gamma_k) = 1$ but $\nu_{m(u)}(B) = 0$; since $\nu_m(\Gamma_k) = 1$, by VKα*, $\nu_{m(s)}(\Diamond A) = 1$; so by TK($\Diamond$), there is some $w \in W$ such that $m(s)Rw$ and $\nu_w(A) = 1$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, R, \nu \rangle_{m'}$; since t does not appear in Γ_k, it remains that $\nu_{m'}(\Gamma_k) = 1$; and since $m'(s) = m(s)$ and $m'(t) = w$, $\nu_{m'(t)}(A) = 1$ and $\langle m'(s), m'(t) \rangle \in R$; so $\nu_{m'}(\Gamma_k \cup \{A_t, s.t\}) = 1$; so by VK$\alpha$*, $\nu_{m'(u)}(B) = 1$. But since $t \neq u$, $m'(u) = m(u)$; so $\nu_{m(u)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K\alpha} B_u$, which is to say, $\Gamma_k \models^*_{K\alpha} \mathcal{P}_k$.

(AMη) If $\mathcal{P}_k$ arises by AMη, then the picture is like this,

| | $s.t$
| |
i | | A_u
k | A_u

where $i < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption) and is not u, and $\mathcal{P}_k$ is A_u. Where this rule is included in *NK*α, *K*α includes condition η. By assumption, $\Gamma_i \models^*_{K\alpha} A_u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{s.t\}$; so by L2.1, $\Gamma_k \cup \{s.t\} \models^*_{K\alpha} A_u$. Suppose $\Gamma_k \not\models^*_{K\alpha} A_u$; then by VK$\alpha$*, there is a $K\alpha$ interpretation $\langle W, R, \nu \rangle_m$ such that $\nu_m(\Gamma_k) = 1$ but $\nu_{m(u)}(A) = 0$. By condition η, there is a $w \in W$ such that $m(s)Rw$; consider a map m' like m except that $m'(t) = w$, and consider $\langle W, R, \nu \rangle_{m'}$; since t does not appear in Γ_k, it remains that $\nu_{m'}(\Gamma_k) = 1$; and since $m'(s) = m(s)$ and $m'(t) = w$, $\langle m'(s), m'(t) \rangle \in R$; so $\nu_{m'}(\Gamma_k \cup \{s.t\}) = 1$; so by VK$\alpha$*, $\nu_{m'(u)}(A) = 1$. But since $t \neq u$, $m'(u) = m(u)$; so $\nu_{m(u)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K\alpha} A_u$, which is to say, $\Gamma_k \models^*_{K\alpha} \mathcal{P}_k$.

(AMρ) If $\mathcal{P}_k$ arises by AMρ, then the picture is like this,

k | $s.s$

where $\mathcal{P}_k$ is $s.s$. Where this rule is in *NK*α, *K*α includes condition ρ. Suppose $\Gamma_k \not\models^*_{K\alpha} s.s$; then by VK$\alpha$*, there is some $K\alpha$ interpretation $\langle W, R, \nu \rangle_m$ such that $\nu_m(\Gamma_k) = 1$ but $\langle m(s), m(s) \rangle \notin R$. But by condition ρ, for any $x \in W$, $\langle x, x \rangle \in R$; so $\langle m(s), m(s) \rangle \in R$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K\alpha} s.s$, which is to say, $\Gamma_k \models^*_{K\alpha} \mathcal{P}_k$.

(AMσ) If $\mathcal{P}_k$ arises by AMσ, then the picture is like this,

$$\begin{array}{l|l} i & s.t \\ & \\ k & t.s \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is t.s. Where this rule is in *NK*α, *K*α includes condition σ. By assumption, $\Gamma_i \models^*_{K\alpha}$ s.t; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L2.1, $\Gamma_k \models^*_{K\alpha}$ s.t. Suppose $\Gamma_k \not\models^*_{K\alpha}$ t.s; then by VKα*, there is some Kα interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $\langle m(t), m(s)\rangle \notin R$; since $v_m(\Gamma_k) = 1$, by VKα*, $\langle m(s), m(t)\rangle \in R$; and by condition σ, for any $\langle x, y\rangle \in R$, $\langle y, x\rangle \in R$; so $\langle m(t), m(s)\rangle \in R$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K\alpha}$ t.s, which is to say, $\Gamma_k \models^*_{K\alpha} \mathcal{P}_k$.

(AMτ) If $\mathcal{P}_k$ arises by AMτ, then the picture is like this,

$$\begin{array}{l|l} i & s.t \\ j & t.u \\ & \\ k & s.u \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is s.u. Where this rule is in *NK*α, *K*α includes condition τ. By assumption, $\Gamma_i \models^*_{K\alpha}$ s.t and $\Gamma_j \models^*_{K\alpha}$ t.u; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L2.1, $\Gamma_k \models^*_{K\alpha}$ s.t and $\Gamma_k \models^*_{K\alpha}$ t.u. Suppose $\Gamma_k \not\models^*_{K\alpha}$ s.u; then by VKα*, there is some Kα interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $\langle m(s), m(u)\rangle \notin R$; since $v_m(\Gamma_k) = 1$, by VKα*, $\langle m(s), m(t)\rangle \in R$ and $\langle m(t), m(u)\rangle \in R$; and by condition τ, for any $\langle x, y\rangle, \langle y, z\rangle \in R$, $\langle x, z\rangle \in R$; so $\langle m(s), m(u)\rangle \in R$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K\alpha}$ s.u, which is to say, $\Gamma_k \models^*_{K\alpha} \mathcal{P}_k$.

For any i, $\Gamma_i \models^*_{K\alpha} \mathcal{P}_i$.

The argument for *NK*υ is similar (simpler) and so omitted.

THEOREM 2.2 *NK*α *is complete: if* $\Gamma \models_{K\alpha} A$ *then* $\Gamma \vdash_{NK\alpha} A$.

Suppose $\Gamma \models_{K\alpha} A$; then $\Gamma_0 \models^*_{K\alpha} A_0$; we show that $\Gamma_0 \vdash^*_{NK\alpha} A_0$. Again, this reduces to the standard notion. The method of our proof has advantages (especially for the quantificational case) over standard approaches to completeness for modal logic. Roughly, we construct a single set which is maximal and consistent relative to *subscripted* formulas, and use this to specify the model. The resultant proof is thus kept structurally parallel to the classical case. For the following, fix on some particular constraint(s) α. Then definitions of *consistency* etc. are relative to it.

CON Γ is CONSISTENT iff there is no A_s such that $\Gamma \vdash^*_{NK\alpha} A_s$ and $\Gamma \vdash^*_{NK\alpha} \neg A_s$.

L2.2 If s is 0 or appears in Γ, and $\Gamma \nvdash^*_{NK\alpha} \neg P_s$, then $\Gamma \cup \{P_s\}$ is consistent.

Suppose s is 0 or appears in Γ and $\Gamma \nvdash^{*}_{NK\alpha} \neg P_s$ but $\Gamma \cup \{P_s\}$ is inconsistent. Then there is some A_t such that $\Gamma \cup \{P_s\} \vdash^{*}_{NK\alpha} A_t$ and $\Gamma \cup \{P_s\} \vdash^{*}_{NK\alpha} \neg A_t$. But then we can argue,

1	Γ	
2	$\quad P_s$	A (c, $\neg$I)
3	$\quad A_t$	from $\Gamma \cup \{P_s\}$
4	$\quad \neg A_t$	from $\Gamma \cup \{P_s\}$
5	$\neg P_s$	2-4 $\neg$I

where the assumption is allowed insofar as s is either 0 or appears in Γ; so $\Gamma \vdash^{*}_{NK\alpha} \neg P_s$. But this is impossible; reject the assumption: if s is 0 or introduced in Γ and $\Gamma \nvdash^{*}_{NK\alpha} \neg P_s$, then $\Gamma \cup \{P_s\}$ is consistent.

L2.3 There is an enumeration of all the subscripted formulas, $\mathcal{P}_1\ \mathcal{P}_2 \dots$

Proof by construction: Order non-subscripted formulas $A, B, C\dots$ in the usual way. Then form a grid with formulas $A, B, C\dots$ ordered across the top, and subscripts $1, 2, 3\dots$ down the side.

$$
\begin{array}{ccccc}
A_1 & \rightarrow & B_1 & & C_1 \quad \dots \\
 & \swarrow & & \nearrow & \\
A_2 & & B_2 & & C_2 \\
\downarrow & \nearrow & & & \\
A_3 & & B_3 & & C_3 \\
\vdots & & & &
\end{array}
$$

Order the members of the resultant grid, $A_1, B_1, A_2\dots$ moving along the arrows from the upper left corner, down and to the right.[3]

Max Γ is s-maximal iff for any A_s either $\Gamma \vdash^{*}_{NK\alpha} A_s$ or $\Gamma \vdash^{*}_{NK\alpha} \neg A_s$.

Sgt Γ is a scapegoat set iff for every formula of the form $\neg\Box A_s$, if $\Gamma \vdash^{*}_{NK\alpha} \neg\Box A_s$ then there is some t such that $\Gamma \vdash^{*}_{NK\alpha} s.t$ and $\Gamma \vdash^{*}_{NK\alpha} \neg A_t$.

C(Γ') For Γ with unsubscripted formulas and the corresponding Γ_0, we construct Γ' as follows. Set $\Omega_0 = \Gamma_0$. By L2.3, there is an enumeration, $\mathcal{P}_1, \mathcal{P}_2\dots$ of all the subscripted formulas; let $\mathcal{E}_0$ be this enumeration. Then for the first A_s in $\mathcal{E}_{i-1}$ such that s is 0 or included in Ω_{i-1}, let $\mathcal{E}_i$ be like $\mathcal{E}_{i-1}$ but without A_s, and set,

[3]As for rational numbers; see, e.g., [12, §2.1.1].

$\Omega_i = \Omega_{i-1}$ if $\Omega_{i-1} \vdash^*_{NK\alpha} \neg A_s$
$\Omega_{i^*} = \Omega_{i-1} \cup \{A_s\}$ if $\Omega_{i-1} \nvdash^*_{NK\alpha} \neg A_s$

and

$\Omega_i = \Omega_{i^*}$ if A_s is not of the form $\neg\Box P_s$
$\Omega_i = \Omega_{i^*} \cup \{s.t, \neg P_t\}$ if A_s is of the form $\neg\Box P_s$
-where t is the first subscript not included in Ω_{i^*}

then

$\Gamma' = \bigcup_{i \geqslant 0} \Omega_i$

Note that there is always sure to be a subscript t not in Ω_{i^*} insofar as there are infinitely many subscripts, and at any stage only finitely many formulas are added – the only subscripts in the initial Ω_0 being 0. Suppose s is introduced in Γ'; then there is some Ω_i in which it is first introduced; and any formula $\mathcal{P}_j$ in the original enumeration that has subscript s is sure to be "considered" for inclusion at a subsequent stage.

L2.4 For any s included in Γ', Γ' is s-maximal.

Suppose s is included in Γ' but Γ' is not s-maximal. Then there is some A_s such that $\Gamma' \nvdash^*_{NK\alpha} A_s$ and $\Gamma' \nvdash^*_{NK\alpha} \neg A_s$. For any i, each member of Ω_{i-1} is in Γ'; so if $\Omega_{i-1} \vdash^*_{NK\alpha} \neg A_s$ then $\Gamma' \vdash^*_{NK\alpha} \neg A_s$; but $\Gamma' \nvdash^*_{NK\alpha} \neg A_s$; so $\Omega_{i-1} \nvdash^*_{NK\alpha} \neg A_s$; so since s is included in Γ', there is a stage in the construction that sets $\Omega_{i^*} = \Omega_{i-1} \cup \{A_s\}$; so by construction, $A_s \in \Gamma'$; so $\Gamma' \vdash^*_{NK\alpha} A_s$. This is impossible; reject the assumption: Γ' is s-maximal.

L2.5 If Γ_0 is consistent, then each Ω_i is consistent.

Suppose Γ_0 is consistent.

Basis: $\Omega_0 = \Gamma_0$ and Γ_0 is consistent; so Ω_0 is consistent.

Assp: For any i, $0 \leqslant i < k$, Ω_i is consistent.

Show: Ω_k is consistent.

Ω_k is either (i) Ω_{k-1}, or (ii) $\Omega_{k^*} = \Omega_{k-1} \cup \{A_s\}$ or (iii) $\Omega_{k^*} \cup \{s.t, \neg P_t\}$.

(i) Suppose Ω_k is Ω_{k-1}. By assumption, Ω_{k-1} is consistent; so Ω_k is consistent.

(ii) Suppose Ω_k is $\Omega_{k^*} = \Omega_{k-1} \cup \{A_s\}$. Then by construction, s is 0 or in Ω_{k-1} and $\Omega_{k-1} \nvdash^*_{NK\alpha} \neg A_s$; so by L2.2, $\Omega_{k-1} \cup \{A_s\}$ is consistent; so Ω_k is consistent.

(iii) Suppose Ω_k is $\Omega_{k^*} \cup \{s.t, \neg P_t\}$. In this case, as above, Ω_{k*} is consistent and by construction, $\neg\Box P_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are A_u and $\neg A_u$ such that $\Omega_{k^*} \cup \{s.t, \neg P_t\} \vdash^*_{NK\alpha} A_u$ and $\Omega_{k^*} \cup \{s.t, \neg P_t\} \vdash^*_{NK\alpha} \neg A_u$. So reason as follows,

1	Ω_{k^*}		
2	$s.t$		A (g, $\Box$I)
3	$\neg P_t$		A (c, $\neg$E)
4	A_u		from $\Omega_{k^*} \cup \{s.t, \neg P_t\}$
5	$\neg A_u$		from $\Omega_{k^*} \cup \{s.t, \neg P_t\}$
6	P_t		3-5 $\neg$E
7	$\Box P_s$		2-6 $\Box$I

where, by construction, t is not in Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{\text{NK}\alpha} \Box P_s$; but $\neg\Box P_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{\text{NK}\alpha} \neg\Box P_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

For any i, Ω_i is consistent.

L2.6 If Γ_0 is consistent, then Γ' is consistent.

Suppose Γ_0 is consistent, but Γ' is not; from the latter, there is some P_s such that $\Gamma' \vdash^*_{\text{NK}\alpha} P_s$ and $\Gamma' \vdash^*_{\text{NK}\alpha} \neg P_s$. Consider derivations D1 and D2 of these results, and the premises $\mathcal{P}_i \ldots \mathcal{P}_j$ of these derivations. By construction, there is an Ω_k with each of these premises as a member; so D1 and D2 are derivations from Ω_k; so Ω_k is not consistent. But since Γ_0 is consistent, by L2.5, Ω_k is consistent. This is impossible; reject the assumption: if Γ_0 is consistent then Γ' is consistent.

L2.7 If Γ_0 is consistent, then Γ' is a scapegoat set.

Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{\text{NK}\alpha} \neg\Box P_s$. By L2.6, Γ' is consistent; and by the constraints on subscripts, s is included in Γ'. Since Γ' is consistent, $\Gamma' \nvdash^*_{\text{NK}\alpha} \neg\neg\Box P_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{\neg\Box P_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{s.t, \neg P_t\}$; so by construction, $s.t \in \Gamma'$ and $\neg P_t \in \Gamma'$; so $\Gamma' \vdash^*_{\text{NK}\alpha} s.t$ and $\Gamma' \vdash^*_{\text{NK}\alpha} \neg P_t$. So Γ' is a scapegoat set.

C(I) We construct an interpretation $I = \langle W, R, v\rangle$ based on Γ' as follows. Let W have a member w_s corresponding to each subscript s included in Γ'. Then set $\langle w_s, w_t\rangle \in R$ iff $\Gamma' \vdash^*_{\text{NK}\alpha} s.t$ and $v_{w_s}(p) = 1$ iff $\Gamma' \vdash^*_{\text{NK}\alpha} p_s$.

L2.8 If Γ_0 is consistent then for $\langle W, R, v\rangle$ constructed as above, and for any s included in Γ', $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{\text{NK}\alpha} A_s$.

Suppose Γ_0 is consistent and s is included in Γ'. By L2.4, Γ' is s-maximal. By L2.6 and L2.7, Γ' is consistent and a scapegoat set. Now by induction on the number of operators in A_s,

Basis: If A_s has no operators, then it is a parameter p_s and by construction, $v_{w_s}(p) = 1$ iff $\Gamma' \vdash^*_{\text{NK}\alpha} p_s$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{\text{NK}\alpha} A_s$.

Assp: For any i, $0 \leqslant i < k$, if A_s has i operators, then $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{\text{NK}\alpha} A_s$.

Show: If A_s has k operators, then $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NK\alpha} A_s$.

If A_s has k operators, then it is of the form $\neg P_s$, $(P \supset Q)_s$, $(P \wedge Q)_s$, $(P \vee Q)_s$, $(P \equiv Q)_s$, $\Box P_s$ or $\Diamond P_s$ where P and Q have $< k$ operators.

($\neg$) A_s is $\neg P_s$. (i) Suppose $v_{w_s}(A) = 1$; then $v_{w_s}(\neg P) = 1$; so by TK($\neg$), $v_{w_s}(P) = 0$; so by assumption, $\Gamma' \nvdash^*_{NK\alpha} P_s$; so by s-maximality, $\Gamma' \vdash^*_{NK\alpha} \neg P_s$, where this is to say, $\Gamma' \vdash^*_{NK\alpha} A_s$. (ii) Suppose $\Gamma' \vdash^*_{NK\alpha} A_s$; then $\Gamma' \vdash^*_{NK\alpha} \neg P_s$; so by consistency, $\Gamma' \nvdash^*_{NK\alpha} P_s$; so by assumption, $v_{w_s}(P) = 0$; so by TK($\neg$), $v_{w_s}(\neg P) = 1$, where this is to say, $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NK\alpha} A_s$.

($\supset$) A_s is $(P \supset Q)_s$. (i) Suppose $v_{w_s}(A) = 1$ but $\Gamma' \nvdash^*_{NK\alpha} A_s$; then $v_{w_s}(P \supset Q) = 1$ but $\Gamma' \nvdash^*_{NK\alpha} (P \supset Q)_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NK\alpha} \neg(P \supset Q)_s$; from this it follows, by simple derivations, that $\Gamma' \vdash^*_{NK\alpha} P_s$ and $\Gamma' \vdash^*_{NK\alpha} \neg Q_s$; so by consistency, $\Gamma' \nvdash^*_{NK\alpha} Q_s$; so by assumption, $v_{w_s}(P) = 1$ and $v_{w_s}(Q) = 0$; so by TK($\supset$), $v_{w_s}(P \supset Q) = 0$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$ then $\Gamma' \vdash^*_{NK\alpha} A_s$.

(ii) Suppose $\Gamma' \vdash^*_{NK\alpha} A_s$ but $v_{w_s}(A) = 0$; then $\Gamma' \vdash^*_{NK\alpha} (P \supset Q)_s$ but $v_{w_s}(P \supset Q) = 0$. From the latter, by TK($\supset$), $v_{w_s}(P) = 1$ and $v_{w_s}(Q) = 0$; so by assumption, $\Gamma' \vdash^*_{NK\alpha} P_s$ and $\Gamma' \nvdash^*_{NK\alpha} Q_s$; but since $\Gamma' \vdash^*_{NK\alpha} (P \supset Q)_s$ and $\Gamma' \vdash^*_{NK\alpha} P_s$, by ($\supset$E), $\Gamma' \vdash^*_{NK\alpha} Q_s$. This is impossible; reject the assumption: if $\Gamma' \vdash^*_{NK\alpha} A_s$, then $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NK\alpha} A_s$.

($\wedge$)

($\vee$)

($\equiv$)

($\Box$) A_s is $\Box P_s$. (i) Suppose that $v_{w_s}(A) = 1$ but $\Gamma' \nvdash^*_{NK\alpha} A_s$; then $v_{w_s}(\Box P) = 1$ but $\Gamma' \nvdash^*_{NK\alpha} \Box P_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NK\alpha} \neg\Box P_s$; so, since Γ' is a scapegoat set, there is some t such that $\Gamma' \vdash^*_{NK\alpha}$ s.t and $\Gamma' \vdash^*_{NK\alpha} \neg P_t$; from the first, by construction, $\langle w_s, w_t \rangle \in R$; and from the second, by consistency, $\Gamma' \nvdash^*_{NK\alpha} P_t$; so by assumption, $v_{w_t}(P) = 0$; but $w_s R w_t$; so by TK($\Box$), $v_{w_s}(\Box P) = 0$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$, then $\Gamma' \vdash^*_{NK\alpha} A_s$.

(ii) Suppose $\Gamma' \vdash^*_{NK\alpha} A_s$ but $v_{w_s}(A) = 0$; then $\Gamma' \vdash^*_{NK\alpha} \Box P_s$ but $v_{w_s}(\Box P) = 0$. From the latter, by TK($\Box$), there is some $w_t \in W$ such that $w_s R w_t$ and $v_{w_t}(P) = 0$; so by assumption, $\Gamma' \nvdash^*_{NK\alpha} P_t$; but since $w_s R w_t$, by construction, $\Gamma' \vdash^*_{NK\alpha}$ s.t; so by ($\Box$E), $\Gamma' \vdash^*_{NK\alpha} P_t$. This is impossible; reject the assumption: if $\Gamma' \vdash_{NK\alpha} A_s$ then $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NK\alpha} A_s$.

($\Diamond$) A_s is $\Diamond P_s$. (i) Suppose $v_{w_s}(A) = 1$; then $v_{w_s}(\Diamond P) = 1$; so by TK($\Diamond$), there is some $w_t \in W$ such that $w_s R w_t$ and $v_{w_t}(P) = 1$;

so by assumption, $\Gamma' \vdash^*_{NK\alpha} P_t$; but since $w_s R w_t$, by construction, $\Gamma' \vdash^*_{NK\alpha} s.t$; so by ($\Diamond$I), $\Gamma' \vdash^*_{NK\alpha} \Diamond P_s$; so $\Gamma' \vdash^*_{NK\alpha} A_s$.

(ii)Suppose $\Gamma' \vdash^*_{NK\alpha} A_s$; then $\Gamma' \vdash^*_{NK\alpha} \Diamond P_s$; so by (MN), $\Gamma' \vdash^*_{NK\alpha} \neg\Box\neg P_s$; so, since Γ' is a scapegoat set, there is some t such that $\Gamma' \vdash^*_{NK\alpha} s.t$ and $\Gamma' \vdash^*_{NK\alpha} \neg\neg P_t$; so by (DN), $\Gamma' \vdash^*_{NK\alpha} P_t$; so by assumption, $v_{w_t}(P) = 1$; but $\Gamma' \vdash^*_{NK\alpha} s.t$; so by construction, $w_s R w_t$; so by TK($\Diamond$), $v_{w_s}(\Diamond P) = 1$; so $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NK\alpha} A_s$.

For any A_s, $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NK\alpha} A_s$.

L2.9 If Γ_0 is consistent, then $\langle W, R, v\rangle$ constructed as above is a $K\alpha$ interpretation.

In each case, we need to show that the interpretation meets the condition(s) α. Suppose Γ_0 is consistent.

(η) Suppose α includes condition η and that $w_s \in W$. Then, by construction, s is a subscript in Γ'; so by reasoning as follows,

1	Γ'	
2	$\quad s.t$	A (g, AMη)
3	$\quad \top_t$	$\top$ is a tautology
4	$\quad \Diamond\top_s$	2,3 $\Diamond$I
5	$\Diamond\top_s$	2-4 AMη
6	$\neg\Box\neg\top_s$	5 MN

$\Gamma' \vdash^*_{NK\alpha} \neg\Box\neg\top_s$; but by L2.7, Γ' is a scapegoat set; so there is a t such that $\Gamma' \vdash^*_{NK\alpha} s.t$; so by construction, $\langle w_s, w_t\rangle \in R$ and η is satisfied.

(ρ) Suppose α includes condition ρ and $w_s \in W$. Then by construction, s is a subscript in Γ'; so by (AMρ), $\Gamma' \vdash^*_{NK\alpha} s.s$; so by construction, $\langle w_s, w_s\rangle \in R$ and ρ is satisfied.

(σ) Suppose α includes condition σ and $\langle w_s, w_t\rangle \in R$. Then by construction, $\Gamma' \vdash^*_{NK\alpha} s.t$ so by (AMσ), $\Gamma' \vdash^*_{NK\alpha} t.s$; so by construction, $\langle w_t, w_s\rangle \in R$ and σ is satisfied.

(τ) Suppose α includes condition τ and $\langle w_s, w_t\rangle, \langle w_t, w_u\rangle \in R$. Then by construction, $\Gamma' \vdash^*_{NK\alpha} s.t$ and $\Gamma' \vdash^*_{NK\alpha} t.u$; so by (AM$\tau$), $\Gamma' \vdash^*_{NK\alpha} s.u$; so by construction, $\langle w_s, w_u\rangle \in R$ and τ is satisfied.

Map For any $w_s \in W$, set $m(s) = w_s$; otherwise $m(s)$ is arbitrary.

L2.10 If Γ_0 is consistent, then $v_m(\Gamma_0) = 1$.

Suppose Γ_0 is consistent and $A_0 \in \Gamma_0$; then by construction, $A_0 \in \Gamma'$; so $\Gamma' \vdash^*_{NK\alpha} A_0$; so since Γ_0 is consistent, by L2.8, $v_{w_0}(A) = 1$. And similarly for any $A_0 \in \Gamma_0$. But $m(0) = w_0$; so $v_m(\Gamma_0) = 1$.

Main result: Suppose $\Gamma \models_{K\alpha} A$ but $\Gamma \nvdash_{NK\alpha} A$. Then $\Gamma_0 \models^*_{K\alpha} A_0$ but $\Gamma_0 \nvdash^*_{NK\alpha} A_0$. By (DN), if $\Gamma_0 \vdash^*_{NK\alpha} \neg\neg A_0$, then $\Gamma_0 \vdash^*_{NK\alpha} A_0$; so $\Gamma_0 \nvdash^*_{NK\alpha} \neg\neg A_0$; so by L2.2, $\Gamma_0 \cup \{\neg A_0\}$ is consistent; so by L2.9 and L2.10, there is a $K\alpha$ interpretation $\langle W, R, v\rangle_m$ constructed as above such that $v_m(\Gamma_0 \cup \{\neg A_0\}) = 1$; so $v_{m(0)}(\neg A) = 1$; so by TK($\neg$), $v_{m(0)}(A) = 0$; so $v_m(\Gamma_0) = 1$ and $v_{m(0)}(A) = 0$; so by VKα^*, $\Gamma_0 \not\models^*_{K\alpha} A_0$. This is impossible; reject the assumption: if $\Gamma \models_{K\alpha} A$, then $\Gamma \vdash_{NK\alpha} A$.

The argument for *NKυ* is similar, and so omitted.

3 NON-NORMAL MODAL LOGICS: $N\alpha$ (CH. 4)

3.1 LANGUAGE / SEMANTIC NOTIONS

LNα The basic language is the same as for $K\alpha$. The VOCABULARY consists of propositional parameters $p_0, p_1 \ldots$ with the operators, $\neg$, $\wedge$, $\vee$, $\supset$, $\equiv$, $\Box$ and $\Diamond$. Each propositional parameter is a FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$, $(A \equiv B)$, $\Box A$ and $\Diamond A$. In addition, we introduce $(A \dashv B)$ as an abbreviation for $\Box(A \supset B)$.

INα An INTERPRETATION is $\langle W, N, R, v\rangle$ where $N \subseteq W$. N is the set of *normal* worlds. Constraints on access are as for $K\alpha$. Thus, where α is empty or indicates some combination of the following constraints,

η	For any x, there is a y such that xRy	extendability
ρ	for all x, xRx	reflexivity
σ	for all x, y, if xRy then yRx	symmetry
τ	for all x, y, z, if xRy and yRz then xRz	transitivity

$\langle W, N, R, v\rangle$ is an $N\alpha$ interpretation when R meets the constraints from α.

TN For complex expressions,

($\neg$) $v_w(\neg A) = 1$ if $v_w(A) = 0$, and 0 otherwise.

($\wedge$) $v_w(A \wedge B) = 1$ if $v_w(A) = 1$ and $v_w(B) = 1$, and 0 otherwise.

($\vee$) $v_w(A \vee B) = 1$ if $v_w(A) = 1$ or $v_w(B) = 1$, and 0 otherwise.

($\supset$) $v_w(A \supset B) = 1$ if $v_w(A) = 0$ or $v_w(B) = 1$, and 0 otherwise.

($\equiv$) $v_w(A \equiv B) = 1$ if $v_w(A) = v_w(B)$, and 0 otherwise.

($\Diamond$) $v_w(\Diamond A) = 1$ if $w \notin N$ or some $x \in W$ such that wRx has $v_x(A) = 1$, and 0 otherwise.

($\Box$) $v_w(\Box A) = 1$ if $w \in N$ and all $x \in W$ such that wRx have $v_x(A) = 1$, and 0 otherwise.

For a set Γ of formulas, $v_w(\Gamma) = 1$ iff $v_w(A) = 1$ for each $A \in \Gamma$; then,

VNα $\Gamma \models_{N\alpha} A$ iff there is no $N\alpha$ interpretation $\langle W, N, R, v\rangle$ and $w \in N$ such that $v_w(\Gamma) = 1$ and $v_w(A) = 0$.

3.2 NATURAL DERIVATIONS: *NNα*

All the rules are as in *NKα* except that whenever a subscript s.t is introduced for □I or ◇E, either s is 0, or there is an additional premise of the sort $\Box A_s$, or $\neg\Diamond A_s$. The resulting change in constraints on these rules is small.

□I

	s.t
	P_t
$\Box P_s$	

where s is 0 or appears in some accessible $\Box A_s$ or $\neg\Diamond A_s$, and t does not appear in any undischarged premise or assumption

◇E

$\Diamond P_s$	
	s.t
	P_t
	Q_u
Q_u	

where s is 0 or appears in some accessible $\Box A_s$ or $\neg\Diamond A_s$, and t does not appear in any undischarged premise or assumption and is not u

Derived rules carry over from *NKα*. Note that MN remains as well. In addition, the following are derived rules for ⥽I and ⥽E in either *NKα* or *NNα*.

⥽I

	s.t
	P_t
	Q_t
$(P ⥽ Q)_s$	

constraints on s and t as for the corresponding *NN* or *NK* □I rule.

⥽E

$(P ⥽ Q)_s$
s.t
P_t
Q_t

We exhibit the new restrictions by considering derivations to show one part of MN, that $\Diamond P_s \vdash_{NN\alpha} \neg\Box\neg P_s$. In the case where $s \neq 0$, the derivation on the left violates the restriction on ◇E in its last line.

1	$\Diamond P_s$	P
2	s.t	A (*g*, 1 ◇E)
3	P_t	
4	$\Box\neg P_s$	A (*c*, ¬I)
5	$\neg P_t$	2,4 □E
6	P_t	3 R
7	$\neg\Box\neg P_s$	4-6 ¬I
8	$\neg\Box\neg P_s$	1,2-7 ◇E

1	$\Diamond P_s$	P
2	$\Box\neg P_s$	A (*c*, ¬I)
3	s.t	A (*g*, 1 ◇E)
4	P_t	
5	$\Box\neg P_s$	A (*c*, ¬I)
6	$\neg P_t$	3,5 □E
7	P_t	4 R
8	$\neg\Box\neg P_s$	5-7 ¬I
9	$\neg\Box\neg P_s$	2,1,3-8 ◇E
10	$\Box\neg P_s$	2 R
11	$\neg\Box\neg P_s$	2-10 ¬I

Supposing s is 0, each derivation is fine. However, if s is other than 0, on the left, (8) violates the restriction on ◇E, insofar as there is no accessible $\Box P_s$ or $\neg\Diamond P_s$. On the right, we get around the problem by making the assumption for ¬I prior to that for ◇E. Note that, in this case, we *cite* the line with $\Box A_s$ for ◇E. Other derivations for MN go through as in the previous section.

3.3 SOUNDNESS AND COMPLETENESS

Preliminaries: Begin with generalized notions of validity. For a model $\langle W, N, R, v\rangle$, let m be a map from subscripts into W such that $m(0)$ is some member of N. Say $\langle W, N, R, v\rangle_m$ is $\langle W, N, R, v\rangle$ ***with*** map m. Then, where Γ is a set of expressions of our language for derivations, $v_m(\Gamma) = 1$ iff for each $A_s \in \Gamma$, $v_{m(s)}(A) = 1$, and for each $s.t \in \Gamma$, $\langle m(s), m(t)\rangle \in R$. Now expand notions of validity to include subscripted formulas, and alternate expressions as indicated in double brackets.

VNα* $\Gamma \models^*_{N\alpha} A_s$ $[\![s.t]\!]$ iff there is no Nα interpretation $\langle W, N, R, v\rangle_m$ such that $v_m(\Gamma) = 1$ but $v_{m(s)}(A) = 0$ $[\![\langle m(s), m(t)\rangle \notin R]\!]$.

NNα* $\Gamma \vdash^*_{NN\alpha} A_s$ $[\![s.t]\!]$ iff there is an *NN*α derivation of A_s $[\![s.t]\!]$ from the members of Γ.

These notions reduce to the standard ones when all the members of Γ and A have subscript 0 (and so do not include expressions of the sort $s.t$). This is obvious for NNα*. In the other case, there is a $\langle W, N, R, v\rangle_m$ and $w \in N$ that makes all the members of Γ_0 true and A_0 false just in case there *is* a world in N that makes the unsubscripted members of Γ true and A false. For the following, cases omitted are like ones worked, and so left to the reader.

THEOREM 3.1 *NNα is sound: If* $\Gamma \vdash_{NN\alpha} A$ ***then*** $\Gamma \models_{N\alpha} A$.

L3.1 If $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{N\alpha} P_s$ $[\![s.t]\!]$, then $\Gamma' \models^*_{N\alpha} P_s$ $[\![s.t]\!]$.
Reasoning parallel to that for L2.1 of *NK*α.

Main result: For each line in a derivation let $\mathcal{P}_i$ be the expression on line i and Γ_i be the set of all premises and assumptions whose scope includes line i. We set out to show "generalized" soundness: if $\Gamma \vdash^*_{NN\alpha} \mathcal{P}$ then $\Gamma \models^*_{N\alpha} \mathcal{P}$. Suppose $\Gamma \vdash^*_{NN\alpha} \mathcal{P}$. Then there is a derivation of $\mathcal{P}$ from premises in Γ where $\mathcal{P}$ appears under the scope of the premises alone. By induction on line number of this derivation, we show that for each line i of this derivation, $\Gamma_i \models^*_{N\alpha} \mathcal{P}_i$. The case when $\mathcal{P}_i = \mathcal{P}$ is the desired result.

Basis: $\mathcal{P}_1$ is a premise or an assumption A_s $[\![s.t]\!]$. Then $\Gamma_1 = \{A_s\}$ $[\![\{s.t\}]\!]$; so for any $\langle W, N, R, v\rangle_m$, $v_m(\Gamma_1) = 1$ iff $v_{m(s)}(A) = 1$ $[\![\langle m(s), m(t)\rangle \in R]\!]$; so there is no $\langle W, N, R, v\rangle_m$ such that $v_m(\Gamma_1) = 1$ but $v_{m(s)}(A) = 0$ $[\![\langle m(s), m(t)\rangle \notin R]\!]$. So by VN$\alpha$*, $\Gamma_1 \models^*_{N\alpha} A_s$ $[\![s.t]\!]$, where this is just to say, $\Gamma_1 \models^*_{N\alpha} \mathcal{P}_1$.

Assp: For any i, $1 \leqslant i < k$, $\Gamma_i \models^*_{N\alpha} \mathcal{P}_i$.

Show: $\Gamma_k \models^*_{N\alpha} \mathcal{P}_k$.

$\mathcal{P}_k$ is either a premise, an assumption, or arises from previous lines by R, $\supset$I, $\supset$E, $\wedge$I, $\wedge$E, $\neg$I, $\neg$E, $\vee$I, $\vee$E, $\equiv$I, $\equiv$E, $\Box$I, $\Box$E, $\Diamond$I, $\Diamond$E or, depending on the system, AMη, AMρ, AMσ or AMτ. If $\mathcal{P}_k$ is a premise or an assumption, then as in the basis, $\Gamma_k \models^*_{N\alpha} \mathcal{P}_k$. So suppose $\mathcal{P}_k$ arises by one of the rules.

(R)

($\supset$I)

($\supset$E) If $\mathcal{P}_k$ arises by $\supset$E, then the picture is like this,

$$\begin{array}{r|l} i & (A \supset B)_s \\ j & A_s \\ & \\ k & B_s \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is B_s. By assumption, $\Gamma_i \models^*_{N\alpha} (A \supset B)_s$ and $\Gamma_j \models^*_{N\alpha} A_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L3.1, $\Gamma_k \models^*_{N\alpha} (A \supset B)_s$ and $\Gamma_k \models^*_{N\alpha} A_s$. Suppose $\Gamma_k \not\models^*_{N\alpha} B_s$; then by VN$\alpha$*, there is some N$\alpha$ interpretation $\langle W, N, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(B) = 0$; since $v_m(\Gamma_k) = 1$, by VNα*, $v_{m(s)}(A \supset B) = 1$ and $v_{m(s)}(A) = 1$; from the former, by TN($\supset$), $v_{m(s)}(A) = 0$ or $v_{m(s)}(B) = 1$; so $v_{m(s)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{N\alpha} B_s$, which is to say, $\Gamma_k \models^*_{N\alpha} \mathcal{P}_k$.

($\wedge$I)

($\wedge$E)

($\neg$I) If $\mathcal{P}_k$ arises by $\neg$I, then the picture is like this,

$$\begin{array}{r|l} & \quad \underline{A_s} \\ & \\ i & \quad B_t \\ j & \quad \neg B_t \\ k & \neg A_s \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $\neg A_s$. By assumption, $\Gamma_i \models^*_{N\alpha} B_t$ and $\Gamma_j \models^*_{N\alpha} \neg B_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{A_s\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{A_s\}$; so by L3.1, $\Gamma_k \cup \{A_s\} \models^*_{N\alpha} B_t$ and $\Gamma_k \cup \{A_s\} \models^*_{N\alpha} \neg B_t$. Suppose $\Gamma_k \not\models^*_{N\alpha} \neg A_s$; then by VN$\alpha$*, there is an N$\alpha$ interpretation $\langle W, N, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(\neg A) = 0$; so by TN($\neg$), $v_{m(s)}(A) = 1$; so $v_m(\Gamma_k) = 1$ and $v_{m(s)}(A) = 1$; so $v_m(\Gamma_k \cup \{A_s\}) = 1$; so by VN$\alpha$*, $v_{m(t)}(B) = 1$ and $v_{m(t)}(\neg B) = 1$; from the latter, by TN($\neg$), $v_{m(t)}(B) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{N\alpha} \neg A_s$, which is to say, $\Gamma_k \models^*_{N\alpha} \mathcal{P}_k$.

($\neg$E)

($\vee$I)

($\vee$E)

($\equiv$I)

($\equiv$E)

($\Box$I) If $\mathcal{P}_k$ arises by $\Box$I, then the picture is like this,

$$\begin{array}{r|l} & \quad \underline{s.t} \\ & \quad \vdots \\ i & \quad A_t \\ k & \Box A_s \end{array}$$

where $i < k$, s is 0 or introduced in some accessible $\Box P_s$ or $\neg\Diamond P_s$, t does not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $\Box A_s$. By assumption, $\Gamma_i \models^*_{N\alpha} A_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{s.t\}$; so by L3.1, $\Gamma_k \cup \{s.t\} \models^*_{N\alpha} A_t$. Suppose $\Gamma_k \not\models^*_{N\alpha} \Box A_s$; then by VN$\alpha$*, there is an N$\alpha$ interpretation $\langle W, N, R, \nu\rangle_m$ such that $\nu_m(\Gamma_k) = 1$ but $\nu_{m(s)}(\Box A) = 0$. If s is 0, then $m(s) \in N$; if s is introduced in some $\Box P_s$ on accessible line j, then by assumption, $\Gamma_j \models^*_{N\alpha} \Box P_s$; but by the nature of access, $\Gamma_j \subseteq \Gamma_k$; so by L3.1, $\Gamma_k \models^*_{N\alpha} \Box P_s$; so by VN$\alpha$*, $\nu_{m(s)}(\Box P) = 1$; so by TN($\Box$), $m(s) \in N$; if s is introduced in some $\neg\Diamond P_s$ on an accessible line j, then by assumption, $\Gamma_j \models^*_{N\alpha} \neg\Diamond P_s$; but by the nature of access, $\Gamma_j \subseteq \Gamma_k$; so by L3.1, $\Gamma_k \models^*_{N\alpha} \neg\Diamond P_s$; so by VN$\alpha$*, $\nu_{m(s)}(\neg\Diamond P) = 1$; so by TN($\neg$), $\nu_{m(s)}(\Diamond P) = 0$; so by TN($\Diamond$), $m(s) \in N$; in any case, then, $m(s) \in N$. So by TN($\Box$), there is some $w \in W$ such that $m(s)Rw$ and $\nu_w(A) = 0$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, N, R, \nu\rangle_{m'}$; since t does not appear in Γ_k, it remains that $\nu_{m'}(\Gamma_k) = 1$; and since $m'(t) = w$ and $m'(s) = m(s)$, $\langle m'(s), m'(t)\rangle \in R$; so $\nu_{m'}(\Gamma_k \cup \{s.t\}) = 1$; so by VN$\alpha$*, $\nu_{m'(t)}(A) = 1$. But $m'(t) = w$; so $\nu_w(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{N\alpha} \Box A_s$, which is to say, $\Gamma_k \models^*_{N\alpha} \mathcal{P}_k$.

($\Box$E) If $\mathcal{P}_k$ arises by $\Box$E, then the picture is like this,

$$\begin{array}{r|l} i & \Box A_s \\ j & s.t \\ & \vdots \\ k & A_t \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is A_t. By assumption, $\Gamma_i \models^*_{N\alpha} \Box A_s$ and $\Gamma_j \models^*_{N\alpha} s.t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L3.1, $\Gamma_k \models^*_{N\alpha} \Box A_s$ and $\Gamma_k \models^*_{N\alpha} s.t$. Suppose $\Gamma_k \not\models^*_{N\alpha} A_t$; then by VN$\alpha$*, there is some N$\alpha$ interpretation $\langle W, N, R, \nu\rangle_m$ such that $\nu_m(\Gamma_k) = 1$ but $\nu_{m(t)}(A) = 0$; since $\nu_m(\Gamma_k) = 1$, by VNα*, $\nu_{m(s)}(\Box A) = 1$ and $\langle m(s), m(t)\rangle \in R$; from the first of these, by TN($\Box$), any w such that $m(s)Rw$ has $\nu_w(A) = 1$;

so $v_{m(t)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{N\alpha} A_t$, which is to say, $\Gamma_k \models^*_{N\alpha} \mathcal{P}_k$.

($\Diamond$I)

($\Diamond$E) If $\mathcal{P}_k$ arises by $\Diamond$E, then the picture is like this,

```
i | ◇A_s
  |  | A_t
  |  | s.t
  |  |----
  |  |
j |  | B_u
k | B_u
```

where $i, j < k$, s is 0 or introduced in some accessible $\Box P_s$ or $\neg\Diamond P_s$, t does not appear in any member of Γ_k (in any undischarged premise or assumption) and is not u, and $\mathcal{P}_k$ is B_u. By assumption, $\Gamma_i \models^*_{N\alpha} \Diamond A_s$ and $\Gamma_j \models^*_{N\alpha} B_u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k \cup \{A_t, s.t\}$; so by L3.1, $\Gamma_k \models^*_{N\alpha} \Diamond A_s$ and $\Gamma_k \cup \{A_t, s.t\} \models^*_{N\alpha} B_u$. Suppose $\Gamma_k \not\models^*_{N\alpha} B_u$; then by VN$\alpha$*, there is an $N\alpha$ interpretation $\langle W, N, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(u)}(B) = 0$. If s is 0, then $m(s) \in N$; if s is introduced in some $\Box P_s$ on accessible line h, then by assumption, $\Gamma_h \models^*_{N\alpha} \Box P_s$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$; so by L3.1, $\Gamma_k \models^*_{N\alpha} \Box P_s$; so by VN$\alpha$*, $v_{m(s)}(\Box P) = 1$; so by TN($\Box$), $m(s) \in N$; if s is introduced in some $\neg\Diamond P_s$ on an accessible line h, then by assumption, $\Gamma_h \models^*_{N\alpha} \neg\Diamond P_s$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$; so by L3.1, $\Gamma_k \models^*_{N\alpha} \neg\Diamond P_s$; so by VN$\alpha$*, $v_{m(s)}(\neg\Diamond P) = 1$; so by TN($\neg$), $v_{m(s)}(\Diamond P) = 0$; so by TN($\Diamond$), $m(s) \in N$; in any case, then, $m(s) \in N$. Since $v_m(\Gamma_k) = 1$, by VNα*, $v_{m(s)}(\Diamond A) = 1$; so by TN($\Diamond$), since $m(s) \in N$, there is some $w \in W$ such that $m(s)Rw$ and $v_w(A) = 1$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, N, R, v\rangle_{m'}$; since t does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m'(s) = m(s)$ and $m'(t) = w$, $v_{m'(t)}(A) = 1$ and $\langle m'(s), m'(t)\rangle \in R$; so $v_{m'}(\Gamma_k \cup \{A_t, s.t\}) = 1$; so by VN$\alpha$*, $v_{m'(u)}(B) = 1$. But since $t \neq u$, $m'(u) = m(u)$; so $v_{m(u)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{N\alpha} B_u$, which is to say, $\Gamma_k \models^*_{N\alpha} \mathcal{P}_k$.

(AMη) If $\mathcal{P}_k$ arises by AMη, then the picture is like this,

```
  |  | s.t
  |  |----
  |  |
i |  | A_u
k | A_u
```

where $i < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption) and is not u, and $\mathcal{P}_k$ is A_u. Where this rule is included in *NN*α, *N*α includes condition η. By assumption, $\Gamma_i \models^*_{N\alpha} A_u$;

but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{s.t\}$; so by L3.1, $\Gamma_k \cup \{s.t\} \models^*_{N\alpha} A_u$. Suppose $\Gamma_k \not\models^*_{N\alpha} A_u$; then by VN$\alpha$*, there is an N$\alpha$ interpretation $\langle W, N, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(u)}(A) = 0$. By condition η, there is a $w \in W$ such that $m(s)Rw$; consider a map m' like m except that $m'(t) = w$, and consider $\langle W, N, R, v\rangle_{m'}$; since t does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m'(s) = m(s)$ and $m'(t) = w$, $\langle m'(s), m'(t)\rangle \in R$; so $v_{m'}(\Gamma_k \cup \{s.t\}) = 1$; so by VN$\alpha$*, $v_{m'(u)}(A) = 1$. But since $t \neq u$, $m'(u) = m(u)$; so $v_{m(u)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{N\alpha} A_u$, which is to say, $\Gamma_k \models^*_{N\alpha} \mathcal{P}_k$.

(AMρ)

(AMσ)

(AMτ) If $\mathcal{P}_k$ arises by AMτ, then the picture is like this,

i	s.t
j	t.u
k	s.u

where $i, j < k$ and $\mathcal{P}_k$ is s.u. Where this rule is in *NN*α, *N*α includes condition τ. By assumption, $\Gamma_i \models^*_{N\alpha}$ s.t and $\Gamma_j \models^*_{N\alpha}$ t.u; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L3.1, $\Gamma_k \models^*_{N\alpha}$ s.t and $\Gamma_k \models^*_{N\alpha}$ t.u. Suppose $\Gamma_k \not\models^*_{N\alpha}$ s.u; then by VNα*, there is some Nα interpretation $\langle W, N, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $\langle m(s), m(u)\rangle \notin R$; since $v_m(\Gamma_k) = 1$, by VNα*, $\langle m(s), m(t)\rangle \in R$ and $\langle m(t), m(u)\rangle \in R$; and by condition τ, for any $\langle x, y\rangle, \langle y, z\rangle \in R$, $\langle x, z\rangle \in R$; so $\langle m(s), m(u)\rangle \in R$. This is impossible; reject the assumption: $\Gamma_k \models^*_{N\alpha}$ s.u, which is to say, $\Gamma_k \models^*_{N\alpha} \mathcal{P}_k$.

For any i, $\Gamma_i \models^*_{N\alpha} \mathcal{P}_i$.

THEOREM 3.2 *NN*α *is complete: if* $\Gamma \models_{N\alpha} A$ *then* $\Gamma \vdash_{NN\alpha} A$.

Suppose $\Gamma \models_{N\alpha} A$; then $\Gamma_0 \models^*_{N\alpha} A_0$; we show that $\Gamma_0 \vdash^*_{NN\alpha} A_0$. Again, this reduces to the standard notion. For the following, fix on some particular constraint(s) α. Then definitions of *consistency* etc. are relative to it.

Con Γ is CONSISTENT iff there is no A_s such that $\Gamma \vdash^*_{NN\alpha} A_s$ and $\Gamma \vdash^*_{NN\alpha} \neg A_s$.

L3.2 If s is 0 or appears in Γ, and $\Gamma \nvdash^*_{NN\alpha} \neg P_s$, then $\Gamma \cup \{P_s\}$ is consistent.

Suppose s is 0 or appears in Γ and $\Gamma \nvdash^*_{NN\alpha} \neg P_s$ but $\Gamma \cup \{P_s\}$ is inconsistent. Then there is some A_t such that $\Gamma \cup \{P_s\} \vdash^*_{NN\alpha} A_t$ and $\Gamma \cup \{P_s\} \vdash^*_{NN\alpha} \neg A_t$. But then we can argue,

1	Γ	
2	$\quad P_s$	A (c, $\neg$I)
3	$\quad A_t$	from $\Gamma \cup \{P_s\}$
4	$\quad \neg A_t$	from $\Gamma \cup \{P_s\}$
5	$\neg P_s$	2-4 $\neg$I

where the assumption is allowed insofar as s is either 0 or appears in Γ; so $\Gamma \vdash^*_{NN\alpha} \neg P_s$. But this is impossible; reject the assumption: if s is 0 or introduced in Γ and $\Gamma \nvdash^*_{NN\alpha} \neg P_s$, then $\Gamma \cup \{P_s\}$ is consistent.

L3.3 There is an enumeration of all the subscripted formulas, $\mathcal{P}_1\ \mathcal{P}_2 \ldots$

Proof by construction as for L2.3 of *NK*α.

MAX Γ is S-MAXIMAL iff for any A_s either $\Gamma \vdash^*_{NN\alpha} A_s$ or $\Gamma \vdash^*_{NN\alpha} \neg A_s$.

SGT Γ is a SCAPEGOAT set iff for every formula of the form $(\Box P \wedge \neg\Box A)_s$, if $\Gamma \vdash^*_{NN\alpha} (\Box P \wedge \neg\Box A)_s$ then there is some t such that $\Gamma \vdash^*_{NN\alpha}$ s.t and $\Gamma \vdash^*_{NN\alpha} \neg A_t$.

C(Γ') For Γ with unsubscripted formulas and the corresponding Γ_0, we construct Γ' as follows. Set $\Omega_0 = \Gamma_0$. By L3.3, there is an enumeration, $\mathcal{P}_1, \mathcal{P}_2 \ldots$ of all the subscripted formulas; let $\mathcal{E}_0$ be this enumeration. Then for the first A_s in $\mathcal{E}_{i-1}$ such that s is 0 or included in Ω_{i-1}, let $\mathcal{E}_i$ be like $\mathcal{E}_{i-1}$ but without A_s, and set,

$$\Omega_i = \Omega_{i-1} \qquad \text{if } \Omega_{i-1} \vdash^*_{NN\alpha} \neg A_s$$
$$\Omega_{i^*} = \Omega_{i-1} \cup \{A_s\} \qquad \text{if } \Omega_{i-1} \nvdash^*_{NN\alpha} \neg A_s$$

and

$$\Omega_i = \Omega_{i^*} \qquad \text{if } A_s \text{ is not of the form } (\Box Q \wedge \neg\Box P_s)$$
$$\Omega_i = \Omega_{i^*} \cup \{s.t, \neg P_t\} \qquad \text{if } A_s \text{ is of the form } (\Box Q \wedge \neg\Box P)_s$$

-where t is the first subscript not included in Ω_{i^*}

then

$$\Gamma' = \bigcup_{i \geqslant 0} \Omega_i$$

Note that there is always sure to be a subscript t not in Ω_{i^*} insofar as there are infinitely many subscripts, and at any stage only finitely many formulas are added – the only subscripts in the initial Ω_0 being 0. Suppose s is introduced in Γ'; then there is some Ω_i in which it is first introduced; and any formula $\mathcal{P}_j$ in the original enumeration that has subscript s is sure to be "considered" for inclusion at a subsequent stage.

L3.4 For any s included in Γ', Γ' is s-maximal.

Suppose s is included in Γ' but Γ' is not s-maximal. Then there is some A_s such that $\Gamma' \nvdash^*_{NN\alpha} A_s$ and $\Gamma' \nvdash^*_{NN\alpha} \neg A_s$. For any i, each member of Ω_{i-1} is in Γ'; so if $\Omega_{i-1} \vdash^*_{NN\alpha} \neg A_s$ then $\Gamma' \vdash^*_{NN\alpha} \neg A_s$; but $\Gamma' \nvdash^*_{NN\alpha} \neg A_s$; so $\Omega_{i-1} \nvdash^*_{NN\alpha} \neg A_s$; so since s is included in Γ', there is a stage in the construction that sets $\Omega_{i^*} = \Omega_{i-1} \cup \{A_s\}$; so by construction, $A_s \in \Gamma'$; so $\Gamma' \vdash^*_{NN\alpha} A_s$. This is impossible; reject the assumption: Γ' is s-maximal.

L3.5 If Γ_0 is consistent, then each Ω_i is consistent.

Suppose Γ_0 is consistent.

Basis: $\Omega_0 = \Gamma_0$ and Γ_0 is consistent; so Ω_0 is consistent.

Assp: For any $i, 0 \leqslant i < k$, Ω_i is consistent.

Show: Ω_k is consistent.

Ω_k is either (i) Ω_{k-1}, or (ii) $\Omega_{k^*} = \Omega_{k-1} \cup \{A_s\}$ or (iii) $\Omega_{k^*} \cup \{s.t, \neg P_t\}$.

(i) Suppose Ω_k is Ω_{k-1}. By assumption, Ω_{k-1} is consistent; so Ω_k is consistent.

(ii) Suppose Ω_k is $\Omega_{k^*} = \Omega_{k-1} \cup \{A_s\}$. Then by construction, s is 0 or in Ω_{k-1} and $\Omega_{k-1} \nvdash^*_{NN\alpha} \neg A_s$; so by L3.2, $\Omega_{k-1} \cup \{A_s\}$ is consistent; so Ω_k is consistent.

(iii) Suppose Ω_k is $\Omega_{k^*} \cup \{s.t, \neg P_t\}$. In this case, as above, Ω_{k*} is consistent and by construction, $(\Box Q \wedge \neg\Box P)_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are A_u and $\neg A_u$ such that $\Omega_{k^*} \cup \{s.t, \neg P_t\} \vdash^*_{NN\alpha} A_u$ and $\Omega_{k^*} \cup \{s.t, \neg P_t\} \vdash^*_{NN\alpha} \neg A_u$. So reason as follows,

1	Ω_{k^*}	
2	$(\Box Q \wedge \neg\Box P)_s$	from Ω_{k^*}
3	$\Box Q_s$	2 $\wedge$E
4	| $s.t$	A (g, $\Box$I)
5	| | $\neg P_t$	A (c, $\neg$E)
6	| | A_u	from $\Omega_{k^*} \cup \{s.t, \neg P_t\}$
7	| | $\neg A_u$	from $\Omega_{k^*} \cup \{s.t, \neg P_t\}$
8	| P_t	5-7 $\neg$E
9	$\Box P_s$	3,4-8 $\Box$I

where, by construction, t is not in Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NN\alpha} \Box P_s$; but $(\Box Q \wedge \neg\Box P)_s \in \Omega_{k^*}$; so with ($\wedge$E), $\Omega_{k^*} \vdash^*_{NN\alpha} \neg\Box P_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

For any i, Ω_i is consistent.

L3.6 If Γ_0 is consistent, then Γ' is consistent.

Suppose Γ_0 is consistent, but Γ' is not; from the latter, there is some P_s such that $\Gamma' \vdash^*_{NN\alpha} P_s$ and $\Gamma' \vdash^*_{NN\alpha} \neg P_s$. Consider derivations D1 and D2 of these results, and the premises $\mathcal{P}_i \ldots \mathcal{P}_j$ of these derivations. By construction, there is an Ω_k with each of these premises as a member;

so D1 and D2 are derivations from Ω_k; so Ω_k is not consistent. But since Γ_0 is consistent, by L3.5, Ω_k is consistent. This is impossible; reject the assumption: if Γ_0 is consistent then Γ' is consistent.

L3.7 If Γ_0 is consistent, then Γ' is a scapegoat set.

Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{NN\alpha} (\Box Q \wedge \neg\Box P)_s$. By L3.6, Γ' is consistent; and by the constraints on subscripts, s is included in Γ'. Since Γ' is consistent, $\Gamma' \nvdash^*_{NN\alpha} \neg(\Box Q \wedge \neg\Box P)_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{(\Box Q \wedge \neg\Box P)_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{s.t, \neg P_t\}$; so by construction, $s.t \in \Gamma'$ and $\neg P_t \in \Gamma'$; so $\Gamma' \vdash^*_{NN\alpha} s.t$ and $\Gamma' \vdash^*_{NN\alpha} \neg P_t$. So Γ' is a scapegoat set.

C(I) We construct an interpretation $I = \langle W, N, R, v\rangle$ based on Γ' as follows. Let W have a member w_s corresponding to each subscript s included in Γ'. Then set $w_s \in N$ iff there is some Q such that $\Gamma' \vdash^*_{NN\alpha} \Box Q_s$; set $R = \{\langle w_s, w_s\rangle \mid w_s \in (W - N)\} \cup \{\langle w_s, w_t\rangle \mid \Gamma' \vdash^*_{NN\alpha} s.t\}$; and $v_{w_s}(p) = 1$ iff $\Gamma' \vdash^*_{NN\alpha} p_s$.

Note that $w_0 \in N$. By a simple derivation, $\vdash^*_{NN\alpha} \Box\top_0$; so $\Gamma' \vdash^*_{NN\alpha} \Box\top_0$; so $w_0 \in N$.

L3.8 If Γ_0 is consistent then for $\langle W, N, R, v\rangle$ constructed as above, and for any s included in Γ', $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NN\alpha} A_s$.

Suppose Γ_0 is consistent and s is included in Γ'. By L3.4, Γ' is s-maximal. By L3.6 and L3.7, Γ' is consistent and a scapegoat set. Now by induction on the number of operators in A_s,

Basis: If A_s has no operators, then it is a parameter p_s and by construction, $v_{w_s}(p) = 1$ iff $\Gamma' \vdash^*_{NN\alpha} p_s$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NN\alpha} A_s$.

Assp: For any i, $0 \leqslant i < k$, if A_s has i operators, then $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NN\alpha} A_s$.

Show: If A_s has k operators, then $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NN\alpha} A_s$.

If A_s has k operators, then it is of the form $\neg P_s$, $(P \supset Q)_s$, $(P \wedge Q)_s$, $(P \vee Q)_s$, $(P \equiv Q)_s$, $\Box P_s$ or $\Diamond P_s$ where P and Q have $< k$ operators.

($\neg$) A_s is $\neg P_s$. (i) Suppose $v_{w_s}(A) = 1$; then $v_{w_s}(\neg P) = 1$; so by TN($\neg$), $v_{w_s}(P) = 0$; so by assumption, $\Gamma' \nvdash^*_{NN\alpha} P_s$; so by s-maximality, $\Gamma' \vdash^*_{NN\alpha} \neg P_s$, where this is to say, $\Gamma' \vdash^*_{NN\alpha} A_s$. (ii) Suppose $\Gamma' \vdash^*_{NN\alpha} A_s$; then $\Gamma' \vdash^*_{NN\alpha} \neg P_s$; so by consistency, $\Gamma' \nvdash^*_{NN\alpha} P_s$; so by assumption, $v_{w_s}(P) = 0$; so by TN($\neg$), $v_{w_s}(\neg P) = 1$, where this is to say, $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NN\alpha} A_s$.

($\supset$) A_s is $(P \supset Q)_s$. (i) Suppose $v_{w_s}(A) = 1$ but $\Gamma' \nvdash^*_{NN\alpha} A_s$; then $v_{w_s}(P \supset Q) = 1$ but $\Gamma' \nvdash^*_{NN\alpha} (P \supset Q)_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NN\alpha} \neg(P \supset Q)_s$; from this it follows, by simple derivations, that $\Gamma' \vdash^*_{NN\alpha} P_s$ and $\Gamma' \vdash^*_{NN\alpha} \neg Q_s$; so by consistency, $\Gamma' \nvdash^*_{NN\alpha} Q_s$; so by assumption, $v_{w_s}(P) = 1$ and $v_{w_s}(Q) = 0$; so by TN(⊃), $v_{w_s}(P \supset Q) = 0$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$ then $\Gamma' \vdash^*_{NN\alpha} A_s$.

(ii) Suppose $\Gamma' \vdash^*_{NN\alpha} A_s$ but $v_{w_s}(A) = 0$; then $\Gamma' \vdash^*_{NN\alpha} (P \supset Q)_s$ but $v_{w_s}(P \supset Q) = 0$. From the latter, by TN(⊃), $v_{w_s}(P) = 1$ and $v_{w_s}(Q) = 0$; so by assumption, $\Gamma' \vdash^*_{NN\alpha} P_s$ and $\Gamma' \nvdash^*_{NN\alpha} Q_s$; but since $\Gamma' \vdash^*_{NN\alpha} (P \supset Q)_s$ and $\Gamma' \vdash^*_{NN\alpha} P_s$, by (⊃E), $\Gamma' \vdash^*_{NN\alpha} Q_s$. This is impossible; reject the assumption: if $\Gamma' \vdash^*_{NN\alpha} A_s$, then $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NN\alpha} A_s$.

(∧)

(∨)

(≡)

(□) A_s is $\Box P_s$. (i) Suppose $v_{w_s}(A) = 1$ but $\Gamma' \nvdash^*_{NN\alpha} A_s$; then $v_{w_s}(\Box P) = 1$ but $\Gamma' \nvdash^*_{NN\alpha} \Box P_s$. From the former, by TN(□), $w_s \in N$; so by construction, there is some Q such that $\Gamma' \vdash^*_{NN\alpha} \Box Q_s$; from the latter, by s-maximality, $\Gamma' \vdash^*_{NN\alpha} \neg\Box P_s$; so by (∧I), $\Gamma' \vdash^*_{NN\alpha} (\Box Q \wedge \neg\Box P)_s$; so, since Γ' is a scapegoat set, there is some t such that $\Gamma' \vdash^*_{NN\alpha}$ s.t and $\Gamma' \vdash^*_{NN\alpha} \neg P_t$; from the first, by construction, $\langle w_s, w_t\rangle \in R$; and from the second, by consistency, $\Gamma' \nvdash^*_{NN\alpha} P_t$; so by assumption, $v_{w_t}(P) = 0$; but $w_s R w_t$; so by TN(□), $v_{w_s}(\Box P) = 0$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$, then $\Gamma' \vdash^*_{NN\alpha} A_s$.

(ii) Suppose $\Gamma' \vdash^*_{NN\alpha} A_s$ but $v_{w_s}(A) = 0$; then $\Gamma' \vdash^*_{NN\alpha} \Box P_s$ but $v_{w_s}(\Box P) = 0$. From the former, by construction, $w_s \in N$; so with the latter, by TN(□), there is some $w_t \in W$ such that $w_s R w_t$ and $v_{w_t}(P) = 0$; so by assumption, $\Gamma' \nvdash^*_{NN\alpha} P_t$; but since $w_s R w_t$ and $w_s \in N$, by construction, $\Gamma' \vdash^*_{NN\alpha}$ s.t; so by (□E), $\Gamma' \vdash^*_{NN\alpha} P_t$. This is impossible; reject the assumption: if $\Gamma' \vdash_{NN\alpha} A_s$ then $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NN\alpha} A_s$.

(◇) A_s is $\Diamond P_s$. (i) Suppose $v_{w_s}(A) = 1$ but $\Gamma' \nvdash^*_{NN\alpha} A_s$; then $v_{w_s}(\Diamond P) = 1$ but $\Gamma' \nvdash^*_{NN\alpha} \Diamond P_s$; from the latter, by s-maximality, $\Gamma' \vdash^*_{NN\alpha} \neg\Diamond P_s$; so by (MN), $\Gamma' \vdash^*_{NN\alpha} \Box\neg P_s$; so by construction, $w_s \in N$; so, with the former, by TN(◇), there is some $w_t \in W$ such that $w_s R w_t$ and $v_{w_t}(P) = 1$; so by assumption, $\Gamma' \vdash^*_{NN\alpha} P_t$; but since $w_s R w_t$ and $w_s \in N$, by construction, $\Gamma' \vdash^*_{NN\alpha}$ s.t; so by (◇I), $\Gamma' \vdash^*_{NN\alpha} \Diamond P_s$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$ then $\Gamma' \vdash^*_{NN\alpha} A_s$.

(ii) Suppose $\Gamma' \vdash^*_{NN\alpha} A_s$ but $v_{w_s}(A) = 0$; then $\Gamma' \vdash^*_{NN\alpha} \Diamond P_s$ but $v_{w_s}(\Diamond P) = 0$. From the latter, by TN(◇), $w_s \in N$; so by construction, there is some Q such that $\Gamma' \vdash^*_{NN\alpha} \Box Q_s$; from the former, by (MN), $\Gamma' \vdash^*_{NN\alpha} \neg\Box\neg P_s$; so by (∧I), $\Gamma' \vdash^*_{NN\alpha} (\Box Q \wedge$

$\neg\Box\neg P)_s$; so, since Γ' is a scapegoat set, there is some t such that $\Gamma' \vdash^*_{NN\alpha}$ s.t and $\Gamma' \vdash^*_{NN\alpha} \neg\neg P_t$; from the first, by construction, $\langle w_s, w_t \rangle \in R$; from the second, by (DN), $\Gamma' \vdash^*_{NN\alpha} P_t$; so by assumption, $v_{w_t}(P) = 1$; so since $w_s R w_t$ by TN($\Diamond$), $v_{w_s}(\Diamond P) = 1$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$ then $\Gamma' \vdash^*_{NN\alpha} A_s$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NN\alpha} A_s$.

For any A_s, $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NN\alpha} A_s$.

L3.9 If Γ_0 is consistent, then $\langle W, N, R, v \rangle$ constructed as above is an Nα interpretation.

In each case, we need to show that the interpretation meets the condition(s) α. Suppose Γ_0 is consistent.

(η) Suppose α includes condition η and $w_s \in W$. If $w_s \notin N$, then by construction, $\langle w_s, w_s \rangle \in R$ and η is satisfied. So suppose $w_s \in N$. Then by construction, there is some Q such that $\Gamma' \vdash^*_{NN\alpha} \Box Q_s$; so by reasoning as follows,

1	Γ'	
2	$\Box Q_s$	from Γ'
3	| s.t	A (*g*, AMη)
4	| $\top_t$	$\top$ is a tautology
5	| $\Diamond\top_s$	3,4 $\Diamond$I
6	$\Diamond\top_s$	3-5 AMη
7	$\neg\Box\neg\top_s$	6 MN
8	$(\Box Q \wedge \neg\Box\neg\top)_s$	2,7 $\wedge$I

$\Gamma' \vdash^*_{NN\alpha} (\Box Q \wedge \neg\Box\neg\top)_s$; but by L3.7, Γ' is a scapegoat set; so there is a t such that $\Gamma' \vdash^*_{NN\alpha}$ s.t; so by construction, $\langle w_s, w_t \rangle \in R$ and η is satisfied.

(ρ) Suppose α includes condition ρ and $w_s \in W$. Then by construction, s is a subscript in Γ'; so by (AMρ), $\Gamma' \vdash^*_{NN\alpha}$ s.s; so by construction, $\langle w_s, w_s \rangle \in R$ and ρ is satisfied.

(σ) Suppose α includes condition σ and $\langle w_s, w_t \rangle \in R$. If $w_s = w_t$ then σ is satisfied automatically. So suppose $w_s \neq w_t$; then by construction, $\Gamma' \vdash^*_{NN\alpha}$ s.t; so by (AMσ), $\Gamma' \vdash^*_{NN\alpha}$ t.s; so by construction, $\langle w_t, w_s \rangle \in R$ and σ is satisfied.

(τ) Suppose α includes condition τ and $\langle w_s, w_t \rangle, \langle w_t, w_u \rangle \in R$. If $w_s = w_t$ or $w_t = w_u$, then τ is satisfied automatically. So suppose $w_s \neq w_t$ and $w_t \neq w_u$; then by construction, $\Gamma' \vdash^*_{NN\alpha}$ s.t and $\Gamma' \vdash^*_{NN\alpha}$ t.u; so by (AMτ), $\Gamma' \vdash^*_{NN\alpha}$ s.u; so by construction, $\langle w_s, w_u \rangle \in R$ and τ is satisfied.

MAP For any $w_s \in W$, set $m(s) = w_s$; otherwise $m(s)$ is arbitrary.

L3.10 If Γ_0 is consistent, then $v_m(\Gamma_0) = 1$.

Suppose Γ_0 is consistent and $A_0 \in \Gamma_0$; then by construction, $A_0 \in \Gamma'$; so $\Gamma' \vdash^*_{NN\alpha} A_0$; so since Γ_0 is consistent, by L3.8, $v_{w_0}(A) = 1$. And similarly for any $A_0 \in \Gamma_0$. But $m(0) = w_0$; so $v_m(\Gamma_0) = 1$.

Main result: Suppose $\Gamma \models_{N\alpha} A$ but $\Gamma \nvdash_{NN\alpha} A$. Then $\Gamma_0 \models^*_{N\alpha} A_0$ but $\Gamma_0 \nvdash^*_{NN\alpha} A_0$. By (DN), if $\Gamma_0 \vdash^*_{NN\alpha} \neg\neg A_0$, then $\Gamma_0 \vdash^*_{NN\alpha} A_0$; so $\Gamma_0 \nvdash^*_{NN\alpha} \neg\neg A_0$; so by L3.2, $\Gamma_0 \cup \{\neg A_0\}$ is consistent; so by L3.9 and L3.10, there is an $N\alpha$ interpretation $\langle W, N, R, v\rangle_m$ constructed as above such that $v_m(\Gamma_0 \cup \{\neg A_0\}) = 1$; so $v_{m(0)}(\neg A) = 1$; so by TN($\neg$), $v_{m(0)}(A) = 0$; so $v_m(\Gamma_0) = 1$ and $v_{m(0)}(A) = 0$; so by VNα*, $\Gamma_0 \not\models^*_{N\alpha} A_0$. This is impossible; reject the assumption: if $\Gamma \models_{N\alpha} A$, then $\Gamma \vdash_{NN\alpha} A$.

4 CONDITIONAL LOGICS: *Cx* (CH. 5)

4.1 LANGUAGE / SEMANTIC NOTIONS

LCx The VOCABULARY consists of propositional parameters $p_0, p_1 \ldots$ with the operators, $\neg$, $\wedge$, $\vee$, $\supset$, $\equiv$, $\Box$, $\Diamond$ and $>$. Each propositional parameter is a FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$, $(A \equiv B)$, $\Box A$, $\Diamond A$ and $(A > B)$.

ICx Where $\mathfrak{I}$ is the set of all formulas in the language, an INTERPRETATION is $\langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle$ where W is a set of worlds, and v assigns 0 or 1 to parameters at worlds. The middle term is a *set* of access relations: for any formula A, there is an access relation R_A which says which worlds are A-accessible from any w. Say $f_A(w) = \{x \in W \mid wR_Ax\}$, and $[A] = \{w \mid v_w(A) = 1\}$. Then, where *x* is empty or indicates some combination of the following constraints,

(1) $f_A(w) \subseteq [A]$

(2) If $w \in [A]$, then $w \in f_A(w)$

(3) If $[A] \neq \phi$, then $f_A(w) \neq \phi$

(4) If $f_A(w) \subseteq [B]$ and $f_B(w) \subseteq [A]$, then $f_A(w) = f_B(w)$

(5) If $f_A(w) \cap [B] \neq \phi$, then $f_{A\wedge B}(w) \subseteq f_A(w)$

(6) If $x \in f_A(w)$ and $y \in f_A(w)$, then $x = y$

(7) If $x \in [A]$, and $y \in f_A(x)$, then $x = y$

$\langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle$ is a *Cx* interpretation when it meets the constraints from *x*. System C has none of the extra constraints; *C+* is C with constraints (1) - (2); *CS* is C with constraints (1) - (5); *C1* is C with constraints (1) - (5) and (7); *C2* is C with constraints (1) - (5) and (6).

TC For complex expressions,

($\neg$) $v_w(\neg A) = 1$ if $v_w(A) = 0$, and 0 otherwise.

($\wedge$) $v_w(A \wedge B) = 1$ if $v_w(A) = 1$ and $v_w(B) = 1$, and 0 otherwise.

($\vee$) $v_w(A \vee B) = 1$ if $v_w(A) = 1$ or $v_w(B) = 1$, and 0 otherwise.

($\supset$) $v_w(A \supset B) = 1$ if $v_w(A) = 0$ or $v_w(B) = 1$, and 0 otherwise.

($\equiv$) $v_w(A \equiv B) = 1$ if $v_w(A) = v_w(B)$, and 0 otherwise.

$(\Diamond)_\upsilon$ $v_w(\Diamond A) = 1$ if some $x \in W$ has $v_x(A) = 1$, and 0 otherwise.

$(\Box)_\upsilon$ $v_w(\Box A) = 1$ if all $x \in W$ have $v_x(A) = 1$, and 0 otherwise.

($>$) $v_w(A > B) = 1$ iff all $x \in W$ such that wR_Ax have $v_x(B) = 1$.

For a set Γ of formulas, $v_w(\Gamma) = 1$ iff $v_w(A) = 1$ for each $A \in \Gamma$; then,

VCx $\Gamma \models_{Cx} A$ iff there is no *Cx* interpretation $\langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle$ and $w \in W$ such that $v_w(\Gamma) = 1$ and $v_w(A) = 0$.

4.2 NATURAL DERIVATIONS: *NCx*

Derivation systems *NCx* take over $\neg$, $\supset$, $\wedge$, $\vee$, $\equiv$, $\Box$ and $\Diamond$ rules from *NK*υ. Thus modal rules are,

$\Box$Iυ
| | $\top_t$ (assumption)
| | P_t
| $\Box P_s$

where t does not appear in any undischarged premise or assumption

$\Diamond$Eυ
| $\Diamond P_s$
| | P_t (assumption)
| | Q_u
| Q_u

where t does not appear in any undischarged premise or assumption and is not u

$\Box$Eυ
| $\Box P_s$
| P_t

$\Diamond$Iυ
| P_t
| $\Diamond P_s$

For $>$, let there be new subscripted expressions of the sort $A_{s/t}$ – which intuitively say $w_sR_Aw_t$. Expressions of this sort do not interact with other formulas except as follows (and so do not interact with rules of *NK*υ):

$>$I
| | $P_{s/t}$ (assumption)
| | Q_t
| $(P > Q)_s$

where t does not appear in any undischarged premise or assumption

$\not>$E
| $\neg(P > Q)_s$
| | $P_{s/t}$
| | $\neg Q_t$ (assumptions)
| | R_u
| R_u

where t does not appear in any undischarged premise or assumption and is not u

$>$E
| $(P > Q)_s$
| $P_{s/t}$
| Q_t

$\not>$I
| $P_{s/t}$
| $\neg Q_t$
| $\neg(P > Q)_s$

Corresponding to constraints (1) - (7) are AMP1, AMP2, AMS1, AMS2, AMS3, AMRS, and two forms of AMDL. For AMRS $\mathcal{A}_{(t)}$ is an expression of the sort Q_t, $Q_{t/v}$, $Q_{v/t}$ or $Q_{t/t}$ with a subscript t, and $\mathcal{A}_{(u)}$ is like $\mathcal{A}_{(t)}$ except that some instance(s) of t are replaced by u. And similarly for AMDL.

$$\text{AMP1}\;\begin{array}{|l}P_{s/t}\\ \\ P_t\end{array}\qquad \text{AMP2}\;\begin{array}{|l}P_t\\ \\ P_{t/t}\end{array}\qquad \text{AMS1}\;\begin{array}{|l}\Diamond P_s\\ \begin{array}{|l}P_{s/t}\\ \hline \\ Q_u\end{array}\\ Q_u\end{array}\qquad \text{AMS2}\;\begin{array}{|l}(P > Q)_s\\ (Q > P)_s\\ P_{s/t}\\ \\ Q_{s/t}\end{array}$$

where t does not appear in any undischarged premise or assumption and is not u

$$\text{AMS3}\;\begin{array}{|l}\neg(P > \neg Q)_s\\ (P \wedge Q)_{s/t}\\ \\ P_{s/t}\end{array}\qquad \text{AMRS}\;\begin{array}{|l}P_{s/t}\\ P_{s/u}\\ \mathcal{A}_{(t)}\\ \\ \mathcal{A}_{(u)}\end{array}\qquad \text{AMDL}\;\begin{array}{|l}P_s\\ P_{s/t}\\ \mathcal{A}_{(t)}\\ \\ \mathcal{A}_{(s)}\end{array}\quad\begin{array}{|l}P_s\\ P_{s/t}\\ \mathcal{A}_{(s)}\\ \\ \mathcal{A}_{(t)}\end{array}$$

In these systems, every subscript is 0, appears in a premise, or appears in the t-place of an assumption for $\Box$Iυ, $\Diamond$Eυ, $>$I, $\not>$E or AMS1. Intuitively there are ***plus*** rules, rules for the ***sphere*** conception, and rules for the Stalnaker and Lewis alternatives. *NC* includes just the rules of *NK*υ plus $>$I, $>$E, $\not>$I and $\not>$E (but, as below, the latter two are derived). Then,

NC+ has the rules of *NC* plus AMP1, AMP2

NCS has the rules of *NC* plus AMP1, AMP2, AMS1, AMS2, AMS3

NC1 has the rules of *NC* plus AMP1, AMP2, AMS1, AMS2, AMS3, AMDL

NC2 has the rules of *NC* plus AMP1, AMP2, AMS1, AMS2, AMS3, AMRS

Where Γ is a set of unsubscripted formulas, let Γ_0 be those same formulas each with subscript 0. Then,

NCx $\Gamma \vdash_{NCx} A$ iff there is an *NCx* derivation of A_0 from Γ_0.

Derived rules carry over from *NK*α. In addition, as first examples, $\not>$I and $\not>$E are derived rules in *NC*, and so in any *NCx*.

$\not>$I

1	$P_{s/t}$	P
2	$\neg Q_t$	P
3	$(P > Q)_s$	A (c, ¬I)
4	Q_t	1,3 >E
5	$\neg Q_t$	2 R
6	$\neg(P > Q)_s$	3-5 ¬I

$\not>$E

1	$\neg(P > Q)_s$	P
2	$\neg R_u$	A (c, ¬E)
3	$P_{s/t}$	A (g, >I)
4	$\neg Q_t$	A (c, ¬E)
	$\vdots$	from 1,3,4
5	R_u	as for $\not>$E
6	$\neg R_u$	2 R
7	Q_t	4-6 ¬E
8	$(P > Q)_s$	3-7 >I
9	$\neg(P > Q)_s$	1 R
10	R_u	2-9 ¬E

As final examples, here is a case in *NCS* using AMs3 and then again in *NC2* but without appeal to AMs3 (so that AMs3 is not necessary in *NC2* for the result). This last case is a bit messy, but should nicely illustrate use of the rules.

$A > B, \neg(A > \neg C) \vdash_{NCS} (A \wedge C) > B$

1	$(A > B)_0$	P
2	$\neg(A > \neg C)_0$	P
3	$(A \wedge C)_{0/1}$	A (g, >I)
4	$A_{0/1}$	2,3 AMs3
5	B_1	1,4 >E
6	$[(A \wedge C) > B]_0$	3-5 >I

$A > B, \neg(A > \neg C) \vdash_{NC2} (A \wedge C) > B$

1	$(A > B)_0$	P
2	$\neg(A > \neg C)_0$	P
3	$A_{0/1}$	A (g, 2 $\not>$E)
4	$\neg\neg C_1$	
5	$(A \wedge C)_{0/2}$	A (g, >I)
6	$(A \wedge C)_{0/3}$	A (g, >I)
7	$(A \wedge C)_3$	6 AMPi
8	A_3	7 ∧E
9	$[(A \wedge C) > A]_0$	6-8 >I
10	$A_{0/3}$	A (g, >I)
11	A_3	10 AMPi
12	$\neg\neg C_3$	3,10,4 AMRS
13	C_3	12 DN
14	$(A \wedge C)_3$	11,13 ∧I
15	$[A > (A \wedge C)]_0$	10-14 >I
16	$A_{0/2}$	9,15,5 AMs2
17	B_2	1,16 >E
18	$[(A \wedge C) > B]_0$	5-17 >I
19	$[(A \wedge C) > B]_0$	2,3-18 $\not>$E

The derivation on the left is a simple application of AMs3. On the right, we go for the final goal by $\not>$E.[4] The real work is getting $A_{0/2}$ so that we can use >E with (1). And we go for this by getting the conditionals that feed into AMs2, given that we already have $(A \wedge C)_{0/2}$.

[4] As, given strategies from [12, chapter 6], we would jump on ∨E, ∃E or ◇E when available.

4.3 SOUNDNESS AND COMPLETENESS

Preliminaries: Begin with generalized notions of validity. For a model $\langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle$, let m be a map from subscripts into W. Say $\langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle_m$ is $\langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle$ ***with*** map m. Then, where Γ is a set of expressions of our language for derivations, $v_m(\Gamma) = 1$ iff for each $A_s \in \Gamma$, $v_{m(s)}(A) = 1$, and for each $A_{s/t} \in \Gamma$, $m(t) \in f_A(m(s))$. Now expand notions of validity to include subscripted formulas, and alternate expressions as indicated in double brackets.

VCx* $\Gamma \models^*_{Cx} A_s$ $[\![A_{s/t}]\!]$ iff there is no *Cx* interpretation $\langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle_m$ such that $v_m(\Gamma) = 1$ but $v_{m(s)}(A) = 0$ $[\![m(t) \notin f_A(m(s))]\!]$.

NCx* $\Gamma \vdash^*_{NCx} A_s$ $[\![A_{s/t}]\!]$ iff there is an *NCx* derivation of A_s $[\![A_{s/t}]\!]$ from the members of Γ.

These notions reduce to the standard ones when all the members of Γ and A have subscript 0 (and so do not include expressions of the sort $A_{s/t}$). This is obvious for NCx*. In the other case, there is a $\langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle_m$ and $w \in W$ that makes all the members of Γ_0 true and A_0 false just in case there ***is*** a world in W that makes the unsubscripted members of Γ true and A false. For the following, cases omitted are like ones worked, and so left to the reader.

THEOREM 4.1 *NCx is sound: **If*** $\Gamma \vdash_{NCx} A$ ***then*** $\Gamma \models_{Cx} A$.

L4.1 If $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{Cx} P_s$ $[\![P_{s/t}]\!]$, then $\Gamma' \models^*_{Cx} P_s$ $[\![P_{s/t}]\!]$.
Reasoning parallel to that for L2.1 of *NK*α.

Main result: For each line in a derivation let $\mathcal{P}_i$ be the expression on line i and Γ_i be the set of all premises and assumptions whose scope includes line i. We set out to show "generalized" soundness: if $\Gamma \vdash^*_{NCx} \mathcal{P}$ then $\Gamma \models^*_{Cx} \mathcal{P}$. Suppose $\Gamma \vdash^*_{NCx} \mathcal{P}$. Then there is a derivation of $\mathcal{P}$ from premises in Γ where $\mathcal{P}$ appears under the scope of the premises alone. By induction on line number of this derivation, we show that for each line i of this derivation, $\Gamma_i \models^*_{Cx} \mathcal{P}_i$. The case when $\mathcal{P}_i = \mathcal{P}$ is the desired result.

Basis: $\mathcal{P}_1$ is a premise or an assumption A_s $[\![A_{s/t}]\!]$. Then $\Gamma_1 = \{A_s\}$ $[\![\{A_{s/t}\}]\!]$; so for any $\langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle_m$, $v_m(\Gamma_1) = 1$ iff $v_{m(s)}(A) = 1$ $[\![m(t) \in f_A(m(s))]\!]$; so there is no $\langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle_m$ such that $v_m(\Gamma_1) = 1$ but $v_{m(s)}(A) = 0$ $[\![m(t) \notin f_A(m(s))]\!]$. So by VCx*, $\Gamma_1 \models^*_{Cx} A_s$ $[\![A_{s/t}]\!]$, where this is just to say, $\Gamma_1 \models^*_{Cx} \mathcal{P}_1$.

Assp: For any $i, 1 \leqslant i < k, \Gamma_i \models^*_{Cx} \mathcal{P}_i$.

Show: $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

$\mathcal{P}_k$ is either a premise, an assumption, or arises from previous lines by R, $\supset$I, $\supset$E, $\wedge$I, $\wedge$E, $\neg$I, $\neg$E, $\vee$I, $\vee$E, $\equiv$I, $\equiv$E, $\Box$Iυ, $\Box$Eυ, $\Diamond$Iυ, $\Diamond$Eυ, $>$I, $>$E or, depending on the system, AMP1, AMP2, AMS1, AMS2, AMS3, AMRS or AMDL. If $\mathcal{P}_k$ is a premise or an assumption, then as in the basis, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$. So suppose $\mathcal{P}_k$ arises by one of the rules.

(R)

($\supset$I)

($\supset$E) If $\mathcal{P}_k$ arises by $\supset$E, then the picture is like this,

i	$(A \supset B)_s$
j	A_s
k	B_s

where $i, j < k$ and $\mathcal{P}_k$ is B_s. By assumption, $\Gamma_i \models^*_{Cx} (A \supset B)_s$ and $\Gamma_j \models^*_{Cx} A_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L4.1, $\Gamma_k \models^*_{Cx} (A \supset B)_s$ and $\Gamma_k \models^*_{Cx} A_s$. Suppose $\Gamma_k \not\models^*_{Cx} B_s$; then by VCx*, there is some *Cx* interpretation $\langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(B) = 0$; since $v_m(\Gamma_k) = 1$, by VCx*, $v_{m(s)}(A \supset B) = 1$ and $v_{m(s)}(A) = 1$; from the former, by TC($\supset$), $v_{m(s)}(A) = 0$ or $v_{m(s)}(B) = 1$; so $v_{m(s)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} B_s$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

($\wedge$I)

($\wedge$E)

($\neg$I) If $\mathcal{P}_k$ arises by $\neg$I, then the picture is like this,

		A_s
i		B_t
j		$\neg B_t$
k	$\neg A_s$	

where $i, j < k$ and $\mathcal{P}_k$ is $\neg A_s$. By assumption, $\Gamma_i \models^*_{Cx} B_t$ and $\Gamma_j \models^*_{Cx} \neg B_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{A_s\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{A_s\}$; so by L4.1, $\Gamma_k \cup \{A_s\} \models^*_{Cx} B_t$ and $\Gamma_k \cup \{A_s\} \models^*_{Cx} \neg B_t$. Suppose $\Gamma_k \not\models^*_{Cx} \neg A_s$; then by VCx*, there is a *Cx* interpretation $\langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(\neg A) = 0$; so by TC($\neg$), $v_{m(s)}(A) = 1$; so $v_m(\Gamma_k) = 1$ and $v_{m(s)}(A) = 1$; so $v_m(\Gamma_k \cup \{A_s\}) = 1$; so by VCx*, $v_{m(t)}(B) = 1$ and $v_{m(t)}(\neg B) = 1$; from the latter, by TC($\neg$), $v_{m(t)}(B) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} \neg A_s$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

($\neg$E)

($\vee$I)

($\vee$E)

($\equiv$I)

($\equiv$E)

($\Box$Iυ) If $\mathcal{P}_k$ arises by $\Box$Iυ, then the picture is like this,

$$\begin{array}{r|l} & \quad \underline{\top_t} \\ & \quad \vdots \\ i & \quad A_t \\ k & \Box A_s \end{array}$$

where $i < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $\Box A_s$. By assumption, $\Gamma_i \models^*_{Cx} A_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{\top_t\}$; so by L4.1, $\Gamma_k \cup \{\top_t\} \models^*_{Cx} A_t$. Suppose $\Gamma_k \not\models^*_{Cx} \Box A_s$; then by VCx*, there is a *Cx* interpretation $\langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(\Box A) = 0$; so by TC($\Box$)$_\upsilon$, there is some $w \in W$ such that $v_w(A) = 0$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle_{m'}$; since t does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and, at any world, $v_{m'(t)}(\top) = 1$; so $v_{m'}(\Gamma_k \cup \{\top_t\}) = 1$; so by VCx*, $v_{m'(t)}(A) = 1$. But $m'(t) = w$; so $v_w(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} \Box A_s$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

($\Box$Eυ) If $\mathcal{P}_k$ arises by $\Box$Eυ, then the picture is like this,

$$\begin{array}{r|l} i & \Box A_s \\ & \vdots \\ k & A_t \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is A_t. By assumption, $\Gamma_i \models^*_{Cx} \Box A_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L4.1, $\Gamma_k \models^*_{Cx} \Box A_s$. Suppose $\Gamma_k \not\models^*_{Cx} A_t$; then by VCx*, there is some *Cx* interpretation $\langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(t)}(A) = 0$; since $v_m(\Gamma_k) = 1$, by VCx*, $v_{m(s)}(\Box A) = 1$; so by TC($\Box$)$_\upsilon$, any w has $v_w(A) = 1$; so $v_{m(t)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} A_t$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

($\Diamond$Iυ)

($\Diamond$Eυ) If $\mathcal{P}_k$ arises by $\Diamond$Eυ, then the picture is like this,

$$\begin{array}{r|l} i & \Diamond A_s \\ & \quad \underline{A_t} \\ & \quad \vdots \\ j & \quad B_u \\ k & B_u \end{array}$$

where $i, j < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption) and is not u, and $\mathcal{P}_k$ is B_u. By assumption, $\Gamma_i \models^*_{Cx} \Diamond A_s$ and $\Gamma_j \models^*_{Cx} B_u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k \cup \{A_t\}$; so by L4.1, $\Gamma_k \models^*_{Cx} \Diamond A_s$ and $\Gamma_k \cup \{A_t\} \models^*_{Cx} B_u$. Suppose $\Gamma_k \not\models^*_{Cx} B_u$; then by VCx*, there is a *Cx* interpretation $\langle W, \{R_A \mid A \in \mathfrak{I}\}, \nu\rangle_m$ such that $\nu_m(\Gamma_k) = 1$ but $\nu_{m(u)}(B) = 0$. Since $\nu_m(\Gamma_k) = 1$, by VCx*, $\nu_{m(s)}(\Diamond A) = 1$; so by TC($\Diamond$)$_\upsilon$, there is some $w \in W$ such that $\nu_w(A) = 1$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, \{R_A \mid A \in \mathfrak{I}\}, \nu\rangle_{m'}$; since t does not appear in Γ_k, it remains that $\nu_{m'}(\Gamma_k) = 1$; and since $m'(t) = w$, $\nu_{m'(t)}(A) = 1$; so $\nu_{m'}(\Gamma_k \cup \{A_t\}) = 1$; so by VCx*, $\nu_{m'(u)}(B) = 1$. But since $t \neq u$, $m'(u) = m(u)$; so $\nu_{m(u)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} B_u$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

(>I) If $\mathcal{P}_k$ arises by >I, then the picture is like this,

	$A_{s/t}$
	⋮
i	B_t
k	$(A > B)_s$

where $i < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $(A > B)_s$. By assumption, $\Gamma_i \models^*_{Cx} B_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{A_{s/t}\}$; so by L4.1, $\Gamma_k \cup \{A_{s/t}\} \models^*_{Cx} B_t$. Suppose $\Gamma_k \not\models^*_{Cx} (A > B)_s$; then by VCx*, there is a *Cx* interpretation $\langle W, \{R_A \mid A \in \mathfrak{I}\}, \nu\rangle_m$ such that $\nu_m(\Gamma_k) = 1$ but $\nu_{m(s)}(A > B) = 0$; so by TC(>), there is some $w \in W$ such that $m(s) R_A w$ but $\nu_w(B) = 0$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, \{R_A | A \in \mathfrak{I}\}, \nu\rangle_{m'}$; since t does not appear in Γ_k, it remains that $\nu_{m'}(\Gamma_k) = 1$; and since $m'(t) = w$ and $m'(s) = m(s)$, $\langle m'(s), m'(t)\rangle \in R_A$; so $\nu_{m'}(\Gamma_k \cup \{A_{s/t}\}) = 1$; so by VCx*, $\nu_{m'(t)}(B) = 1$. But $m'(t) = w$; so $\nu_w(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} (A > B)_s$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

(>E) If $\mathcal{P}_k$ arises by >E, then the picture is like this,

i	$(A > B)_s$
j	$A_{s/t}$
	⋮
k	B_t

where $i, j < k$ and $\mathcal{P}_k$ is B_t. By assumption, $\Gamma_i \models^*_{Cx} (A > B)_s$ and $\Gamma_j \models^*_{Cx} A_{s/t}$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L4.1, $\Gamma_k \models^*_{Cx} (A > B)_s$ and $\Gamma_k \models^*_{Cx} A_{s/t}$. Suppose $\Gamma_k \not\models^*_{Cx} B_t$; then by VCx*, there is some *Cx* interpretation $\langle W, \{R_A \mid A \in \mathfrak{I}\}, \nu\rangle_m$ such that $\nu_m(\Gamma_k) = 1$ but $\nu_{m(t)}(B) = 0$; since $\nu_m(\Gamma_k) = 1$, by VCx*, $\nu_{m(s)}(A > B) = 1$ and $\langle m(s), m(t)\rangle \in R_A$; from the former, by TC(>), any $w \in W$ such that

$m(s)R_A w$ has $v_w(B) = 1$; so $v_{m(t)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} B_t$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

(AMP1) If $\mathcal{P}_k$ arises by AMP1, then the picture is like this,

$$\begin{array}{r|l} i & A_{s/t} \\ & \\ k & A_t \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is A_t. Where this rule is in *NCx*, *Cx* includes condition (1). By assumption, $\Gamma_i \models^*_{Cx} A_{s/t}$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L4.1, $\Gamma_k \models^*_{Cx} A_{s/t}$. Suppose $\Gamma_k \not\models^*_{Cx} A_t$; then by VCx*, there is some *Cx* interpretation $\langle W, \{R_A | A \in \mathfrak{I}\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(t)}(A) = 0$; since $v_m(\Gamma_k) = 1$, by VCx*, $m(t) \in f_A(m(s))$; so by condition (1), $m(t) \in [A]$; so $v_{m(t)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} A_t$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

(AMP2) If $\mathcal{P}_k$ arises by AMP2, then the picture is like this,

$$\begin{array}{r|l} i & A_t \\ & \\ k & A_{t/t} \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is $A_{t/t}$. Where this rule is in *NCx*, *Cx* includes condition (2). By assumption, $\Gamma_i \models^*_{Cx} A_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L4.1, $\Gamma_k \models^*_{Cx} A_t$. Suppose $\Gamma_k \not\models^*_{Cx} A_{t/t}$; then by VCx*, there is some *Cx* interpretation $\langle W, \{R_A | A \in \mathfrak{I}\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $m(t) \notin f_A(m(t))$; since $v_m(\Gamma_k) = 1$, by VCx*, $v_{m(t)}(A) = 1$; so $m(t) \in [A]$; so by condition (2), $m(t) \in f_A(m(t))$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} A_{t/t}$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

(AMS1) If $\mathcal{P}_k$ arises by AMS1, then the picture is like this,

$$\begin{array}{r|l} i & \Diamond A_s \\ & \quad \underline{|\, A_{s/t}} \\ & \quad | \\ j & \quad |\, B_u \\ k & B_u \end{array}$$

where $i, j < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption) and is not u, and $\mathcal{P}_k$ is B_u. Where this rule is in *NCx*, *Cx* includes condition (3). By assumption, $\Gamma_i \models^*_{Cx} \Diamond A_s$ and $\Gamma_j \models^*_{Cx} B_u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k \cup \{A_{s/t}\}$; so by L4.1, $\Gamma_k \models^*_{Cx} \Diamond A_s$ and $\Gamma_k \cup \{A_{s/t}\} \models^*_{Cx} B_u$. Suppose $\Gamma_k \not\models^*_{Cx} B_u$; then by VCx*, there is a *Cx* interpretation $\langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(u)}(B) = 0$. Since $v_m(\Gamma_k) = 1$, by VCx*, $v_{m(s)}(\Diamond A) = 1$; so by TC($\Diamond$)$_v$, there is some $w \in W$ such that $v_w(A) = 1$; so $w \in [A]$ and $[A] \neq \phi$; so by condition (3), $f_A(m(s)) \neq \phi$; so there is some $x \in f_A(m(s))$. Now consider a map m' like m except that $m'(t) = x$, and consider $\langle W, \{R_A \mid A \in \mathfrak{I}\}, \nu\rangle_{m'}$; since t does not appear in Γ_k, it remains that $\nu_{m'}(\Gamma_k) = 1$; and since $m'(t) = x$ and $m'(s) = m(s)$, $m'(t) \in f_A(m'(s))$; so $\nu_{m'}(\Gamma_k) = 1$ and $\langle m'(s), m'(t)\rangle \in R_A$; so $\nu_{m'}(\Gamma_k \cup \{A_{s/t}\}) = 1$; so by VCx*, $\nu_{m'(u)}(B) = 1$. But since $t \neq u$, $m'(u) = m(u)$; so $\nu_{m(u)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} B_u$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

(AMS2) If $\mathcal{P}_k$ arises by AMS2, then the picture is like this,

$$\begin{array}{r|l} h & (A > B)_s \\ i & (B > A)_s \\ j & A_{s/t} \\ & \\ k & B_{s/t} \end{array}$$

where $h, i, j < k$ and $\mathcal{P}_k$ is $B_{s/t}$. Where this rule is in *NCx*, *Cx* includes condition (4). By assumption, $\Gamma_h \models^*_{Cx} (A > B)_s$, $\Gamma_i \models^*_{Cx} (B > A)_s$ and $\Gamma_j \models^*_{Cx} A_{s/t}$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L4.1, $\Gamma_k \models^*_{Cx} (A > B)_s$, $\Gamma_k \models^*_{Cx} (B > A)_s$, and $\Gamma_k \models^*_{Cx} A_{s/t}$. Suppose $\Gamma_k \not\models^*_{Cx} B_{s/t}$; then by VCx*, there is some *Cx* interpretation $\langle W, \{R_A \mid A \in \mathfrak{I}\}, \nu\rangle_m$ such that $\nu_m(\Gamma_k) = 1$ but $m(t) \notin f_B(m(s))$; since $\nu_m(\Gamma_k) = 1$, by VCx*, $\nu_{m(s)}(A > B) = 1$, $\nu_{m(s)}(B > A) = 1$; and $m(t) \in f_A(m(s))$. Suppose $w \in f_A(m(s))$; then $m(s)R_Aw$ and since $\nu_{m(s)}(A > B) = 1$, by TC(>), $\nu_w(B) = 1$; so $w \in [B]$ and, generalizing, we have that $f_A(m(s)) \subseteq [B]$. Suppose $w \in f_B(m(s))$; then $m(s)R_Bw$ and since $\nu_{m(s)}(B > A) = 1$, by TC(>), $\nu_w(A) = 1$; so $w \in [A]$ and, generalizing, we have that $f_B(m(s)) \subseteq [A]$. So $f_A(m(s)) \subseteq [B]$ and $f_B(m(s)) \subseteq [A]$; so by condition (4), $f_A(m(s)) = f_B(m(s))$; thus since $m(t) \in f_A(m(s))$, $m(t) \in f_B(m(s))$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} B_{s/t}$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

(AMS3) If $\mathcal{P}_k$ arises by AMS3, then the picture is like this,

$$\begin{array}{r|l} i & \neg(A > \neg B)_s \\ j & (A \wedge B)_{s/t} \\ & \\ k & A_{s/t} \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $A_{s/t}$. Where this rule is in *NCx*, *Cx* includes condition (5). By assumption, $\Gamma_i \models^*_{Cx} \neg(A > \neg B)_s$ and $\Gamma_j \models^*_{Cx} (A \wedge B)_{s/t}$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L4.1, $\Gamma_k \models^*_{Cx} \neg(A > \neg B)_s$, and $\Gamma_k \models^*_{Cx} (A \wedge B)_{s/t}$. Suppose $\Gamma_k \not\models^*_{Cx} A_{s/t}$; then by VCx*, there is some *Cx* interpretation $\langle W, \{R_A \mid A \in \mathfrak{I}\}, \nu\rangle_m$ such that $\nu_m(\Gamma_k) = 1$ but $m(t) \notin f_A(m(s))$; since $\nu_m(\Gamma_k) = 1$, by VCx*, $\nu_{m(s)}(\neg(A > \neg B)) = 1$, and $m(t) \in f_{A \wedge B}(m(s))$. Since $\nu_{m(s)}(\neg(A > \neg B)) = 1$, by TC(¬), $\nu_{m(s)}(A > \neg B) = 0$; so by TC(>), there is some

$w \in W$ such that $m(s)R_A w$ and $v_w(\neg B) = 0$; so by TC($\neg$), $v_w(B) = 1$; but $w \in f_A(m(s))$; so $f_A(m(s)) \cap [B] \neq \phi$; so by condition (5), $f_{A\wedge B}(m(s)) \subseteq f_A(m(s))$; so $m(t) \in f_A(m(s))$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} A_{s/t}$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

(AMRS) If $\mathcal{P}_k$ arises by AMRS, then the picture is like this,

$$\begin{array}{r|l} h & A_{s/t} \\ i & A_{s/u} \\ j & \mathcal{Q}_{(t)} \\ & \\ k & \mathcal{Q}_{(u)} \end{array}$$

where $h, i, j < k$ and $\mathcal{P}_k$ is $\mathcal{Q}_{(u)}$. Suppose $\mathcal{Q}_{(t)}$ is some B_t and $\mathcal{Q}_{(u)}$ is B_u. Where this rule is in *NCx*, *Cx* includes condition (6). By assumption, $\Gamma_h \models^*_{Cx} A_{s/t}$, $\Gamma_i \models^*_{Cx} A_{s/u}$ and $\Gamma_j \models^*_{Cx} B_t$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L4.1, $\Gamma_k \models^*_{Cx} A_{s/t}$, $\Gamma_k \models^*_{Cx} A_{s/u}$, and $\Gamma_k \models^*_{Cx} B_t$. Suppose $\Gamma_k \not\models^*_{Cx} B_u$; then by VCx*, there is some *Cx* interpretation $\langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(u)}(B) = 0$; since $v_m(\Gamma_k) = 1$, by VCx*, $m(t) \in f_A(m(s))$, $m(u) \in f_A(m(s))$, and $v_{m(t)}(B) = 1$. With the first two of these, by condition (6), $m(t) = m(u)$; so $v_{m(u)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} B_u$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$. And similarly when $\mathcal{Q}_{(t)}$ is $B_{t/v}$, $B_{v/t}$, or $B_{t/t}$.

(AMDL) If $\mathcal{P}_k$ arises by AMDL, then the picture is like this,

$$\begin{array}{r|l} h & A_{s} \\ i & A_{s/t} \\ j & \mathcal{Q}_{(t)} \\ & \\ k & \mathcal{Q}_{(s)} \end{array} \qquad \text{or} \qquad \begin{array}{r|l} h & A_{s} \\ i & A_{s/t} \\ j & \mathcal{Q}_{(s)} \\ & \\ k & \mathcal{Q}_{(t)} \end{array}$$

where $h, i, j < k$ and, in the left-hand case, $\mathcal{P}_k$ is $\mathcal{Q}_{(s)}$. Suppose $\mathcal{Q}_{(t)}$ is of the sort $B_{t/v}$ and $\mathcal{Q}_{(s)}$ is $B_{s/v}$. Where this rule is in *NCx*, *Cx* includes condition (7). By assumption, $\Gamma_h \models^*_{Cx} A_s$, $\Gamma_i \models^*_{Cx} A_{s/t}$ and $\Gamma_j \models^*_{Cx} B_{t/v}$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L4.1, $\Gamma_k \models^*_{Cx} A_s$, $\Gamma_k \models^*_{Cx} A_{s/t}$, and $\Gamma_k \models^*_{Cx} B_{t/v}$. Suppose $\Gamma_k \not\models^*_{Cx} B_{s/v}$; then by VCx*, there is some *Cx* interpretation $\langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $\langle m(s), m(v)\rangle \notin R_B$; since $v_m(\Gamma_k) = 1$, by VCx*, $v_{m(s)}(A) = 1$, $m(t) \in f_A(m(s))$, and $\langle m(t), m(v)\rangle \in R_B$. From the first of these, $m(s) \in [A]$; so by condition (7), $m(s) = m(t)$; so $\langle m(s), m(v)\rangle \in R_B$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} B_{s/v}$ which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$. And similarly when $\mathcal{Q}_{(t)}$ is B_t, $B_{v/t}$ or $B_{t/t}$. And similarly in the right-hand case.

For any i, $\Gamma_i \models^*_{Cx} \mathcal{P}_i$.

THEOREM 4.2 *NCx is complete: if* $\Gamma \models_{Cx} A$ *then* $\Gamma \vdash_{NCx} A$.

Suppose $\Gamma \models_{Cx} A$; then $\Gamma_0 \models^*_{Cx} A_0$; we show that $\Gamma_0 \vdash^*_{NCx} A_0$. Again, this reduces to the standard notion. For the following, fix on some particular constraint(s) x. Then definitions of *consistency* etc. are relative to it.

CON Γ is CONSISTENT iff there is no A_s such that $\Gamma \vdash^*_{NCx} A_s$ and $\Gamma \vdash^*_{NCx} \neg A_s$.

L4.2 If s is 0 or appears in Γ, and $\Gamma \nvdash^*_{NCx} \neg P_s$, then $\Gamma \cup \{P_s\}$ is consistent.

Reasoning parallel to L2.2 for *NK*α.

L4.3 There is an enumeration of all the subscripted formulas, $\mathcal{P}_1\ \mathcal{P}_2 \ldots$

Proof by construction as for L2.3 for *NK*α.

MAX Γ is S-MAXIMAL iff for any A_s either $\Gamma \vdash^*_{NCx} A_s$ or $\Gamma \vdash^*_{NCx} \neg A_s$.

SGT Γ is a SCAPEGOAT set for $\Box$ iff for every formula of the form $\neg\Box A_s$, if $\Gamma \vdash^*_{NCx} \neg\Box A_s$ then there is some t such that $\Gamma \vdash^*_{NCx} \neg A_t$.

Γ is a SCAPEGOAT set for $>$ iff for any formula of the form $\neg(A > B)_s$, if $\Gamma \vdash^*_{NCx} \neg(A > B)_s$ then there is some t such that $\Gamma \vdash^*_{NCx} A_{s/t}$ and $\Gamma \vdash^*_{NCx} \neg B_t$.

C(Γ') For Γ with unsubscripted formulas and the corresponding Γ_0, we construct Γ' as follows. Set $\Omega_0 = \Gamma_0$. By L4.3, there is an enumeration, $\mathcal{P}_1, \mathcal{P}_2 \ldots$ of all the subscripted formulas; let $\mathcal{E}_0$ be this enumeration. Then for the first A_s in $\mathcal{E}_{i-1}$ such that s is 0 or included in Ω_{i-1}, let $\mathcal{E}_i$ be like $\mathcal{E}_{i-1}$ but without A_s, and set,

$\Omega_i = \Omega_{i-1}$ if $\Omega_{i-1} \vdash^*_{NCx} \neg A_s$

$\Omega_{i^*} = \Omega_{i-1} \cup \{A_s\}$ if $\Omega_{i-1} \nvdash^*_{NCx} \neg A_s$

and

$\Omega_i = \Omega_{i^*}$ if A_s is not of the form $\neg\Box P_s$ or $\neg(P > Q)_s$

$\Omega_i = \Omega_{i^*} \cup \{\neg P_t\}$ if A_s is of the form $\neg\Box P_s$

$\Omega_i = \Omega_{i^*} \cup \{P_{s/t}, \neg Q_t\}$ if A_s is of the form $\neg(P > Q)_s$

-where t is the first subscript not included in Ω_{i^*}

then

$\Gamma' = \bigcup_{i \geqslant 0} \Omega_i$

Note that there is always sure to be a subscript t not in Ω_{i^*} insofar as there are infinitely many subscripts, and at any stage only finitely many formulas are added – the only subscripts in the initial Ω_0 being 0. Suppose s is introduced in Γ'; then there is some Ω_i in which it is first introduced; and any formula $\mathcal{P}_j$ in the original enumeration that has subscript s is sure to be "considered" for inclusion at a subsequent stage.

L4.4 For any s included in Γ', Γ' is s-maximal.

Reasoning parallel to L2.4 for *NK*α.

L4.5 If Γ_0 is consistent, then each Ω_i is consistent.

Suppose Γ_0 is consistent.

Basis: $\Omega_0 = \Gamma_0$ and Γ_0 is consistent; so Ω_0 is consistent.

Assp: For any $i, 0 \leqslant i < k$, Ω_i is consistent.

Show: Ω_k is consistent.

Ω_k is either (i) Ω_{k-1}, (ii) $\Omega_{k^*} = \Omega_{k-1} \cup \{A_s\}$, (iii) $\Omega_{k^*} \cup \{\neg P_t\}$ or (iv) $\Omega_{k^*} \cup \{P_{s/t}, \neg Q_t\}$.

(i) Suppose Ω_k is Ω_{k-1}. By assumption, Ω_{k-1} is consistent; so Ω_k is consistent.

(ii) Suppose Ω_k is $\Omega_{k^*} = \Omega_{k-1} \cup \{A_s\}$. Then by construction, s is 0 or in Ω_{k-1} and $\Omega_{k-1} \nvdash^*_{NCx} \neg A_s$; so by L4.2, $\Omega_{k-1} \cup \{A_s\}$ is consistent; so Ω_k is consistent.

(iii) Suppose Ω_k is $\Omega_{k^*} \cup \{\neg P_t\}$. In this case, as above, Ω_{k*} is consistent and by construction, $\neg\Box P_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are A_u and $\neg A_u$ such that $\Omega_{k^*} \cup \{\neg P_t\} \vdash^*_{NCx} A_u$ and $\Omega_{k^*} \cup \{\neg P_t\} \vdash^*_{NCx} \neg A_u$. So reason as follows,

1	Ω_{k^*}	
2	$\quad \top_t$	A (g, $\Box$Iυ)
3	$\quad\quad \neg P_t$	A (c, $\neg$E)
4	$\quad\quad A_u$	from $\Omega_{k^*} \cup \{\neg P_t\}$
5	$\quad\quad \neg A_u$	from $\Omega_{k^*} \cup \{\neg P_t\}$
6	$\quad P_t$	3-5 $\neg$E
7	$\Box P_s$	2-6 $\Box$Iυ

where, by construction, t is not in Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NCx} \Box P_s$; but $\neg\Box P_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{NCx} \neg\Box P_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

(iv) Suppose Ω_k is $\Omega_{k^*} \cup \{P_{s/t}, \neg Q_t\}$. In this case, as above, Ω_{k*} is consistent and by construction, $\neg(P > Q)_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are A_u and $\neg A_u$ such that $\Omega_{k^*} \cup \{P_{s/t}, \neg Q_t\} \vdash^*_{NCx} A_u$ and $\Omega_{k^*} \cup \{P_{s/t}, \neg Q_t\} \vdash^*_{NCx} \neg A_u$. So reason as follows,

1	Ω_{k^*}	
2	$\quad P_{s/t}$	A (g, $>$I)
3	$\quad\quad \neg Q_t$	A (c, $\neg$E)
4	$\quad\quad A_u$	from $\Omega_{k^*} \cup \{P_{s/t}, \neg Q_t\}$
5	$\quad\quad \neg A_u$	from $\Omega_{k^*} \cup \{P_{s/t}, \neg Q_t\}$
6	$\quad Q_t$	3-5 $\neg$E
7	$(P > Q)_s$	2-6 $>$I

where, by construction, t is not in Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NCx} (P > Q)_s$; but $\neg(P > Q)_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{NCx} \neg(P > Q)_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

For any i, Ω_i is consistent.

L4.6 If Γ_0 is consistent, then Γ' is consistent.

Reasoning parallel to L2.6 for *NKα*.

L4.7 If Γ_0 is consistent, then Γ' is a scapegoat set for $\Box$ and $>$.

For $\Box$. Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{NCx} \neg\Box P_s$. By L4.6, Γ' is consistent; and by the constraints on subscripts, s is included in Γ'. Since Γ' is consistent, $\Gamma' \nvdash^*_{NCx} \neg\neg\Box P_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{\neg\Box P_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{\neg P_t\}$; so by construction, $\neg P_t \in \Gamma'$; so $\Gamma' \vdash^*_{NCx} \neg P_t$. So Γ' is a scapegoat set for $\Box$.

For $>$. Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{NCx} \neg(P > Q)_s$. By L4.6, Γ' is consistent; and by the constraints on subscripts, s is included in Γ'. Since Γ' is consistent, $\Gamma' \nvdash^*_{NCx} \neg\neg(P > Q)_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{\neg(P > Q)_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{P_{s/t}, \neg Q_t\}$; so by construction, $P_{s/t} \in \Gamma'$ and $\neg Q_t \in \Gamma'$; so $\Gamma' \vdash^*_{NCx} P_{s/t}$ and $\Gamma' \vdash^*_{NCx} \neg Q_t$. So Γ' is a scapegoat set for $>$.

C(I) We construct an interpretation $I = \langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle$ based on Γ' as follows. Let W have a member w_s corresponding to each subscript s included in Γ', except that in *C1*, if there is some A such that $\Gamma' \vdash^*_{NC1} A_s$ and $\Gamma' \vdash^*_{NC1} A_{s/t}$ then $w_s = w_t$, and in *C2*, if there is some A such that $\Gamma' \vdash^*_{NC2} A_{s/t}$ and $\Gamma' \vdash^*_{NC2} A_{s/u}$ then $w_t = w_u$ (we could do this, in the usual way, by establishing equivalence classes from members of W). Then $\langle w_s, w_t\rangle \in R_A$ iff $\Gamma' \vdash^*_{NCx} A_{s/t}$; and $v_{w_s}(p) = 1$ iff $\Gamma' \vdash^*_{NCx} p_s$.

Note that the specification is consistent for *C1* and *C2*: Say $\mathcal{P}_{(s)}$ is some p_s, $P_{s/v}$, $P_{v/s}$ or $P_{s/s}$. (i) Suppose $w_s = w_t$ and $\Gamma' \vdash^*_{NC1} \mathcal{P}_{(s)}$. Since $w_s = w_t$ there is some A such that $\Gamma' \vdash^*_{NC1} A_s$ and $\Gamma' \vdash^*_{NC1} A_{s/t}$; so by AMDL, $\Gamma' \vdash^*_{NC1} \mathcal{P}_{(t)}$. And similarly if $w_s = w_t$ and $\Gamma' \vdash^*_{NC1} \mathcal{P}_{(t)}$, then $\Gamma' \vdash^*_{NC1} \mathcal{P}_{(s)}$. (ii) Suppose $w_t = w_u$ and $\Gamma' \vdash^*_{NC2} \mathcal{P}_{(t)}$. Since $w_t = w_u$, there is some A such that $\Gamma' \vdash^*_{NC2} A_{s/t}$ and $\Gamma' \vdash^*_{NC2} A_{s/u}$; so by AMRS, $\Gamma' \vdash^*_{NC2} \mathcal{P}_{(u)}$. And similarly if $w_t = w_u$ and $\Gamma' \vdash^*_{NC2} \mathcal{P}_{(u)}$, then $\Gamma' \vdash^*_{NC2} \mathcal{P}_{(t)}$.

L4.8 If Γ_0 is consistent then for $\langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle$ constructed as above, and for any s included in Γ', $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NCx} A_s$.

Suppose Γ_0 is consistent and s is included in Γ'. By L4.4, Γ' is s-maximal. By L4.6 and L4.7, Γ' is consistent and a scapegoat set for $\Box$ and $>$. Now by induction on the number of operators in A_s, *Basis:* If A_s has no operators, then it is a parameter p_s and by construction, $v_{w_s}(p) = 1$ iff $\Gamma' \vdash^*_{NCx} p_s$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NCx} A_s$.

Assp: For any i, $0 \leqslant i < k$, if A_s has i operators, then $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NCx} A_s$.

Show: If A_s has k operators, then $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NCx} A_s$.

If A_s has k operators, then it is of the form $\neg P_s$, $(P \supset Q)_s$, $(P \wedge Q)_s$, $(P \vee Q)_s$, $(P \equiv Q)_s$, $\Box P_s$, $\Diamond P_s$ or $(P > Q)_s$ where P and Q have $< k$ operators.

($\neg$) A_s is $\neg P_s$. (i) Suppose $v_{w_s}(A) = 1$; then $v_{w_s}(\neg P) = 1$; so by TC($\neg$), $v_{w_s}(P) = 0$; so by assumption, $\Gamma' \nvdash^*_{NCx} P_s$; so by s-maximality, $\Gamma' \vdash^*_{NCx} \neg P_s$, where this is to say, $\Gamma' \vdash^*_{NCx} A_s$. (ii) Suppose $\Gamma' \vdash^*_{NCx} A_s$; then $\Gamma' \vdash^*_{NCx} \neg P_s$; so by consistency, $\Gamma' \nvdash^*_{NCx} P_s$; so by assumption, $v_{w_s}(P) = 0$; so by TC($\neg$), $v_{w_s}(\neg P) = 1$, where this is to say, $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NCx} A_s$.

($\supset$)

($\wedge$)

($\vee$)

($\equiv$)

($\Box$) A_s is $\Box P_s$. (i) Suppose $v_{w_s}(A) = 1$ but $\Gamma' \nvdash^*_{NCx} A_s$; then $v_{w_s}(\Box P) = 1$ but $\Gamma' \nvdash^*_{NCx} \Box P_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NCx} \neg\Box P_s$; so, since Γ' is a scapegoat set for $\Box$, there is some t such that $\Gamma' \vdash^*_{NCx} \neg P_t$; so by consistency, $\Gamma' \nvdash^*_{NCx} P_t$; so by assumption, $v_{w_t}(P) = 0$; so by TC($\Box$)$_\upsilon$, $v_{w_s}(\Box P) = 0$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$, then $\Gamma' \vdash^*_{NCx} A_s$.

(ii) Suppose $\Gamma' \vdash^*_{NCx} A_s$ but $v_{w_s}(A) = 0$; then $\Gamma' \vdash^*_{NCx} \Box P_s$ but $v_{w_s}(\Box P) = 0$. From the the latter, by TC($\Box$)$_\upsilon$, there is some $w_t \in W$ such that $v_{w_t}(P) = 0$; so by assumption, $\Gamma' \nvdash^*_{NCx} P_t$; but since $w_t \in W$, by construction, t appears in Γ' so by ($\Box$Eυ), $\Gamma' \vdash^*_{NCx} P_t$. This is impossible; reject the assumption: if $\Gamma' \vdash_{NCx} A_s$ then $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NCx} A_s$.

($\Diamond$)

($>$) A_s is $(P > Q)_s$. Suppose $v_{w_s}(A) = 1$ but $\Gamma' \nvdash^*_{NCx} A_s$; then $v_{w_s}(P > Q) = 1$ but $\Gamma' \nvdash^*_{NCx} (P > Q)_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NCx} \neg(P > Q)_s$; so, since Γ' is a scapegoat set for $>$, there is some t such that $\Gamma' \vdash^*_{NCx} P_{s/t}$ and $\Gamma' \vdash^*_{NCx} \neg Q_t$; from the first, by construction, $\langle w_s, w_t \rangle \in R_P$; and from the second, by consistency, $\Gamma' \nvdash^*_{NCx} Q_t$; so by assumption, $v_{w_t}(Q) = 0$; so by TC($>$), $v_{w_s}(P > Q) = 0$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$, then $\Gamma' \vdash^*_{NCx} A_s$.

(ii) Suppose $\Gamma' \vdash^*_{NCx} A_s$ but $v_{w_s}(A) = 0$; then $\Gamma' \vdash^*_{NCx} (P > Q)_s$ but $v_{w_s}(P > Q) = 0$. From the the latter, by TC($>$), there is some $w_t \in W$ such that $\langle w_s, w_t \rangle \in R_P$ and $v_{w_t}(Q) = 0$; from the first of these, by construction, $\Gamma' \vdash^*_{NCx} P_{s/t}$; and from the second, by assumption, $\Gamma' \nvdash^*_{NCx} Q_t$; but by (>E), $\Gamma' \vdash^*_{NCx} Q_t$. This is impossible; reject the assumption: if $\Gamma' \vdash_{NCx} A_s$ then $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NCx} A_s$.

For any A_s, $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NCx} A_s$.

L4.9 If Γ_0 is consistent, then $\langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle$ constructed as above is a *Cx* interpretation.

In each case, we need to show that the interpretation meets the condition(s) x. Suppose Γ_0 is consistent.

(1) If (1) is in *Cx*, then AMP1 is in *NCx*. Suppose $w_t \in f_A(w_s)$; then $\langle w_s, w_t \rangle \in R_A$; so by construction, $\Gamma' \vdash^*_{NCx} A_{s/t}$; so by AMP1, $\Gamma' \vdash^*_{NCx} A_t$; so by L4.8, $v_{w_t}(A) = 1$; so $w_t \in [A]$. So $f_A(w_s) \subseteq [A]$.

(2) If (2) is in *Cx* then AMP2 is in *NCx*. Suppose $w_s \in [A]$; then $v_{w_s}(A) = 1$; so by L4.8, $\Gamma' \vdash^*_{NCx} A_s$; so by AMP2, $\Gamma' \vdash^*_{NCx} A_{s/s}$; so by construction, $\langle w_s, w_s \rangle \in R_A$; so $w_s \in f_A(w_s)$.

(3) If (3) is in *Cx* then AMS1 is in *NCx*. Suppose $[A] \neq \phi$ but $f_A(w_s) = \phi$. From the former, there is some $w_t \in W$ such that $v_{w_t}(A) = 1$; so by L4.8, $\Gamma' \vdash^*_{NCx} A_t$; so by ($\Diamond$I$\upsilon$), $\Gamma' \vdash^*_{NCx} \Diamond A_s$. From the latter, there is no w_u such that $w_s R_A w_u$; so there is no w_u such that $w_s R_A w_u$ and $v_{w_u}(B) = 0$, and there is no w_u such that $w_s R_A w_u$ and $v_{w_u}(\neg B) = 0$; so by TC(>), $v_{w_s}(A > B) = 1$ and $v_{w_s}(A > \neg B) = 1$; so by L4.8, $\Gamma' \vdash^*_{NCx} (A > B)_s$ and $\Gamma' \vdash^*_{NCx} (A > \neg B)_s$. So reason as follows,

1	Γ'	
2	$\Diamond A_s$	from Γ'
3	$(A > B)_s$	from Γ'
4	$(A > \neg B)_s$	from Γ'
5	$A_{s/t}$	A (*g*, 2 AMS1)
6	$\Diamond A_s$	A (*c*, ¬I)
7	B_t	3,5 >E
8	$\neg B_t$	4,5 >E
9	$\neg\Diamond A_s$	6-8 ¬I
10	$\neg\Diamond A_s$	2,5-9 AMS1

So $\Gamma' \vdash^*_{NCx} \neg\Diamond A_s$; and since by L4.6, Γ' is consistent, $\Gamma' \nvdash^*_{NCx} \Diamond A_s$. This is impossible; reject the assumption: if $[A] \neq \phi$, then $f_A(w_s) \neq \phi$.

(4) If (4) is in *Cx* then AMS2 is in *NCx*. Suppose $f_A(w_s) \subseteq [B]$ and $f_B(w_s) \subseteq [A]$. Then any $x \in W$ such that $w_s R_A x$ has

$v_x(B) = 1$ and any $y \in W$ such that $w_s R_B y$ has $v_y(A) = 1$; so by TC(>), $v_{w_s}(A > B) = 1$ and $v_{w_s}(B > A) = 1$; so by L4.8, $\Gamma' \vdash^*_{NCx} (A > B)_s$ and $\Gamma' \vdash^*_{NCx} (B > A)_s$. Suppose $w_t \in f_A(w_s)$; then by construction, $\Gamma' \vdash^*_{NCx} A_{s/t}$; so by AMS2, $\Gamma' \vdash^*_{NCx} B_{s/t}$; so by construction, $w_t \in f_B(w_s)$. Suppose $w_t \in f_B(w_s)$; then by construction, $\Gamma' \vdash^*_{NCx} B_{s/t}$; so by AMS2, $\Gamma' \vdash^*_{NCx} A_{s/t}$; so by construction, $w_t \in f_A(w_s)$. So $f_A(w_s) = f_B(w_s)$.

(5) If (5) is in *Cx* then AMS3 is in *NCx*. Suppose $f_A(w_s) \cap [B] \neq \phi$ but $f_{A\wedge B}(w_s) \not\subseteq f_A(w_s)$. From the former, there is some $w_t \in f_A(w_s)$ such that $v_{w_t}(B) = 1$; so by TC(¬), $v_{w_t}(\neg B) = 0$; so by TC(>), $v_{w_s}(A > \neg B) = 0$; so by TC(¬), $v_{w_s}(\neg(A > \neg B)) = 1$; so by L4.8, $\Gamma' \vdash^*_{NCx} \neg(A > \neg B)_s$. From the latter, there is some w_u such that $w_u \in f_{A\wedge B}(w_s)$ but $w_u \notin f_A(w_s)$. From the first of these, by construction, $\Gamma' \vdash^*_{NCx} (A \wedge B)_{s/u}$; so by AMS3, $\Gamma' \vdash^*_{NCx} A_{s/u}$; so by construction, $w_u \in f_A(w_s)$. This is impossible; reject the assumption: if $f_A(w_s) \cap [B] \neq \phi$ then $f_{A\wedge B}(w_s) \subseteq f_A(w_s)$.

(6) Suppose (6) is in *Cx*, $w_t \in f_A(w_s)$ and $w_u \in f_A(w_s)$. Then by construction, $\Gamma' \vdash^*_{NCx} A_{s/t}$ and $\Gamma' \vdash^*_{NCx} A_{s/u}$; and by construction, since we are in *C2*, $w_t = w_u$.

(7) Suppose (7) is in *Cx*, $w_s \in [A]$ and $w_t \in f_A(w_s)$. Since $w_s \in [A]$, $v_{w_s}(A) = 1$; so by L4.8, $\Gamma' \vdash^*_{NCx} A_s$; and since $w_t \in f_A(w_s)$, by construction, $\Gamma' \vdash^*_{NCx} A_{s/t}$. So by construction, since we are in *C1*, $w_s = w_t$.

MAP For any $w_s \in W$, set $m(s) = w_s$; otherwise $m(s)$ is arbitrary.

L4.10 If Γ_0 is consistent, then $v_m(\Gamma_0) = 1$.

Reasoning parallel to L2.10 for *NKα*.

Main result: Suppose $\Gamma \vDash_{Cx} A$ but $\Gamma \nvdash_{NCx} A$. Then $\Gamma_0 \vDash^*_{Cx} A_0$ but $\Gamma_0 \nvdash^*_{NCx} A_0$. By (DN), if $\Gamma_0 \vdash^*_{NCx} \neg\neg A_0$, then $\Gamma_0 \vdash^*_{NCx} A_0$; so $\Gamma_0 \nvdash^*_{NCx} \neg\neg A_0$; so by L4.2, $\Gamma_0 \cup \{\neg A_0\}$ is consistent; so by L4.9 and L4.10, there is a *Cx* interpretation $\langle W, \{R_A \mid A \in \mathfrak{I}\}, v\rangle_m$ constructed as above such that $v_m(\Gamma_0 \cup \{\neg A_0\}) = 1$; so $v_{m(0)}(\neg A) = 1$; so by TC(¬), $v_{m(0)}(A) = 0$; so $v_m(\Gamma_0) = 1$ and $v_{m(0)}(A) = 0$; so by VCx*, $\Gamma_0 \nvDash^*_{Cx} A_0$. This is impossible; reject the assumption: if $\Gamma \vDash_{Cx} A$, then $\Gamma \vdash_{NCx} A$.

5 INTUITIONISTIC LOGIC: *IL* (CH. 6)

5.1 LANGUAGE / SEMANTIC NOTIONS

LIL The VOCABULARY consists of propositional parameters $p_0, p_1 \ldots$ with the operators, $\wedge$, $\vee$, $\neg$, and $\sqsupset$. Each propositional parameter is a FORMULA; if A and B are formulas, so are $(A \wedge B)$, $(A \vee B)$, $\neg A$, and $(A \sqsupset B)$.

IIL An INTERPRETATION is a triple $\langle W, R, v\rangle$ where W is a set of worlds, R is a subset of $W^2 = W \times W$, and v is a function such that for any $w \in W$ and p, $v_w(p) = 1$ or $v_w(p) = 0$. For $x, y, z \in W$, an interpretation is subject to the conditions,

ρ	for all x, xRx	reflexivity
τ	for all x, y, z, if xRy and yRz then xRz	transitivity
h	for any parameter p, if $v_x(p) = 1$, and xRy, then $v_y(p) = 1$	heredity

We think of worlds as representing a state of information at a given time. $v_w(p) = 1$ when p is proved at state w. The heredity condition guarantees that what is proved at one stage remains proved at the next. Notice that $v_w(p) = 0$ does not indicate that p is *false* – but rather that p *isn't proved*.

TIL For complex expressions,

($\wedge$) $v_w(A \wedge B) = 1$ if $v_w(A) = 1$ and $v_w(B) = 1$, and 0 otherwise.

($\vee$) $v_w(A \vee B) = 1$ if $v_w(A) = 1$ or $v_w(B) = 1$, and 0 otherwise.

($\neg$) $v_w(\neg A) = 1$ if all $x \in W$ such that wRx have $v_x(A) = 0$, and 0 otherwise.

($\sqsupset$) $v_w(A \sqsupset B) = 1$ if all $x \in W$ such that wRx have either $v_x(A) = 0$ or $v_x(B) = 1$, and 0 otherwise.

For a set Γ of formulas, $v_w(\Gamma) = 1$ iff $v_w(A) = 1$ for each $A \in \Gamma$; then,

VIL $\Gamma \models_{IL} A$ iff there is no *IL* interpretation $\langle W, R, v\rangle$ and $w \in W$ such that $v_w(\Gamma) = 1$ and $v_w(A) = 0$.

5.2 NATURAL DERIVATIONS: *NIL*

Augment the language for intuionistic logic to include expressions with subscripts and expressions of the sort $s.t$ as for *NK*α, along with a unary operator, $\sim$. Intuitively, $\sim A$ indicates that A is not (yet) proven. There is one new rule for the heredity condition. Otherwise, rules are as in *NK*$\rho\tau$ with $\sim$ like $\neg$, and rules for $\sqsupset$ and $\neg$ on the analogy of $\dashv$ and $\Box\neg$.

R | P_s
 | P_s

H | P_s
 | $s.t$
 | P_t

where P includes no instance of $\sim$

$\wedge$I | P_s
 | Q_s
 | $(P \wedge Q)_s$

$\wedge$E | $(P \wedge Q)_s$
 | P_s

$\wedge$E | $(P \wedge Q)_s$
 | Q_s

∨I | P_s / $(P \vee Q)_s$

∨I | P_s / $(Q \vee P)_s$

∨E | $(P \vee Q)_s$ / | P_s … R_t / | Q_s … R_t / R_t

$\sim$I | | P_s … Q_t, $\sim Q_t$ / $\sim P_s$

$\sim$E | | $\sim P_s$ … Q_t, $\sim Q_t$ / P_s

$\sqsupset$I | | $s.t$, P_t … Q_t / $(P \sqsupset Q)_s$

where t does not appear in any undischarged premise or assumption

$\sqsupset$E | $(P \sqsupset Q)_s$, $s.t$, P_t / Q_t

AMρ | / $s.s$

$\neg$I | | $s.t$ … $\sim P_t$ / $\neg P_s$

where t does not appear in any undischarged premise or assumption

$\neg$E | $\neg P_s$, $s.t$ / $\sim P_t$

AMτ | $s.t$, $t.u$ / $s.u$

Every subscript is 0, appears in a premise, or appears in the t-place of an accessible assumption for $\sqsupset$I or $\neg$I. Where the members of Γ and A are formulas in the original language for intuitionistic logic (without subscripts and without $\sim$), let let the members of Γ_0 be the formulas in Γ, each with subscript 0. Then,

NIL $\Gamma \vdash_{NIL} A$ iff there is an *NIL* derivation of A_0 from the members of Γ_0.

As examples, here are instances of the more interesting standard axioms for intuitionistic logic. Note that our account of a derivation guarantees that $\sim$ is not an operator in any of A, B, or C.

A1 $\vdash_{NIL} A \sqsupset (B \sqsupset A)$

1	0.1	A (g, $\sqsupset$I)
2	A_1	
3	1.2	A (g, $\sqsupset$I)
4	B_2	
5	A_2	2,3 H
6	$(B \sqsupset A)_1$	3-5 $\sqsupset$I
7	$[A \sqsupset (B \sqsupset A)]_0$	1-6 $\sqsupset$I

A2 $\vdash_{\text{NIL}} (A \sqsupset B) \sqsupset [(A \sqsupset (B \sqsupset C)) \sqsupset (A \sqsupset C)]$

1	0.1	A (g, $\sqsupset$I)
2	$(A \sqsupset B)_1$	
3	1.2	A (g, $\sqsupset$I)
4	$(A \sqsupset (B \sqsupset C))_2$	
5	2.3	A (g, $\sqsupset$I)
6	A_3	
7	1.3	3,5 AMτ
8	B_3	2,7,6 $\sqsupset$E
9	$(B \sqsupset C)_3$	4,5,6 $\sqsupset$E
10	3.3	AMρ
11	C_3	9,10,8 $\sqsupset$E
12	$(A \sqsupset C)_2$	5-11 $\sqsupset$I
13	$[(A \sqsupset (B \sqsupset C)) \sqsupset (A \sqsupset C)]_1$	3-12 $\sqsupset$I
14	$((A \sqsupset B) \sqsupset [(A \sqsupset (B \sqsupset C)) \sqsupset (A \sqsupset C)])_0$	1-13 $\sqsupset$I

A3 $\vdash_{\text{NIL}} A \sqsupset (B \sqsupset (A \wedge B))$

A4 $\vdash_{\text{NIL}} (A \wedge B) \sqsupset A$

A5 $\vdash_{\text{NIL}} (A \wedge B) \sqsupset B$

A6 $\vdash_{\text{NIL}} A \sqsupset (A \vee B)$

A7 $\vdash_{\text{NIL}} B \sqsupset (A \vee B)$

A8 $\vdash_{\text{NIL}} (A \sqsupset C) \sqsupset [(B \sqsupset C) \sqsupset ((A \vee B) \sqsupset C)]$

A9 $\vdash_{\text{NIL}} (A \sqsupset B) \sqsupset [(A \sqsupset \rightharpoondown B) \sqsupset \rightharpoondown A]$

1	0.1	A (g, $\sqsupset$I)
2	$(A \sqsupset B)_1$	
3	1.2	A (g, $\sqsupset$I)
4	$(A \sqsupset \rightharpoondown B)_2$	
5	2.3	A (g, $\rightharpoondown$I)
6	A_3	A (c, ~I)
7	1.3	3,5 AMτ
8	B_3	2,7,6 $\sqsupset$E
9	$\rightharpoondown B_3$	4,5,6 $\sqsupset$E
10	3.3	AMρ
11	$\sim B_3$	9,10 $\rightharpoondown$E
12	$\sim A_3$	6-11 ~I
13	$\rightharpoondown A_2$	5-12 $\rightharpoondown$I
14	$[(A \sqsupset \rightharpoondown B) \sqsupset \rightharpoondown A]_1$	3-13 $\sqsupset$I
15	$((A \sqsupset B) \sqsupset [(A \sqsupset \rightharpoondown B) \sqsupset \rightharpoondown A])_0$	1-14 $\sqsupset$I

A10 $\vdash_{NIL} \neg A \sqsupset (A \sqsupset B)$

1	0.1	A (g, $\sqsupset$I)
2	$\neg A_1$	
3	1.2	A (g, $\sqsupset$I)
4	A_2	
5	$\sim B_2$	A (c, $\sim$E)
6	A_2	4 R
7	$\sim A_2$	2,3 $\neg$E
8	B_2	5-7 $\sim$E
9	$(A \sqsupset B)_1$	3-8 $\sqsupset$I
10	$[\neg A \sqsupset (A \sqsupset B)]_0$	1-9 $\sqsupset$I

A system with these axioms and MP (which we already have by AMρ with $\sqsupset$E) turns into classical logic if A10 is replaced by double negation, $\neg\neg A \sqsupset A$. But we cannot prove $\neg\neg A \sqsupset A$ (or at least we cannot if our derivation system is sound).

5.3 Soundness and Completeness

Preliminaries: Begin with generalized notions of validity to include expressions with subscripts and operator '$\sim$'. First, as a supplement to TIL,

TIL ($\sim$) $v_w(\sim A) = 1$ if $v_w(A) = 0$, and 0 otherwise.

For a model $\langle W, R, v\rangle$, let m be a map from subscripts into W. Say $\langle W, R, v\rangle_m$ is $\langle W, R, v\rangle$ ***with*** map m. Then, where Γ is a set of expressions of our language for derivations, $v_m(\Gamma) = 1$ iff for each $A_s \in \Gamma$, $v_{m(s)}(A) = 1$, and for each $s.t \in \Gamma$, $\langle m(s), m(t)\rangle \in R$. Now expand notions of validity to include subscripted formulas, and alternate expressions as indicated in double brackets.

VIL* $\Gamma \models^*_{IL} A_s$ $[\![s.t]\!]$ iff there is no *IL* interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma) = 1$ but $v_{m(s)}(A) = 0$ $[\![\langle m(s), m(t)\rangle \notin R]\!]$.

NIL* $\Gamma \vdash^*_{NIL} A_s$ $[\![s.t]\!]$ iff there is an *NIL* derivation of A_s $[\![s.t]\!]$ from the members of Γ.

These notions reduce to the standard ones when all the members of Γ and A have subscript 0 (and so do not include expressions of the sort $s.t$) and do not include '$\sim$'. For the following, cases omitted are like ones worked, and so left to the reader.

THEOREM 5.1 *NIL is sound: If* $\Gamma \vdash_{NIL} A$ ***then*** $\Gamma \models_{IL} A$.

L5.1 If $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{IL} P_s$ $[\![s.t]\!]$, then $\Gamma' \models^*_{IL} P_s$ $[\![s.t]\!]$.

Reasoning parallel to that for L2.1 of *NKα*.

Main result: For each line in a derivation let $\mathcal{P}_i$ be the expression on line i and Γ_i be the set of all premises and assumptions whose scope includes line i. We set out to show "generalized" soundness: if $\Gamma \vdash^*_{\text{NIL}} \mathcal{P}$ then $\Gamma \vDash^*_{\text{IL}} \mathcal{P}$. As above, this reduces to the standard result when $\mathcal{P}$ and all the members of Γ are formulas with subscript 0 and do not include '$\sim$'. Suppose $\Gamma \vdash^*_{\text{NIL}} \mathcal{P}$. Then there is a derivation of $\mathcal{P}$ from premises in Γ where $\mathcal{P}$ appears under the scope of the premises alone. By induction on line number of this derivation, we show that for each line i of this derivation, $\Gamma_i \vDash^*_{\text{IL}} \mathcal{P}_i$. The case when $\mathcal{P}_i = \mathcal{P}$ is the desired result.

Basis: $\mathcal{P}_1$ is a premise or an assumption A_s $[\![s.t]\!]$. Then $\Gamma_1 = \{A_s\}$ $[\![\{s.t\}]\!]$; so for any $\langle W, R, v\rangle_m$, $v_m(\Gamma_1) = 1$ iff $v_{m(s)}(A) = 1$ $[\![\langle m(s), m(t)\rangle \in R]\!]$; so there is no $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_1) = 1$ but $v_{m(s)}(A) = 0$ $[\![\langle m(s), m(t)\rangle \notin R]\!]$. So by VIL*, $\Gamma_1 \vDash^*_{\text{IL}} A_s$ $[\![s.t]\!]$, where this is just to say, $\Gamma_1 \vDash^*_{\text{IL}} \mathcal{P}_1$.

Assp: For any $i, 1 \leqslant i < k, \Gamma_i \vDash^*_{\text{IL}} \mathcal{P}_i$.

Show: $\Gamma_k \vDash^*_{\text{IL}} \mathcal{P}_k$.

$\mathcal{P}_k$ is either a premise, an assumption, or arises from previous lines by R, $\wedge$I, $\wedge$E, $\vee$I, $\vee$E, $\sim$I, $\sim$E, $\neg$I, $\neg$E, $\sqsupset$I, $\sqsupset$E, AMρ, AMτ or H. If $\mathcal{P}_k$ is a premise or an assumption, then as in the basis, $\Gamma_k \vDash^*_{\text{IL}} \mathcal{P}_k$. So suppose $\mathcal{P}_k$ arises by one of the rules.

(R)

($\wedge$I)

($\wedge$E)

($\vee$I)

($\vee$E)

($\sim$I) If $\mathcal{P}_k$ arises by $\sim$I, then the picture is like this,

$$\begin{array}{r|l} & \quad\begin{array}{|l} A_s \\ \hline \\ \\ B_t \\ \sim B_t \end{array} \\ k & \sim A_s \end{array}$$

(lines: A_s; i: B_t; j: $\sim B_t$; k: $\sim A_s$)

where $i, j < k$ and $\mathcal{P}_k$ is $\sim A_s$. By assumption, $\Gamma_i \vDash^*_{\text{IL}} B_t$ and $\Gamma_j \vDash^*_{\text{IL}} \sim B_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{A_s\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{A_s\}$; so by L5.1, $\Gamma_k \cup \{A_s\} \vDash^*_{\text{IL}} B_t$ and $\Gamma_k \cup \{A_s\} \vDash^*_{\text{IL}} \sim B_t$. Suppose $\Gamma_k \nvDash^*_{\text{IL}} \sim A_s$; then by VIL*, there is an *IL* interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(\sim A) = 0$; so by TIL($\sim$), $v_{m(s)}(A) = 1$; so $v_m(\Gamma_k) = 1$ and $v_{m(s)}(A) = 1$; so $v_m(\Gamma_k \cup \{A_s\}) = 1$; so by VIL*, $v_{m(t)}(B) = 1$ and $v_{m(t)}(\sim B) = 1$; from the latter, by TIL($\sim$), $v_{m(t)}(B) = 0$. This is impossible; reject the assumption: $\Gamma_k \vDash^*_{\text{IL}} \sim A_s$, which is to say, $\Gamma_k \vDash^*_{\text{IL}} \mathcal{P}_k$.

(~E)

($\neg$I) If $\mathcal{P}_k$ arises by $\neg$I, then the picture is like this,

$$\begin{array}{r|l}
 & \quad | \; s.t \\
 & \quad | \; \vdots \\
i & \quad | \; {\sim}A_t \\
k & \neg A_s
\end{array}$$

where $i < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $\neg A_s$. By assumption, $\Gamma_i \models^*_{IL} {\sim}A_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{s.t\}$; so by L5.1, $\Gamma_k \cup \{s.t\} \models^*_{IL} {\sim}A_t$. Suppose $\Gamma_k \not\models^*_{IL} \neg A_s$; then by VIL*, there is an *IL* interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(\neg A) = 0$; so by TIL($\neg$), there is some $w \in W$ such that $m(s)Rw$ and $v_w(A) = 1$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, R, v\rangle_{m'}$; since t does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m'(t) = w$ and $m'(s) = m(s)$, $\langle m'(s), m'(t)\rangle \in R$; so $v_{m'}(\Gamma_k \cup \{s.t\}) = 1$; so by VIL*, $v_{m'(t)}({\sim}A) = 1$; so by TIL(~), $v_{m'}(A) = 0$. But $m'(t) = w$; so $v_w(A) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{IL} \neg A_s$, which is to say, $\Gamma_k \models^*_{IL} \mathcal{P}_k$.

($\neg$E) If $\mathcal{P}_k$ arises by $\neg$E, then the picture is like this,

$$\begin{array}{r|l}
i & \neg A_s \\
j & s.t \\
 & \vdots \\
k & {\sim}A_t
\end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is ${\sim}A_t$. By assumption, $\Gamma_i \models^*_{IL} \neg A_s$ and $\Gamma_j \models^*_{IL} s.t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L5.1, $\Gamma_k \models^*_{IL} \neg A_s$ and $\Gamma_k \models^*_{IL} s.t$. Suppose $\Gamma_k \not\models^*_{IL} {\sim}A_t$; then by VIL*, there is some *IL* interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(t)}({\sim}A) = 0$; so by TIL(~), $v_{m(t)}(A) = 1$. Since $v_m(\Gamma_k) = 1$, by VIL*, $v_{m(s)}(\neg A) = 1$ and $\langle m(s), m(t)\rangle \in R$; from the first of these, by TIL($\neg$), any w such that $m(s)Rw$ has $v_w(A) = 0$; so $v_{m(t)}(A) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{IL} A_t$, which is to say, $\Gamma_k \models^*_{IL} \mathcal{P}_k$.

($\sqsupset$I) If $\mathcal{P}_k$ arises by $\sqsupset$I, then the picture is like this,

$$\begin{array}{r|l}
 & \quad | \; s.t \\
 & \quad | \; A_t \\
 & \quad | \; \vdots \\
i & \quad | \; B_t \\
k & (A \sqsupset B)_s
\end{array}$$

where $i < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $(A \sqsupset B)_s$. By assumption, $\Gamma_i \models^*_{IL}$

B_t; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{s.t, A_t\}$; so by L5.1, $\Gamma_k \cup \{s.t, A_t\} \models^*_{IL} B_t$. Suppose $\Gamma_k \not\models^*_{IL} (A \sqsupset B)_s$; then by VIL*, there is an *IL* interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(A \sqsupset B) = 0$; so by TIL($\sqsupset$), there is some $w \in W$ such that $m(s)Rw$ with $v_w(A) = 1$ and $v_w(B) = 0$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, R, v\rangle_{m'}$; since t does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; since $m'(t) = w$ and $m'(s) = m(s)$, $v_{m'(t)}(A) = 1$ and $\langle m'(s), m'(t)\rangle \in R$; so $v_{m'}(\Gamma_k \cup \{s.t, A_t\}) = 1$; so by VIL*, $v_{m'(t)}(B) = 1$. But $m'(t) = w$; so $v_w(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{IL} (A \sqsupset B)_s$, which is to say, $\Gamma_k \models^*_{IL} \mathcal{P}_k$.

($\sqsupset$E)

(AMρ)

(AMτ) If $\mathcal{P}_k$ arises by AMτ, then the picture is like this,

i	s.t
j	t.u
k	s.u

where $i, j < k$ and $\mathcal{P}_k$ is s.u. By assumption, $\Gamma_i \models^*_{IL}$ s.t and $\Gamma_j \models^*_{IL}$ t.u; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L5.1, $\Gamma_k \models^*_{IL}$ s.t and $\Gamma_k \models^*_{IL}$ t.u. Suppose $\Gamma_k \not\models^*_{IL}$ s.u; then by VIL*, there is some *IL* interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $\langle m(s), m(u)\rangle \notin R$; since $v_m(\Gamma_k) = 1$, by VIL*, $\langle m(s), m(t)\rangle \in R$ and $\langle m(t), m(u)\rangle \in R$; but *IL* includes condition τ; so for any $\langle x, y\rangle, \langle y, z\rangle \in R$, $\langle x, z\rangle \in R$; so $\langle m(s), m(u)\rangle \in R$. This is impossible; reject the assumption: $\Gamma_k \models^*_{IL}$ s.u, which is to say, $\Gamma_k \models^*_{IL} \mathcal{P}_k$.

(H) If $\mathcal{P}_k$ arises by H, then the picture is like this,

i	A_s
j	s.t
k	A_t

where $i, j < k$, A has no instance of '$\sim$' and $\mathcal{P}_k$ is A_t. By assumption, $\Gamma_i \models^*_{IL} A_s$ and $\Gamma_j \models^*_{IL}$ s.t; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L5.1, $\Gamma_k \models^*_{IL} A_s$ and $\Gamma_k \models^*_{IL}$ s.t. Suppose $\Gamma_k \not\models^*_{IL} A_t$; then by VIL*, there is some *IL* interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(t)}(A) = 0$; since $v_m(\Gamma_k) = 1$, by VIL*, $v_{m(s)}(A) = 1$ and $\langle m(s), m(t)\rangle \in R$.

Now, by induction on the number of operators in A, we show that for A without '$\sim$', if $v_x(A) = 1$ and xRy, then $v_y(A) = 1$. Suppose xRy.

Basis: Suppose A is a parameter p and $v_x(A) = 1$; then $v_x(p) = 1$; so by condition h, $v_y(p) = 1$; so $v_y(A) = 1$.

Assp: For $0 \leqslant i < k$, if A has i operators and $v_x(A) = 1$, then $v_y(A) = 1$.

Show: If A has k operators and $v_x(A) = 1$, then $v_y(A) = 1$.

If A has k operators and no instance of '$\sim$' then it is of the form, $P \wedge Q$, $P \vee Q$, $\neg P$, or $P \sqsupset Q$, where P and Q have $< k$ operators.

($\wedge$) Suppose A is $P \wedge Q$ and $v_x(A) = 1$; then $v_x(P \wedge Q) = 1$; so by TIL($\wedge$), $v_x(P) = 1$ and $v_x(Q) = 1$; so by assumption, $v_y(P) = 1$ and $v_y(Q) = 1$; so by TIL($\wedge$), $v_y(P \wedge Q) = 1$; so $v_y(A) = 1$.

($\vee$) Suppose A is $P \vee Q$ and $v_x(A) = 1$; then $v_x(P \vee Q) = 1$; so by TIL($\vee$), $v_x(P) = 1$ or $v_x(Q) = 1$; so by assumption, $v_y(P) = 1$ or $v_y(Q) = 1$; so by TIL($\vee$), $v_y(P \vee Q) = 1$; so $v_y(A) = 1$.

($\neg$) Suppose A is $\neg P$ and $v_x(A) = 1$ but $v_y(A) = 0$; then $v_x(\neg P) = 1$ but $v_y(\neg P) = 0$. From the former, by TIL($\neg$), any w such that xRw has $v_w(P) = 0$. From the latter, by TIL($\neg$), there is some $z \in W$ such that yRz and $v_z(P) = 1$. But xRy and yRz so by τ, xRz; so $v_z(P) = 0$. This is impossible; reject the assumption: if $v_x(A) = 1$, then $v_y(A) = 1$.

($\sqsupset$) Suppose A is $P \sqsupset Q$ and $v_x(A) = 1$ but $v_y(A) = 0$; then $v_x(P \sqsupset Q) = 1$ but $v_y(P \sqsupset Q) = 0$. From the former, by TIL($\sqsupset$), any w such that xRw has $v_w(P) = 0$ or $v_w(Q) = 1$. From the latter, by TIL($\sqsupset$), there is some $z \in W$ such that yRz where $v_z(P) = 1$ and $v_z(Q) = 0$. But xRy and yRz so by τ, xRz; so $v_z(P) = 0$ or $v_z(Q) = 1$. This is impossible; reject the assumption: if $v_x(A) = 1$, then $v_y(A) = 1$.

For any such A, if $v_x(A) = 1$, then $v_y(A) = 1$.

So, returning to the case for (H), $v_{m(t)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{IL} A_t$, which is to say, $\Gamma_k \models^*_{IL} \mathcal{P}_k$.

For any i, $\Gamma_i \models^*_{IL} \mathcal{P}_i$.

THEOREM 5.2 *NIL is complete: if* $\Gamma \models_{IL} A$ *then* $\Gamma \vdash_{NIL} A$.

Suppose $\Gamma \models_{IL} A$; then $\Gamma_0 \models^*_{IL} A_0$; we show that $\Gamma_0 \vdash^*_{NIL} A_0$. Again, this reduces to the standard notion.

CON Γ is CONSISTENT iff there is no A_s such that $\Gamma \vdash^*_{NIL} A_s$ and $\Gamma \vdash^*_{NIL} {\sim}A_s$.

L5.2 If s is 0 or appears in Γ, and $\Gamma \nvdash^*_{NIL} {\sim}P_s$, then $\Gamma \cup \{P_s\}$ is consistent.

Suppose s is 0 or appears in Γ and $\Gamma \nvdash^*_{NIL} {\sim}P_s$ but $\Gamma \cup \{P_s\}$ is inconsistent. Then there is some A_t such that $\Gamma \cup \{P_s\} \vdash^*_{NIL} A_t$ and $\Gamma \cup \{P_s\} \vdash^*_{NIL} {\sim}A_t$. But then we can argue,

1	Γ	
2	$\quad P_s$	A (*c*, ~I)
3	$\quad A_t$	from $\Gamma \cup \{P_s\}$
4	$\quad \sim A_t$	from $\Gamma \cup \{P_s\}$
5	$\sim P_s$	2-4 ~I

where the assumption is allowed insofar as s is either 0 or appears in Γ; so $\Gamma \vdash^*_{\text{NIL}} \sim P_s$. But this is impossible; reject the assumption: if s is 0 or introduced in Γ and $\Gamma \nvdash^*_{\text{NIL}} \sim P_s$, then $\Gamma \cup \{P_s\}$ is consistent.

L5.3 There is an enumeration of all the subscripted formulas, $\mathcal{P}_1\ \mathcal{P}_2 \ldots$

Proof by construction as for L2.3 of *NK*α.

MAX Γ is S-MAXIMAL iff for any A_s either $\Gamma \vdash^*_{\text{NIL}} A_s$ or $\Gamma \vdash^*_{\text{NIL}} \sim A_s$.

SGT Γ is a SCAPEGOAT set for $\neg$ iff for every formula of the form $\sim\neg A_s$, if $\Gamma \vdash^*_{\text{NIL}} \sim\neg A_s$ then there is some t such that $\Gamma \vdash^*_{\text{NIL}}$ s.t and $\Gamma \vdash^*_{\text{NIL}} A_t$.

Γ is a SCAPEGOAT set for $\sqsupset$ iff for every formula of the form $\sim(A \sqsupset B)_s$, if $\Gamma \vdash^*_{\text{NIL}} \sim(A \sqsupset B)_s$ then there is some t such that $\Gamma \vdash^*_{\text{NIL}}$ s.t, $\Gamma \vdash^*_{\text{NIL}} A_t$ and $\Gamma \vdash^*_{\text{NIL}} \sim B_t$.

C(Γ') For Γ with unsubscripted formulas and the corresponding Γ_0, we construct Γ' as follows. Set $\Omega_0 = \Gamma_0$. By L5.3, there is an enumeration, $\mathcal{P}_1, \mathcal{P}_2 \ldots$ of all the subscripted formulas; let $\mathcal{E}_0$ be this enumeration. Then for the first A_s in $\mathcal{E}_{i-1}$ such that s is 0 or included in Ω_{i-1}, let $\mathcal{E}_i$ be like $\mathcal{E}_{i-1}$ but without A_s, and set,

$\Omega_i = \Omega_{i-1}$	if $\Omega_{i-1} \vdash^*_{\text{NIL}} \sim A_s$
$\Omega_{i^*} = \Omega_{i-1} \cup \{A_s\}$	if $\Omega_{i-1} \nvdash^*_{\text{NIL}} \sim A_s$

and

$\Omega_i = \Omega_{i^*}$	if A_s is not of the form $\sim\neg P_s$ or $\sim(P \sqsupset Q)_s$
$\Omega_i = \Omega_{i^*} \cup \{s.t, P_t\}$	if A_s is of the form $\sim\neg P_s$
$\Omega_i = \Omega_{i^*} \cup \{s.t, P_t, \sim Q_t\}$	if A_s is of the form $\sim(P \sqsupset Q)_s$

-where t is the first subscript not included in Ω_{i^*}

then

$$\Gamma' = \bigcup_{i \geqslant 0} \Omega_i$$

Note that there is always sure to be a subscript t not in Ω_{i^*} insofar as there are infinitely many subscripts, and at any stage only finitely many formulas are added – the only subscripts in the initial Ω_0 being 0. Suppose s is introduced in Γ'; then there is some Ω_i in which it is first introduced; and any formula $\mathcal{P}_j$ in the original enumeration that has subscript s is sure to be "considered" for inclusion at a subsequent stage.

L5.4 For any s included in Γ', Γ' is s-maximal.

Suppose s is included in Γ' but Γ' is not s-maximal. Then there is some A_s such that $\Gamma' \nvdash^*_{NIL} A_s$ and $\Gamma' \nvdash^*_{NIL} \sim A_s$. For any i, each member of Ω_{i-1} is in Γ'; so if $\Omega_{i-1} \vdash^*_{NIL} \sim A_s$ then $\Gamma' \vdash^*_{NIL} \sim A_s$; but $\Gamma' \nvdash^*_{NIL} \sim A_s$; so $\Omega_{i-1} \nvdash^*_{NIL} \sim A_s$; so since s is included in Γ', there is a stage in the construction that sets $\Omega_{i^*} = \Omega_{i-1} \cup \{A_s\}$; so by construction, $A_s \in \Gamma'$; so $\Gamma' \vdash^*_{NIL} A_s$. This is impossible; reject the assumption: Γ' is s-maximal.

L5.5 If Γ_0 is consistent, then each Ω_i is consistent.

Suppose Γ_0 is consistent.

Basis: $\Omega_0 = \Gamma_0$ and Γ_0 is consistent; so Ω_0 is consistent.

Assp: For any i, $0 \leqslant i < k$, Ω_i is consistent.

Show: Ω_k is consistent.

Ω_k is either (i) Ω_{k-1}, or (ii) $\Omega_{k^*} = \Omega_{k-1} \cup \{A_s\}$, (iii) $\Omega_{k^*} \cup \{s.t, P_t\}$ or (iv) $\Omega_{k^*} \cup \{s.t, P_t, \sim Q_t\}$.

(i) Suppose Ω_k is Ω_{k-1}. By assumption, Ω_{k-1} is consistent; so Ω_k is consistent.

(ii) Suppose Ω_k is $\Omega_{k^*} = \Omega_{k-1} \cup \{A_s\}$. Then by construction, s is 0 or in Ω_{k-1} and $\Omega_{k-1} \nvdash^*_{NIL} \sim A_s$; so by L5.2, $\Omega_{k-1} \cup \{A_s\}$ is consistent; so Ω_k is consistent.

(iii) Suppose Ω_k is $\Omega_{k^*} \cup \{s.t, P_t\}$. In this case, as above, Ω_{k*} is consistent and by construction, $\sim\neg P_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are A_u and $\sim A_u$ such that $\Omega_{k^*} \cup \{s.t, P_t\} \vdash^*_{NIL} A_u$ and $\Omega_{k^*} \cup \{s.t, P_t\} \vdash^*_{NIL} \sim A_u$. So reason as follows,

1	Ω_{k^*}	
2	$s.t$	A (g, $\neg$I)
3	P_t	A (c, $\sim$I)
4	A_u	from $\Omega_{k^*} \cup \{s.t, P_t\}$
5	$\sim A_u$	from $\Omega_{k^*} \cup \{s.t, P_t\}$
6	$\sim P_t$	3-5 $\sim$I
7	$\neg P_s$	2-6 $\neg$I

where, by construction, t is not in Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NIL} \neg P_s$; but $\sim\neg P_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{NIL} \sim\neg P_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

(iv) Suppose Ω_k is $\Omega_{k^*} \cup \{s.t, P_t, \sim Q_t\}$. In this case, as above, Ω_{k*} is consistent and by construction, $\sim(P \sqsupset Q)_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are A_u and $\sim A_u$ such that $\Omega_{k^*} \cup \{s.t, P_t, \sim Q_t\} \vdash^*_{NIL} A_u$ and $\Omega_{k^*} \cup \{s.t, P_t, \sim Q_t\} \vdash^*_{NIL} \sim A_u$. So reason as follows,

1	Ω_{k^*}	
2	$s.t$	A (g, $\sqsupset$I)
3	P_t	
4	$\sim Q_t$	A (c, $\sim$E)
5	A_u	from $\Omega_{k^*} \cup \{s.t, P_t, \sim Q_t\}$
6	$\sim A_u$	from $\Omega_{k^*} \cup \{s.t, P_t, \sim Q_t\}$
7	Q_t	4-6 $\sim$E
8	$(P \sqsupset Q)_s$	2-7 $\sqsupset$I

where, by construction, t is not in Ω_{k^*}. So $\Omega_{k^*} \vdash^{*}_{\text{NIL}} (P \sqsupset Q)_s$; but $\sim(P \sqsupset Q)_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^{*}_{\text{NIL}} \sim(P \sqsupset Q)_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

For any i, Ω_i is consistent.

L5.6 If Γ_0 is consistent, then Γ' is consistent.

Reasoning parallel to L2.6 for *NK*α.

L5.7 If Γ_0 is consistent, then Γ' is a scapegoat set for $\rightharpoondown$ and $\sqsupset$.

For $\rightharpoondown$. Suppose Γ_0 is consistent and $\Gamma' \vdash^{*}_{\text{NIL}} \sim\rightharpoondown P_s$. By L5.6, Γ' is consistent; and by the constraints on subscripts, s is included in Γ'. Since Γ' is consistent, $\Gamma' \nvdash^{*}_{\text{NIL}} \sim\sim\rightharpoondown P_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{\sim\rightharpoondown P_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{s.t, P_t\}$; so by construction, $s.t \in \Gamma'$ and $P_t \in \Gamma'$; so $\Gamma' \vdash^{*}_{\text{NIL}} s.t$ and $\Gamma' \vdash^{*}_{\text{NIL}} P_t$. So Γ' is a scapegoat set for $\rightharpoondown$.

For $\sqsupset$. Suppose Γ_0 is consistent and $\Gamma' \vdash^{*}_{\text{NIL}} \sim(P \sqsupset Q)_s$. By L5.6, Γ' is consistent; and by the constraints on subscripts, s is included in Γ'. Since Γ' is consistent, $\Gamma' \nvdash^{*}_{\text{NIL}} \sim\sim(P \sqsupset Q)_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{\sim(P \sqsupset Q)_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{s.t, P_t, \sim Q_t\}$; so by construction, $s.t \in \Gamma'$, $P_t \in \Gamma'$ and $\sim Q_t \in \Gamma'$; so $\Gamma' \vdash^{*}_{\text{NIL}} s.t$, $\Gamma' \vdash^{*}_{\text{NIL}} P_t$ and $\Gamma' \vdash^{*}_{\text{NIL}} \sim Q_t$. So Γ' is a scapegoat set for $\sqsupset$.

C(I) We construct an interpretation $I = \langle W, R, v\rangle$ based on Γ' as follows. Let W have a member w_s corresponding to each subscript s included in Γ'. Then set $\langle w_s, w_t\rangle \in R$ iff $\Gamma' \vdash^{*}_{\text{NIL}} s.t$, and $v_{w_s}(p) = 1$ iff $\Gamma' \vdash^{*}_{\text{NIL}} p_s$.

L5.8 If Γ_0 is consistent then for $\langle W, R, v\rangle$ constructed as above, and for any s included in Γ', $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^{*}_{\text{NIL}} A_s$.

Suppose Γ_0 is consistent and s is included in Γ'. By L5.4, Γ' is s-maximal. By L5.6 and L5.7, Γ' is consistent and a scapegoat set for $\rightharpoondown$ and $\sqsupset$. Now by induction on the number of operators in A_s,

Basis: If A_s has no operators, then it is a parameter p_s and by construction, $v_{w_s}(p) = 1$ iff $\Gamma' \vdash^{*}_{\text{NIL}} p_s$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^{*}_{\text{NIL}} A_s$.

Assp: For any i, $0 \leqslant i < k$, if A_s has i operators, then $\nu_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NIL} A_s$.

Show: If A_s has k operators, then $\nu_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NIL} A_s$.

If A_s has k operators, then it is of the form $\sim P_s$, $(P \wedge Q)_s$, $(P \vee Q)_s$, $(P \sqsupset Q)_s$, or $\neg P_s$ where P and Q have $< k$ operators.

($\sim$) A_s is $\sim P_s$. (i) Suppose $\nu_{w_s}(A) = 1$; then $\nu_{w_s}(\sim P) = 1$; so by TIL($\sim$), $\nu_{w_s}(P) = 0$; so by assumption, $\Gamma' \nvdash^*_{NIL} P_s$; so by s-maximality, $\Gamma' \vdash^*_{NIL} \sim P_s$, where this is to say, $\Gamma' \vdash^*_{NIL} A_s$. (ii) Suppose $\Gamma' \vdash^*_{NIL} A_s$; then $\Gamma' \vdash^*_{NIL} \sim P_s$; so by consistency, $\Gamma' \nvdash^*_{NIL} P_s$; so by assumption, $\nu_{w_s}(P) = 0$; so by TIL($\sim$), $\nu_{w_s}(\sim P) = 1$, where this is to say, $\nu_{w_s}(A) = 1$. So $\nu_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NIL} A_s$.

($\wedge$)

($\vee$)

($\sqsupset$)

($\neg$) A_s is $\neg P_s$. (i) Suppose $\nu_{w_s}(A) = 1$ but $\Gamma' \nvdash^*_{NIL} A_s$; then $\nu_{w_s}(\neg P) = 1$ but $\Gamma' \nvdash^*_{NIL} \neg P_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NIL} \sim\neg P_s$; so, since Γ' is a scapegoat set for $\neg$, there is some t such that $\Gamma' \vdash^*_{NIL} s.t$ and $\Gamma' \vdash^*_{NIL} P_t$; from the first, by construction, $\langle w_s, w_t \rangle \in R$; and from the second, by assumption, $\nu_{w_t}(P) = 1$; so by TIL($\neg$), $\nu_{w_s}(\neg P) = 0$. This is impossible; reject the assumption: if $\nu_{w_s}(A) = 1$, then $\Gamma' \vdash^*_{NIL} A_s$.

(ii) Suppose $\Gamma' \vdash^*_{NIL} A_s$ but $\nu_{w_s}(A) = 0$; then $\Gamma' \vdash^*_{NIL} \neg P_s$ but $\nu_{w_s}(\neg P) = 0$. From the latter, by TIL($\neg$), there is some $w_t \in W$ such that $w_s R w_t$ and $\nu_{w_t}(P) = 1$; so by assumption, $\Gamma' \vdash^*_{NIL} P_t$; but since $w_s R w_t$, by construction, $\Gamma' \vdash^*_{NIL} s.t$; so by ($\neg$E), $\Gamma' \vdash^*_{NIL} \sim P_t$; so by consistency, $\Gamma' \nvdash^*_{NIL} P_t$. This is impossible; reject the assumption: if $\Gamma' \vdash_{NIL} A_s$ then $\nu_{w_s}(A) = 1$. So $\nu_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NIL} A_s$.

For any A_s, $\nu_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NIL} A_s$.

L5.9 If Γ_0 is consistent, then $\langle W, R, \nu \rangle$ constructed as above is an *IL* interpretation.

For this, we need to show that the interpretation meets the ρ, τ and h conditions.

(ρ) Suppose $w_s \in W$. Then by construction, s is a subscript in Γ'; so by (AMρ), $\Gamma' \vdash^*_{NIL} s.s$; so by construction, $\langle w_s, w_s \rangle \in R$ and ρ is satisfied.

(τ)

(*b*) Suppose $v_{w_s}(p) = 1$ and $w_s R w_t$. Then by construction, $\Gamma' \vdash^*_{NIL} p_s$ and $\Gamma' \vdash^*_{NIL}$ s.t; so by (H), $\Gamma' \vdash^*_{NIL} p_t$; so by construction, $v_{w_t}(p) = 1$.

Map For any $w_s \in W$, set $m(s) = w_s$; otherwise $m(s)$ is arbitrary.

L5.10 If Γ_0 is consistent, then $v_m(\Gamma_0) = 1$.

Reasoning parallel to L2.10 for *NKα*.

Main result: Suppose $\Gamma \models_{IL} A$ but $\Gamma \nvdash_{NIL} A$. Then $\Gamma_0 \models^*_{IL} A_0$ but $\Gamma_0 \nvdash^*_{NIL} A_0$. By a simple derivation, if $\Gamma_0 \vdash^*_{NIL} {\sim}{\sim}A_0$, then $\Gamma_0 \vdash^*_{NIL} A_0$; so $\Gamma_0 \nvdash^*_{NIL} {\sim}{\sim}A_0$; so by L5.2, $\Gamma_0 \cup \{{\sim}A_0\}$ is consistent; so by L5.9 and L5.10, there is an *IL* interpretation $\langle W, R, v\rangle_m$ constructed as above such that $v_m(\Gamma_0 \cup \{{\sim}A_0\}) = 1$; so $v_{m(0)}({\sim}A) = 1$; so by TIL(~), $v_{m(0)}(A) = 0$; so $v_m(\Gamma_0) = 1$ and $v_{m(0)}(A) = 0$; so by VIL*, $\Gamma_0 \not\models^*_{IL} A_0$. This is impossible; reject the assumption: if $\Gamma \models_{IL} A$, then $\Gamma \vdash_{NIL} A$.

6 many-valued logics: *Mx* (ch. 7,8)

This section develops derivations for the systems for which Priest supplies tableaux in his text: (classical logic), *K3*, *LP* and *FDE*. Thus there are no derivations for his semantically described *L3* and *RM3*.

6.1 language / semantic notions

LMx The language consists of propositional parameters $p_0, p_1 \ldots$ with the operators, $\neg$, $\wedge$, $\vee$, and $\supset$. Each propositional parameter is a formula; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, and $(A \supset B)$. $A \equiv B$ abbreviates $(A \supset B) \wedge (B \supset A)$.

IMx An interpretation is a function v which assigns to each propositional parameter some subset of $\{0, 1\}$; so $v(p)$ is ϕ, $\{1\}$, $\{0\}$ or $\{1, 0\}$. Intuitively, $v(p)$ is true iff $1 \in v(p)$ and $v(p)$ is false iff $0 \in v(p)$. Where x is empty or includes some combination of the following constraints,

exc	for no p are both $0 \in v(p)$ and $1 \in v(p)$	exclusion
exh	for any p, either $1 \in v(p)$ or $0 \in v(p)$	exhaustion

v is an *Mx* interpretation when it meets the constraints from x. *MCL* has both *exc* and *exh*, *MK3* just *exc*, *MLP* just *exh*, and *MFD* neither *exc* nor *exh* (these are classical logic, and Priest's *K3*, *LP* and *FDE*).

TM For complex expressions,

($\neg$) $1 \in v(\neg A)$ iff $0 \in v(A)$; $0 \in v(\neg A)$ iff $1 \in v(A)$.

($\wedge$) $1 \in v(A \wedge B)$ iff $1 \in v(A)$ and $1 \in v(B)$; $0 \in v(A \wedge B)$ iff $0 \in v(A)$ or $0 \in v(B)$.

(∨) $1 \in v(A \vee B)$ iff $1 \in v(A)$ or $1 \in v(B)$; $0 \in v(A \vee B)$ iff $0 \in v(A)$ and $0 \in v(B)$.

(⊃) $1 \in v(A \supset B)$ iff $0 \in v(A)$ or $1 \in v(B)$; $0 \in v(A \supset B)$ iff $1 \in v(A)$ and $0 \in v(B)$.

For a set Γ of formulas, $1 \in v(\Gamma)$ iff $1 \in v(A)$ for each $A \in \Gamma$; then,

VMx $\Gamma \models_{Mx} A$ iff there is no *Mx* interpretation v such that $1 \in v(\Gamma)$ but $1 \notin v(A)$.

This account is adequate to the (superficially) different presentations in these chapters of Priest. For the multivalued approach: classical logic has values {0}, {1}, with {1} designated; *K3* has ϕ, {0}, {1}, with {1} designated; *LP* has {0}, {1}, {0, 1}, with {1} and {0, 1} designated; and *FDE* has ϕ, {0}, {1}, {0, 1}, with {1} and {0, 1} designated. For the relational approach, we identify the relation as set membership. And a v as above maps to a Routley interpretation with $v_w(p) = 1$ iff $1 \in v(p)$, and $v_{w^*}(p) = 0$ iff $0 \in v(p)$.[5] Then, in each case, conditions for truth and validity are as above.

6.2 natural derivations: *NMx*

Introduce expressions of the sort A and $\overline{A}$. Intuitively $\overline{A}$ indicates that A is *not false*. Let \A\ and /A/ represent either A or $\overline{A}$ where what is represented is constant in a given context, but \A\ and /A/ are opposite. And similarly for //A// and \\A\\, though there need be no fixed relation between overlines on \A\ and \\A\\. Except for a pair of new rules corresponding to conditions *exc* and *exh*, derivation rules mirror ones for classical logic. (≡I) and (≡E) are now derived.

D | P
  | $\overline{P}$

U | $\overline{P}$
  | P

R | /P/
  | /P/

¬I | | /P/
   | | //Q//
   | | \\¬Q\\
   | \¬P\

¬E | | /¬P/
   | | //Q//
   | | \\¬Q\\
   | \P\

∧I | /P/
   | /Q/
   | /P ∧ Q/

∧E | /P ∧ Q/
   | /P/

∧E | /P ∧ Q/
   | /Q/

[5] For this, see [4, sections 8.5.8, 8.7.17 and 8.7.18] along with L6.0 below.

∨I | /P/
| /P ∨ Q/

∨I | /P/
| /Q ∨ P/

∨E | /P ∨ Q/
| | /P/
| | //R//
| | /Q/
| | //R//
| //R//

⊃I | | /P/
| | \Q\
| \P ⊃ Q\

⊃E | \P ⊃ Q\
| /P/
| \Q\

≡I | | /P/
| | \Q\
| | /Q/
| | \P\
| \P ≡ Q\

≡E | \P ≡ Q\
| /P/
| \Q\

≡E | \P ≡ Q\
| /Q/
| \P\

NMCL has all the rules. *NMK3* has the I- and E-rules for ¬, ∧, ∨, ⊃ with (R) and (D) (for truth ***down***). *NMLP* has the I- and E-rules for ¬, ∧, ∨, ⊃ with (R) and (U) (for truth ***up***). *NMFD* has just the I- and E-rules for ¬, ∧, ∨, ⊃ with (R). Where the members of Γ and A are expressions without overlines,

NMx $\Gamma \vdash_{\mathrm{NMx}} A$ iff there is an *NMx* derivation of A from the members of Γ.

Two-way derived rules carry over from *CL* with consistent overlines. Thus, e.g.,

Impl /P ⊃ Q/ ◁▷ /¬P ∨ Q/
/¬P ⊃ Q/ ◁▷ /P ∨ Q/

MT, NB and DS appear in the forms,

MT | /P ⊃ Q/
| \¬Q\
| /¬P/

NB | /P ≡ Q/
| \¬P\
| /¬Q/

| /P ≡ Q/
| \¬Q\
| /¬P/

DS | /P ∨ Q/
| \¬P\
| /Q/

| /P ∨ Q/
| \¬Q\
| /P/

As examples, here are derivations, cast to show the general forms, for MT and the second form of DS.

$/P \supset Q/, \backslash\neg Q\backslash \vdash_{NMx} /\neg P/$

1	$/P \supset Q/$	P
2	$\backslash\neg Q\backslash$	P
3	$\backslash P\backslash$	A (c, $\neg$I)
4	$/Q/$	1,3 $\supset$E
5	$\backslash\neg Q\backslash$	2 R
6	$/\neg P/$	3-5 $\neg$I

$/P \vee Q/, \backslash\neg Q\backslash \vdash_{NMx} /P/$

1	$/P \vee Q/$	P
2	$\backslash\neg Q\backslash$	P
3	$/P/$	A (g, 1 $\vee$E)
4	$/P/$	3 R
5	$/Q/$	A (g, 1 $\vee$E)
6	$\backslash\neg P\backslash$	A (c, $\neg$E)
7	$/Q/$	5 R
8	$\backslash\neg Q\backslash$	2 R
9	$/P/$	6-8 $\neg$E
10	$/P/$	1,3-4,5-9 $\vee$E

And for some particular results requiring (D) and (U), here are demonstrations of standard rule and axioms for classical logic, making use of the full rule set (see, e.g. [12, chapter 3]).

MP $A, A \supset B \vdash_{NM_{CL}} B$

1	A	P
2	$A \supset B$	P
3	$\overline{A}$	1 D
4	B	2,3 $\supset$E

A1 $\vdash_{NM_{CL}} A \supset (B \supset A)$

1	$\overline{A}$	A (g, $\supset$I)
2	$\overline{B}$	A (g, $\supset$I)
3	A	1 U
4	$B \supset A$	2-3 $\supset$I
5	$A \supset (B \supset A)$	1-4 $\supset$I

A2 $\vdash_{NM_{CL}} [A \supset (B \supset C)] \supset [(A \supset B) \supset (A \supset C)]$

1	$\overline{A \supset (B \supset C)}$	A (g, $\supset$I)
2	$\overline{A \supset B}$	A (g, $\supset$I)
3	$\overline{A}$	A (g, $\supset$I)
4	$A \supset B$	2 U
5	B	3,4 $\supset$E
6	$A \supset (B \supset C)$	1 U
7	$B \supset C$	3,6 $\supset$E
8	$\overline{B}$	5 D
9	C	7,8 $\supset$E
10	$A \supset C$	3-9 $\supset$I
11	$(A \supset B) \supset (A \supset C)$	2-10 $\supset$I
12	$[A \supset (B \supset C)] \supset [(A \supset B) \supset (A \supset C)]$	1-11 $\supset$I

A3 $\vdash_{NM_{CL}} (\neg A \supset \neg B) \supset [(\neg A \supset B) \supset A]$

1	$\overline{\neg A \supset \neg B}$	A (*g*, ⊃I)
2	$\quad \overline{\neg A \supset B}$	A (*g*, ⊃I)
3	$\quad\quad \overline{\neg A}$	A (*c*, ¬E)
4	$\quad\quad \neg A$	3 U
5	$\quad\quad \overline{B}$	2,4 ⊃E
6	$\quad\quad \overline{\neg B}$	1,4 ⊃E
7	$\quad\quad \neg B$	6 U
8	$\quad A$	3-7 ¬E
9	$(\neg A \supset B) \supset A$	2-8 ⊃I
10	$(\neg A \supset \neg B) \supset [(\neg A \supset B) \supset A]$	1-9 ⊃I

Of course, there is not much point going back-and-forth between overline and non-overline expressions in the full classical system. But these examples should illustrate the rules. And overlines matter for the other systems.

6.3 SOUNDNESS AND COMPLETENESS

Preliminaries: Begin with generalized notions of truth and validity to include expressions with overlines. First, ***holding*** as a generalization of TM. Say $/A/$ ***holds*** iff $h(A) = 1$ and otherwise ***fails***. As usual, for the following, cases omitted are like ones worked, and so left to the reader.

HM (B) $h(p) = 1$ iff $1 \in v(p)$, and otherwise $h(p) = 0$; $h(\overline{p}) = 1$ iff $0 \notin v(p)$, and otherwise $h(\overline{p}) = 0$.

(¬) $h(/\neg A/) = 1$ iff $h(\backslash A \backslash) = 0$, and otherwise $h(/\neg A/) = 0$.

(∧) $h(/A \wedge B/) = 1$ iff $h(/A/) = 1$ and $h(/B/) = 1$, and otherwise $h(/A \wedge B/) = 0$.

(∨) $h(/A \vee B/) = 1$ iff $h(/A/) = 1$ or $h(/B/) = 1$, and otherwise $h(/A \vee B/) = 0$.

(⊃) $h(/A \supset B/) = 1$ iff $h(\backslash A \backslash) = 0$ or $h(/B/) = 1$, and otherwise $h(/A \supset B/) = 0$.

This formulation nicely mirrors the original classical definition TCL. And h and v are related as one would expect.

L6.0 For any *Mx* interpretation v and corresponding h, $h(A) = 1$ iff $1 \in v(A)$, and $h(\overline{A}) = 1$ iff $0 \notin v(A)$.

Basis: If A has no operators, then it is a parameter p. By HM(B), $h(p) = 1$ iff $1 \in v(p)$ and $h(\overline{p}) = 1$ iff $0 \notin v(p)$; so $h(A) = 1$ iff $1 \in v(A)$, and $h(\overline{A}) = 1$ iff $0 \notin v(A)$.

Assp: For $0 \leqslant i < k$, if A has i operators, then $h(A) = 1$ iff $1 \in v(A)$, and $h(\overline{A}) = 1$ iff $0 \notin v(A)$.

Show: If A has k operators, then $h(A) = 1$ iff $1 \in v(A)$, and $h(\overline{A}) = 1$ iff $0 \notin v(A)$.

If A has k operators, then it is of the form, $\neg P$, $P \wedge Q$, $P \vee Q$, or $P \supset Q$ where P and Q have $< k$ operators.

($\neg$) Suppose A is $\neg P$. By HM($\neg$), $h(\neg P) = 1$ iff $h(\overline{P}) = 0$; by assumption, iff $0 \in v(P)$; by TM($\neg$) iff $1 \in v(\neg P)$. By HM($\neg$), $h(\overline{\neg P}) = 1$ iff $h(P) = 0$; by assumption, iff $1 \notin v(P)$; by TM($\neg$) iff $0 \notin v(\neg P)$. So $h(A) = 1$ iff $1 \in v(A)$, and $h(\overline{A}) = 1$ iff $0 \notin v(A)$.

($\wedge$) Suppose A is $P \wedge Q$. By HM($\wedge$), $h(P \wedge Q) = 1$ iff $h(P) = 1$ and $h(Q) = 1$; by assumption, iff $1 \in v(P)$ and $1 \in v(Q)$; by TM($\wedge$) iff $1 \in v(P \wedge Q)$. By HM($\wedge$), $h(\overline{P \wedge Q}) = 1$ iff $h(\overline{P}) = 1$ and $h(\overline{Q}) = 1$; by assumption, iff $0 \notin v(P)$ and $0 \notin v(Q)$; by TM($\wedge$) iff $0 \notin v(P \wedge Q)$. So $h(A) = 1$ iff $1 \in v(A)$, and $h(\overline{A}) = 1$ iff $0 \notin v(A)$.

($\vee$)

($\supset$) Suppose A is $P \supset Q$. By HM($\supset$), $h(P \supset Q) = 1$ iff $h(\overline{P}) = 0$ or $h(Q) = 1$; by assumption, iff $0 \in v(P)$ or $1 \in v(Q)$; by TM($\supset$) iff $1 \in v(P \supset Q)$. By HM($\supset$), $h(\overline{P \supset Q}) = 1$ iff $h(P) = 0$ or $h(\overline{Q}) = 1$; by assumption, iff $1 \notin v(P)$ or $0 \notin v(Q)$; by TM($\supset$) iff $0 \notin v(P \supset Q)$. So $h(A) = 1$ iff $1 \in v(A)$, and $h(\overline{A}) = 1$ iff $0 \notin v(A)$.

For any A, $h(A) = 1$ iff $1 \in v(A)$, and $h(\overline{A}) = 1$ iff $0 \notin v(A)$.

So A holds iff $1 \in v(A)$, and otherwise fails; and $\overline{A}$ holds iff $0 \notin v(A)$, and otherwise fails. This permits natural generalizations for notions of validity. For any v, where Γ is a set of expressions with or without overlines, say $h(\Gamma) = 1$ iff $h(/A/) = 1$ for each $/A/ \in \Gamma$. Then,

VMx* $\Gamma \vDash^{*}_{Mx} /A/$ iff there is no *Mx* interpretation v and corresponding h such that $h(\Gamma) = 1$ but $h(/A/) = 0$.

NMx* $\Gamma \vdash^{*}_{NMx} /A/$ iff there is an *NMx* derivation of $/A/$ from the members of Γ.

These notions reduce to the standard ones when all the members of Γ and $/A/$ are without overlines. This is obvious for NMx*. And similarly, we have $h(A) = 1$ iff $1 \in v(A)$; so VMx* collapses to VMx.

THEOREM 6.1 *NMx is sound: If* $\Gamma \vdash_{NMx} A$ *then* $\Gamma \vDash_{Mx} A$.

L6.1 If $\Gamma \subseteq \Gamma'$ and $\Gamma \vDash^{*}_{Mx} /P/$, then $\Gamma' \vDash^{*}_{Mx} /P/$.

Suppose $\Gamma \subseteq \Gamma'$ and $\Gamma \vDash^{*}_{Mx} /P/$, but $\Gamma' \nvDash^{*}_{Mx} /P/$. From the latter, by VMx*, there is some v and h such that $h(\Gamma') = 1$ but $h(/P/) = 0$. But since $h(\Gamma') = 1$ and $\Gamma \subseteq \Gamma'$, $h(\Gamma) = 1$; so $h(\Gamma) = 1$ but $h(/P/) = 0$; so by VMx*, $\Gamma \not\models^*_{Mx} /P/$. This is impossible; reject the assumption: if $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{Mx} /P/$, then $\Gamma' \models^*_{Mx} /P/$.

Main result: For each line in a derivation let $\mathcal{P}_i$ be the formula on line i (with or without overlines) and set Γ_i equal to the set of all premises and assumptions whose scope includes line i. We set out to show "generalized" soundness: if $\Gamma \vdash^*_{NMx} /A/$ then $\Gamma \models^*_{Mx} /A/$. As above, this reduces to the standard result when the members of Γ and A are without overlines. Suppose $\Gamma \vdash^*_{NMx} /A/$. Then there is a derivation of $/A/$ from premises in Γ where $/A/$ appears under the scope of the premises alone. By induction on line number of this derivation, we show that for each line i of this derivation, $\Gamma_i \models^*_{Mx} \mathcal{P}_i$. The case when $\mathcal{P}_i = /A/$ is the desired result.

Basis: $\mathcal{P}_1$ is a premise or an assumption $/A/$. Then $\Gamma_1 = \{/A/\}$; so $h(\Gamma_1) = 1$ iff $h(/A/) = 1$; so there is no h such that $h(\Gamma_1) = 1$ but $h(/A/) = 0$. So by VMx*, $\Gamma_1 \models^*_{Mx} /A/$, where this is just to say, $\Gamma_1 \models^*_{Mx} \mathcal{P}_1$.

Assp: For any $i, 1 \leqslant i < k, \Gamma_i \models^*_{Mx} \mathcal{P}_i$.

Show: $\Gamma_k \models^*_{Mx} \mathcal{P}_k$.

$\mathcal{P}_k$ is either a premise, an assumption, or arises from previous lines by R, ⊃I, ⊃E, ∧I, ∧E, ¬I, ¬E, ∨I, ∨E or, depending on the system, D or U. If $\mathcal{P}_k$ is a premise or an assumption, then as in the basis, $\Gamma_k \models^*_{Mx} \mathcal{P}_k$. So suppose $\mathcal{P}_k$ arises by one of the rules.

(R)

(⊃I) If $\mathcal{P}_k$ arises by ⊃I, then the picture is like this,

		\A\
j		/B/
k	/A ⊃ B/	

where $j < k$ and $\mathcal{P}_k$ is $/A \supset B/$. By assumption, $\Gamma_j \models^*_{Mx} /B/$; and by the nature of access, $\Gamma_j \subseteq \Gamma_k \cup \{\backslash A \backslash\}$; so by L6.1, $\Gamma_k \cup \{\backslash A \backslash\} \models^*_{Mx} /B/$. Suppose $\Gamma_k \not\models^*_{Mx} /A \supset B/$; then by VMx*, there is some ν and h such that $h(\Gamma_k) = 1$ but $h(/A \supset B/) = 0$; from the latter, by HM(⊃), $h(\backslash A \backslash) = 1$ and $h(/B/) = 0$; so $h(\Gamma_k) = 1$ and $h(\backslash A \backslash) = 1$; so $h(\Gamma_k \cup \{\backslash A \backslash\}) = 1$; so by VMx*, $h(/B/) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Mx} /A \supset B/$, which is to say, $\Gamma_k \models^*_{Mx} \mathcal{P}_k$.

(⊃E) If $\mathcal{P}_k$ arises by ⊃E, then the picture is like this,

i	/A ⊃ B/
j	\A\
k	/B/

where $i, j < k$ and $\mathcal{P}_k$ is $/B/$. By assumption, $\Gamma_i \models^*_{Mx} /A \supset B/$ and $\Gamma_j \models^*_{Mx} \backslash A\backslash$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L6.1, $\Gamma_k \models^*_{Mx} /A \supset B/$ and $\Gamma_k \models^*_{Mx} \backslash A\backslash$. Suppose $\Gamma_k \not\models^*_{Mx} /B/$; then by VMx*, there is some ν and h such that $h(\Gamma_k) = 1$ but $h(/B/) = 0$; since $h(\Gamma_k) = 1$, by VMx*, $h(/A \supset B/) = 1$ and $h(\backslash A\backslash) = 1$; from the former, by HM($\supset$), $h(\backslash A\backslash) = 0$ or $h(/B/) = 1$; so $h(/B/) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Mx} /B/$, which is to say, $\Gamma_k \models^*_{Mx} \mathcal{P}_k$.

($\wedge$I)

($\wedge$E)

($\neg$I) If $\mathcal{P}_k$ arises by $\neg$I, then the picture is like this,

	$/A/$
i	$/\!/B/\!/$
j	$\backslash\!\backslash\neg B\backslash\!\backslash$
k	$\backslash\neg A\backslash$

where $i, j < k$ and $\mathcal{P}_k$ is $\backslash\neg A\backslash$. By assumption, $\Gamma_i \models^*_{Mx} /\!/B/\!/$ and $\Gamma_j \models^*_{Mx} \backslash\!\backslash\neg B\backslash\!\backslash$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{/A/\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{/A/\}$; so by L6.1, $\Gamma_k \cup \{/A/\} \models^*_{Mx} /\!/B/\!/$ and $\Gamma_k \cup \{/A/\} \models^*_{Mx} \backslash\!\backslash\neg B\backslash\!\backslash$. Suppose $\Gamma_k \not\models^*_{Mx} \backslash\neg A\backslash$; then by VMx*, there is some ν and h such that $h(\Gamma_k) = 1$ but $h(\backslash\neg A\backslash) = 0$; from the latter, by HM($\neg$), $h(/A/) = 1$; so $h(\Gamma_k) = 1$ and $h(/A/) = 1$; so $h(\Gamma_k \cup \{/A/\}) = 1$; so by VMx*, $h(/\!/B/\!/) = 1$ and $h(\backslash\!\backslash\neg B\backslash\!\backslash) = 1$; from the latter, by HM($\neg$), $h(/\!/B/\!/) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Mx} \backslash\neg A\backslash$, which is to say, $\Gamma_k \models^*_{Mx} \mathcal{P}_k$.

($\neg$E)

($\vee$I)

($\vee$E)

(D) If $\mathcal{P}_k$ arises by D, then the picture is like this,

i	A
k	$\overline{A}$

where $i < k$ and $\mathcal{P}_k$ is $\overline{A}$. Where this rule is included in *NMx*, *Mx* has condition *exc*, so no interpretation has $\nu(p) = \{1, 0\}$. By assumption, $\Gamma_i \models^*_{Mx} A$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L6.1, $\Gamma_k \models^*_{Mx} A$. Suppose $\Gamma_k \not\models^*_{Mx} \overline{A}$; then by VMx*, there is some ν and h such that $h(\Gamma_k) = 1$ but $h(\overline{A}) = 0$; since $h(\Gamma_k) = 1$, by VMx*, $h(A) = 1$. But for these interpretations, for any A, if $h(A) = 1$ then $h(\overline{A}) = 1$.

Basis: A is a parameter p. Suppose $h(A) = 1$; then $h(p) = 1$; so by HM(B), $1 \in v(p)$; so by *exc*, $0 \notin v(p)$; so by HM(B), $h(\overline{p}) = 1$; so $h(\overline{A}) = 1$.

Assp: For any i, $0 \leqslant i < k$, if A has i operators, and $h(A) = 1$, then $h(\overline{A}) = 1$.

Show: If A has k operators, and $h(A) = 1$, then $h(\overline{A}) = 1$.

If A has k operators, then A is of the form, $\neg P$, $P \wedge Q$, $P \vee Q$, or $P \supset Q$, where P and Q have $< k$ operators.

($\neg$) A is $\neg P$. Suppose $h(A) = 1$; then $h(\neg P) = 1$; so by HM($\neg$), $h(\overline{P}) = 0$; so by assumption, $h(P) = 0$; so by HM($\neg$), $h(\overline{\neg P}) = 1$, which is to say, $h(\overline{A}) = 1$.

($\wedge$) A is $P \wedge Q$. Suppose $h(A) = 1$; then $h(P \wedge Q) = 1$; so by HM($\wedge$), $h(P) = 1$ and $h(Q) = 1$; so by assumption, $h(\overline{P}) = 1$ and $h(\overline{Q}) = 1$; so by HM($\wedge$), $h(\overline{P \wedge Q}) = 1$, which is to say $h(\overline{A}) = 1$.

($\vee$)

($\supset$) A is $P \supset Q$. Suppose $h(A) = 1$; then $h(P \supset Q) = 1$; so by HM($\supset$), $h(\overline{P}) = 0$ or $h(Q) = 1$; so by assumption, $h(P) = 0$ or $h(\overline{Q}) = 1$; so by HM($\supset$), $h(\overline{P \supset Q}) = 1$, which is to say $h(\overline{A}) = 1$.

For any A, if $h(A) = 1$, then $h(\overline{A}) = 1$.

So, returning to the case for (D), $h(\overline{A}) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Mx} \overline{A}$, which is to say, $\Gamma_k \models^*_{Mx} \mathcal{P}_k$.

(U) If $\mathcal{P}_k$ arises by U, then the picture is like this,

$$\begin{array}{r|l} i & \overline{A} \\ & \\ k & A \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is A. Where this rule is included in *NMx*, *Mx* has condition *exh*, so no interpretation has $v(p) = \phi$. By assumption, $\Gamma_i \models^*_{Mx} \overline{A}$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L6.1, $\Gamma_k \models^*_{Mx} \overline{A}$. Suppose $\Gamma_k \not\models^*_{Mx} A$; then by VMx*, there is some v and h such that $h(\Gamma_k) = 1$ but $h(A) = 0$; since $h(\Gamma_k) = 1$, by VMx*, $h(\overline{A}) = 1$. But for these interpretations, for any A, if $h(\overline{A}) = 1$ then $h(A) = 1$.

Basis: A is a parameter p. Suppose $h(\overline{A}) = 1$; then $h(\overline{p}) = 1$; so by HM(B), $0 \notin v(p)$; so by *exh*, $1 \in v(p)$; so by HM(B), $h(p) = 1$; so $h(A) = 1$.

Assp: For any i, $0 \leqslant i < k$, if A has i operators, and $h(\overline{A}) = 1$, then $h(A) = 1$.

Show: If A has k operators, and $h(\overline{A}) = 1$, then $h(A) = 1$.

If A has k operators, then A is of the form, $\neg P$, $P \wedge Q$, $P \vee Q$, or $P \supset Q$, where P and Q have $< k$ operators.

(¬) A is $\neg P$. Suppose $h(\overline{A}) = 1$; then $h(\overline{\neg P}) = 1$; so by HM(¬), $h(P) = 0$; so by assumption, $h(\overline{P}) = 0$; so by HM(¬), $h(\neg P) = 1$, which is to say, $h(A) = 1$.

(∧) A is $P \wedge Q$. Suppose $h(\overline{A}) = 1$; then $h(\overline{P \wedge Q}) = 1$; so by HM(∧), $h(\overline{P}) = 1$ and $h(\overline{Q}) = 1$; so by assumption, $h(P) = 1$ and $h(Q) = 1$; so by HM(∧), $h(P \wedge Q) = 1$, which is to say $h(A) = 1$.

(∨)

(⊃) A is $P \supset Q$. Suppose $h(\overline{A}) = 1$; then $h(\overline{P \supset Q}) = 1$; so by HM(⊃), $h(P) = 0$ or $h(\overline{Q}) = 1$; so by assumption, $h(\overline{P}) = 0$ or $h(Q) = 1$; so by HM(⊃), $h(P \supset Q) = 1$, which is to say $h(A) = 1$.

For any A, if $h(\overline{A}) = 1$, then $h(A) = 1$.

So, returning to the case for (U), $h(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Mx} A$, which is to say, $\Gamma_k \models^*_{Mx} \mathcal{P}_k$.

For any i, $\Gamma_i \models^*_{Mx} A_i$.

THEOREM 6.2 *NMx is complete: if* $\Gamma \models_{Mx} A$ *then* $\Gamma \vdash_{NMx} A$.

Suppose $\Gamma \models_{Mx} A$; then $\Gamma \models^*_{Mx} A$; we show that $\Gamma \vdash^*_{NMx} A$. Again, this reduces to the standard notion when there are no overlines. Fix on some particular constraint(s) x. Then definitions of *consistency* etc. are relative to it.

CON Γ is CONSISTENT iff there is no A such that $\Gamma \vdash^*_{NMx} /A/$ and $\Gamma \vdash^*_{NMx} \backslash\neg A\backslash$.

L6.2 If $\Gamma \nvdash^*_{NMx} \backslash\neg P\backslash$, then $\Gamma \cup \{/P/\}$ is consistent.

Suppose $\Gamma \nvdash^*_{NMx} \backslash\neg P\backslash$ but $\Gamma \cup \{/P/\}$ is inconsistent. Then there is some A such that $\Gamma \cup \{/P/\} \vdash^*_{NMx} /\!/A/\!/$ and $\Gamma \cup \{/P/\} \vdash^*_{NMx} \backslash\backslash\neg A\backslash\backslash$. But then we can argue,

1	Γ	
2	$/P/$	A (*c*, ¬I)
3	$/\!/A/\!/$	from $\Gamma \cup \{/P/\}$
4	$\backslash\backslash\neg A\backslash\backslash$	from $\Gamma \cup \{/P/\}$
5	$\backslash\neg P\backslash$	2-4 ¬I

So $\Gamma \vdash^*_{NMx} \backslash\neg P\backslash$. But this is impossible; reject the assumption: if $\Gamma \nvdash^*_{NMx} \backslash\neg P\backslash$, then $\Gamma \cup \{/P/\}$ is consistent.

L6.3 There is an enumeration of all the formulas, $\mathcal{P}_1, \mathcal{P}_2 \ldots$

Proof by construction. A simple approach is to order $A_1, A_2 \ldots$ in the usual way, and let the final enumeration be, $A_1, \overline{A}_1, A_2, \overline{A}_2 \ldots$.

MAX Γ is MAXIMAL iff for any A either $\Gamma \vdash^*_{NMx} /A/$ or $\Gamma \vdash^*_{NMx} \backslash\neg A\backslash$.

C(Γ') We construct a Γ' from Γ as follows. Set $\Omega_0 = \Gamma$. By L6.3, there is an enumeration, $\mathcal{P}_1, \mathcal{P}_2 \ldots$ of all the formulas; for any $\mathcal{P}_i = /A/$ in this series set,

$$\Omega_i = \Omega_{i-1} \quad \text{if } \Omega_{i-1} \vdash^*_{NMx} \backslash\neg A\backslash$$
$$\Omega_i = \Omega_{i-1} \cup \{/A/\} \quad \text{if } \Omega_{i-1} \nvdash^*_{NMx} \backslash\neg A\backslash$$

then

$$\Gamma' = \bigcup_{i \geqslant 0} \Omega_i$$

L6.4 Γ' is maximal.

Suppose Γ' is not maximal. Then there is some $\mathcal{P}_i = /A/$ such that $\Gamma' \nvdash^*_{NMx} /A/$ and $\Gamma' \nvdash^*_{NMx} \backslash\neg A\backslash$. For any i, each member of Ω_{i-1} is in Γ'; so if $\Omega_{i-1} \vdash^*_{NMx} \backslash\neg A\backslash$ then $\Gamma' \vdash^*_{NMx} \backslash\neg A\backslash$; but $\Gamma' \nvdash^*_{NMx} \backslash\neg A\backslash$; so $\Omega_{i-1} \nvdash^*_{NMx} \backslash\neg A\backslash$; so by construction, $\Omega_i = \Omega_{i-1} \cup \{/A/\}$; so by construction, $/A/ \in \Gamma'$; so $\Gamma' \vdash^*_{NMx} /A/$. This is impossible; reject the assumption: Γ' is maximal.

L6.5 If Γ is consistent, then each Ω_i is consistent.

Suppose Γ is consistent.

Basis: $\Omega_0 = \Gamma$ and Γ is consistent; so Ω_0 is consistent.

Assp: For any $i, 0 \leqslant i < k$, Ω_i is consistent.

Show: Ω_k is consistent.

Ω_k is either Ω_{k-1} or $\Omega_{k-1} \cup \{/A/\}$. Suppose the former; by assumption, Ω_{k-1} is consistent; so Ω_k is consistent. Suppose the latter; then by construction, $\Omega_{k-1} \nvdash^*_{NMx} \backslash\neg A\backslash$; so by L6.2, $\Omega_{k-1} \cup \{/A/\}$ is consistent; so Ω_k is consistent.

For any i, Ω_i is consistent.

L6.6 If Γ is consistent, then Γ' is consistent.

Suppose Γ is consistent, but Γ' is not; from the latter, there is some P such that $\Gamma' \vdash^*_{NMx} /P/$ and $\Gamma' \vdash^*_{NMx} \backslash\neg P\backslash$. Consider derivations D1 and D2 of these results and the premises of these derivations. Where $\mathcal{P}_i$ is the last of these premises in the enumeration of formulas, by the construction of Γ', each of the premises must be a member of Ω_i; so D1 and D2 are derivations from Ω_i; so Ω_i is not consistent. But since Γ is consistent, by L6.5, Ω_i is consistent. This is impossible; reject the assumption: if Γ is consistent then Γ' is consistent.

C(ν) We construct an interpretation ν based on Γ' as follows. For any parameter p, set $1 \in \nu(p)$ iff $\Gamma' \vdash^*_{NMx} p$, and $0 \in \nu(p)$ iff $\Gamma' \nvdash^*_{NMx} \overline{p}$.

L6.7 If Γ is consistent then for any A, $h(/A/) = 1$ iff $\Gamma' \vdash^*_{NMx} /A/$.

Suppose Γ is consistent. By L6.4, Γ' is maximal; by L6.6, Γ' is consistent. Now by induction on the number of operators in A, *Basis:* If A has no operators, then it is a parameter p or $\overline{p}$. By construction, $\Gamma' \vdash^*_{NMx} p$ iff $1 \in \nu(p)$; by HM(B), iff $h(p) = 1$. Similarly, by construction, $\Gamma' \nvdash^*_{NMx} \overline{p}$ iff $0 \in \nu(p)$; by HM(B), iff $h(\overline{p}) \neq 1$. So $h(/p/) = 1$ iff $\Gamma' \vdash^*_{NMx} /p/$, which is to say, $h(/A/) = 1$ iff $\Gamma' \vdash^*_{NMx} /A/$.

Assp: For any i, $0 \leqslant i < k$, if A has i operators, then $h(/A/) = 1$ iff $\Gamma' \vdash^*_{NMx} /A/$.

Show: If A has k operators, then $h(/A/) = 1$ iff $\Gamma' \vdash^*_{NMx} /A/$.

If A has k operators, then it is of the form $\neg P$, $P \wedge Q$, $P \vee Q$ or $P \supset Q$ where P and Q have $< k$ operators.

($\neg$) A is $\neg P$. (i) Suppose $h(/A/) = 1$; then $h(/\neg P/) = 1$; so by HM($\neg$), $h(\backslash P\backslash) = 0$; so by assumption, $\Gamma' \nvdash^*_{NMx} \backslash P\backslash$; so by maximality, $\Gamma' \vdash^*_{NMx} /\neg P/$, where this is to say, $\Gamma' \vdash^*_{NMx} /A/$. (ii) Suppose $\Gamma' \vdash^*_{NMx} /A/$; then $\Gamma' \vdash^*_{NMx} /\neg P/$; so by consistency, $\Gamma' \nvdash^*_{NMx} \backslash P\backslash$; so by assumption, $h(\backslash P\backslash) = 0$; so by HM($\neg$), $h(/\neg P/) = 1$, where this is to say, $h(/A/) = 1$. So $h(/A/) = 1$ iff $\Gamma' \vdash^*_{NMx} /A/$.

($\wedge$)

($\vee$)

($\supset$) A is $P \supset Q$. (i) Suppose $h(/A/) = 1$ but $\Gamma' \nvdash^*_{NMx} /A/$; then $h(/P \supset Q/) = 1$ but $\Gamma' \nvdash^*_{NMx} /P \supset Q/$. From the latter, by maximality, $\Gamma' \vdash^*_{NMx} \backslash\neg(P \supset Q)\backslash$; from this it follows, by the following derivations,

1	$\backslash\neg(P \supset Q)\backslash$	P
2	$\quad /\neg P/$	A (c, $\neg$E)
3	$\quad\quad \backslash P\backslash$	A (g, $\supset$I)
4	$\quad\quad\quad \backslash\neg Q\backslash$	A (c, $\neg$E)
5	$\quad\quad\quad \backslash P\backslash$	3 R
6	$\quad\quad\quad /\neg P/$	2 R
7	$\quad\quad /Q/$	4-6 $\neg$E
8	$\quad /P \supset Q/$	3-7 $\supset$I
9	$\quad \backslash\neg(P \supset Q)\backslash$	1 R
10	$\backslash P\backslash$	2-9 $\neg$E

1	$\backslash\neg(P \supset Q)\backslash$	P
2	$\quad /Q/$	A (c, $\neg$I)
3	$\quad\quad \backslash P\backslash$	A (g, $\supset$I)
4	$\quad\quad /Q/$	2 R
5	$\quad /P \supset Q/$	3-4 $\supset$I
6	$\quad \backslash\neg(P \supset Q)\backslash$	1 R
7	$\backslash\neg Q\backslash$	2-6 $\neg$I

that $\Gamma' \vdash^*_{NMx} \backslash P\backslash$ and $\Gamma' \vdash^*_{NMx} \backslash\neg Q\backslash$; so by consistency, $\Gamma' \nvdash^*_{NMx} /Q/$; so by assumption, $h(\backslash P\backslash) = 1$ and $h(/Q/) = 0$; so by HM($\supset$), $h(/P \supset Q/) = 0$. This is impossible; reject the assumption: if $h(/A/) = 1$ then $\Gamma' \vdash^*_{NMx} /A/$.

(ii) Suppose $\Gamma' \vdash^*_{NMx} /A/$ but $h(/A/) = 0$; then $\Gamma' \vdash^*_{NMx} /P \supset Q/$ but $h(/P \supset Q/) = 0$. From the latter, by HM($\supset$), $h(\backslash P\backslash) = 1$ and $h(/Q/) = 0$; so by assumption, $\Gamma' \vdash^*_{NMx} \backslash P\backslash$ and $\Gamma' \nvdash^*_{NMx} /Q/$; but since $\Gamma' \vdash^*_{NMx} /P \supset Q/$ and $\Gamma' \vdash^*_{NMx} \backslash P\backslash$, by ($\supset$E), $\Gamma' \vdash^*_{NMx} /Q/$.

This is impossible; reject the assumption: if $\Gamma' \vdash^*_{NMx} /A/$, then $h(/A/) = 1$. So $h(/A/) = 1$ iff $\Gamma' \vdash^*_{NMx} /A/$.

———

For any A, $h(/A/) = 1$ iff $\Gamma' \vdash^*_{NMx} /A/$.

L6.8 If Γ is consistent, then ν constructed as above is an *Mx* interpretation.

For this, we need to show that the relevant constraints are met. Suppose Γ is consistent; by L6.4, Γ′ is maximal; by L6.6, Γ′ is consistent.

(*exc*) For systems *MCL* and *MK3* with $\nu(p) \neq \{1,0\}$, (D) is in *NKx*. Suppose $\nu(p) = \{1,0\}$; then $1 \in \nu(p)$ and $0 \in \nu(p)$; so by construction, $\Gamma' \vdash^*_{NMx} p$ and $\Gamma' \nvdash^*_{NMx} \overline{p}$; from the latter, by maximality, $\Gamma' \vdash^*_{NMx} \neg p$; so by (D), $\Gamma' \vdash^*_{NMx} \overline{\neg p}$; so Γ′ is inconsistent. This is impossible; reject the assumption: $\nu(p) \neq \{1,0\}$.

(*exh*) For systems *MCL* and *MLP* with $\nu(p) \neq \phi$, (U) is in *NKx*. Suppose $\nu(p) = \phi$; then $1 \notin \nu(p)$ and $0 \notin \nu(p)$; so by construction, $\Gamma' \nvdash^*_{NMx} p$ and $\Gamma' \vdash^*_{NMx} \overline{p}$; from the former, by maximality, $\Gamma' \vdash^*_{NMx} \overline{\neg p}$; so by (U), $\Gamma' \vdash^*_{NMx} \neg p$; so Γ′ is inconsistent. This is impossible; reject the assumption: $\nu(p) \neq \phi$.

L6.9 If Γ is consistent, then $h(\Gamma) = 1$.

Suppose Γ is consistent and $/A/ \in \Gamma$; then by construction, $/A/ \in \Gamma'$; so $\Gamma' \vdash^*_{NMx} /A/$; so since Γ is consistent, by L6.7, $h(/A/) = 1$. And similarly for any $/A/ \in \Gamma$. So $h(\Gamma) = 1$.

Main result: Suppose $\Gamma \models_{Mx} A$ but $\Gamma \nvdash_{NMx} A$. Then $\Gamma \models^*_{Mx} A$ but $\Gamma \nvdash^*_{NMx} A$. By (DN), if $\Gamma \vdash^*_{NMx} \neg\neg A$, then $\Gamma \vdash^*_{NMx} A$; so $\Gamma \nvdash^*_{NMx} \neg\neg A$; so by L6.2, $\Gamma \cup \{\overline{\neg A}\}$ is consistent; so by L6.8 and L6.9, there is an *Mx* interpretation ν with corresponding h constructed as above such that $h(\Gamma \cup \{\overline{\neg A}\}) = 1$; so $h(\overline{\neg A}) = 1$; so by HM(¬), $h(A) = 0$; so $h(\Gamma) = 1$ and $h(A) = 0$; so by VMx*, $\Gamma \not\models^*_{Mx} A$. This is impossible; reject the assumption: if $\Gamma \models_{Mx} A$, then $\Gamma \vdash_{NMx} A$.

7 BASIC RELEVANT LOGIC: υX (CH. 9)

7.1 LANGUAGE / SEMANTIC NOTIONS

This section is developed directly in terms introduced for "expanded" notions of validity in demonstration of soundness and completeness in section 6. Apart from that discussion, the notions should be roughly familiar from derivations in that section.

LυX The VOCABULARY consists of propositional parameters $p_0, p_1 \ldots$ with the operators, ¬, ∧, ∨, and →. Each propositional parameter is a FORMULA; if A and B are formulas, so are ¬A, (A∧B), (A∨B), and (A → B). $A \supset B$ abbreviates $\neg A \vee B$, and $A \equiv B$ abbreviates $(A \supset B) \wedge (B \supset A)$.

This time, from the start, for any formula A, we allow A and $\overline{A}$, where as before /A/ and \A\ (//A// and \\A\\) represent one or the other (and similarly for N and $\overline{N}$ immediately below).

IυX An INTERPRETATION is $\langle W, N, \overline{N}, h\rangle$ where W is a set of worlds, and $N, \overline{N} \subseteq W$ are normal worlds for truth and non-falsity respectively; h is a function such that for any $w \in W$, $h_w(/p/) = 1$ or $h_w(/p/) = 0$, and for any w not in /N/, $h_w(/A \rightarrow B/) = 1$ or $h_w(/A \rightarrow B/) = 0$. So h makes assignments directly to experssions of the sort $/A \rightarrow B/$ at worlds not in /N/. Say /A/ ***holds*** at w if $h_w(/A/) = 1$ and otherwise ***fails***. Interpretations may also be subject to the constraints,

K $\quad N = \overline{N} = W$

4 $\quad N = \overline{N}$

The K systems are subject to constraint (K), the 4 systems to (4). Of course, (K) implies (4); so it is enough that interpretations for υK_4 and υK_* are subject to (K); υN_4 is subject to (4), and υN_* to neither. With restriction K, h reduces to a simple assignment to parameters at worlds.

Hυ For expressions not assigned a value directly,

- ($\neg$) $h_w(/\neg A/) = 1$ if $h_w(\backslash A\backslash) = 0$, and 0 otherwise.
- ($\wedge$) $h_w(/A \wedge B/) = 1$ if $h_w(/A/) = 1$ and $h_w(/B/) = 1$, and 0 otherwise.
- ($\vee$) $h_w(/A \vee B/) = 1$ if $h_w(/A/) = 1$ or $h_w(/B/) = 1$, and 0 otherwise.
- $(\rightarrow)_4$ For $w \in /N/$, $h_w(/A \rightarrow B/) = 1$ iff there is no $x \in W$ such that $h_x(A) = 1$ and $h_x(/B/) = 0$.
- $(\rightarrow)_*$ For $w \in /N/$, $h_w(/A \rightarrow B/) = 1$ iff there is no $x \in W$ such that $h_x(//A//) = 1$ and $h_x(//B//) = 0$.

The 4-systems υN_4 and υK_4 take $H\upsilon(\rightarrow)_4$; the star systems υN_* and υK_* take $H\upsilon(\rightarrow)_*$. Where Γ does not include formulas with overlines, $h_w(\Gamma) = 1$ iff $h_w(A) = 1$ for each $A \in \Gamma$; then,

VυX $\Gamma \models_{\upsilon X} A$ iff there is no υX interpretation $\langle W, N, \overline{N}, h\rangle$ and $w \in N$ such that $h_w(\Gamma) = 1$ and $h_w(A) = 0$.

As in the previous section, the single account is meant to accommodate different presentations in Priest, and help exhibit their differences. In particular, as for the previous section, given constraint (4), an interpretation $\langle W, N, \overline{N}, h\rangle$ corresponds to a relational $\langle W, N, \rho\rangle$, where $h_w(A) = 1$ iff A bears relation ρ (which, as in the previous section, may be set membership) to 1 at w, and $h_w(\overline{A}) = 1$ iff A does not bear ρ to 0 at w. And an interpretation $\langle W, N, \overline{N}, h\rangle$

corresponds to a star interpretation $\langle W, N, *, v\rangle$ where $h_w(A) = 1$ iff $v_w(A) = 1$ and $h_w(\overline{A}) = 1$ iff $v_{w^*}(A) = 1$.[6]

7.2 NATURAL DERIVATIONS: NυX

Allow expressions with both integer subscripts and overlines. I- and E- rules for $\neg$, $\wedge$, $\vee$, $\supset$ and $\equiv$ are a natural combination of rules for *NKυ* and *NFDE*, with rules for $\supset$ and $\equiv$ now derived.

R
$/P/_s$
$/P/_s$

$\neg$I
| $/P/_s$
| $/\!/Q/\!/_t$
| $\backslash\!\backslash\neg Q\backslash\!\backslash_t$
$\backslash\neg P\backslash_s$

$\neg$E
| $/\neg P/_s$
| $/\!/Q/\!/_t$
| $\backslash\!\backslash\neg Q\backslash\!\backslash_t$
$\backslash P\backslash_s$

$\wedge$I
$/P/_s$
$/Q/_s$
$/P \wedge Q/_s$

$\wedge$E
$/P \wedge Q/_s$
$/P/_s$

$\wedge$E
$/P \wedge Q/_s$
$/Q/_s$

$\vee$I
$/P/_s$
$/P \vee Q/_s$

$\vee$I
$/P/_s$
$/Q \vee P/_s$

$\vee$E
$/P \vee Q/_s$
| $/P/_s$
| $/\!/R/\!/_t$
| $/Q/_s$
| $/\!/R/\!/_t$
$/\!/R/\!/_t$

$\supset$I
| $/P/_s$
| $\backslash Q\backslash_s$
$\backslash P \supset Q\backslash_s$

$\supset$E
$\backslash P \supset Q\backslash_s$
$/P/_s$
$\backslash Q\backslash_s$

$\equiv$I
| $/P/_s$
| $\backslash Q\backslash_s$
| $/Q/_s$
| $\backslash P\backslash_s$
$\backslash P \equiv Q\backslash_s$

$\equiv$E
$\backslash P \equiv Q\backslash_s$
$/P/_s$
$\backslash Q\backslash_s$

$\equiv$E
$\backslash P \equiv Q\backslash_s$
$/Q/_s$
$\backslash P\backslash_s$

The different derivation systems of this section add to these from,

[6] For the latter, given a star interpretation $\langle W, N, *, v\rangle$ consider an υX_* interpretation $\langle W', N', \overline{N}', h\rangle$ with a $w' \in W'$ corresponding to each $w \in W$. And for an υX_* interpretation $\langle W', N', \overline{N}', h\rangle$ consider a star interpretation $\langle W, N, *, v\rangle$ with a w and $w^* \in W$ corresponding to each $w' \in W'$. Then set $x' \in N'$ iff $x \in N$; $x' \in \overline{N}'$ iff $x^* \in N$; $h_{x'}(p) = 1$ iff $v_x(p) = 1$; $h_{x'}(\overline{p}) = 1$ iff $v_{x^*}(p) = 1$; for $x' \notin N'$, $h_{x'}(P \to Q) = 1$ iff $v_x(P \to Q) = 1$; and for $x' \notin \overline{N}'$, $h_{x'}(\overline{P \to Q}) = 1$ iff $v_{x^*}(P \to Q) = 1$. Then the result follows by a simple induction (for a related demonstration, see the proof of L7.0 below).

$\to$I4	P_t (assumption)		
	$/Q/_t$		
	$/P \to Q/_s$		
where t does not appear in any undischarged premise or assumption			
$\to$E4	$/P \to Q/_s$		
	P_t		
	$/Q/_t$		
$\to$I$*$	$//P//_t$ (assumption)		
	$//Q//_t$		
	$/P \to Q/_s$		
where t does not appear in any undischarged premise or assumption			
$\to$E$*$	$/P \to Q/_s$		
	$//P//_t$		
	$//Q//_t$		

For the star-rules, $//P//_t$ and $//Q//_t$ may be either P_t and Q_t, or $\overline{P}_t$ and $\overline{Q}_t$. Consider a constraint (n) which requires that $s = 0$ for application of $\to$I and $\to$E, and a stronger constraint (s) which requires that $/P \to Q/_s$ for these rules is of the sort $(P \to Q)_0$ with subscript 0 and without overline. Then,

NυK$_4$ adds $\to$I4 and $\to$E4

NυN$_4$ adds $\to$I4 and $\to$E4 with constraint (n)

NυK$_*$ adds $\to$I$*$ and $\to$E$*$

NυN$_*$ adds $\to$I$*$ and $\to$E$*$ with constraint (s)

In these systems, every subscript is 0, appears in a premise, or appears in the t-place of an accessible assumption for $\to$I. Where the members of Γ and A are without overlines or subscripts, let Γ_0 be the members of Γ, each with subscript 0. Then,

NυX $\Gamma \vdash_{\text{N}\upsilon\text{X}} A$ iff there is an NυX derivation of A_0 from Γ_0.

Derived rules are as one would expect. Two-way derived rules carry over from *CL* with overlines and subscripts constant throughout. Thus, e.g.,

Impl $/P \supset Q/_s \;\triangleleft\triangleright\; /\neg P \vee Q/_s$
$/\neg P \supset Q/_s \;\triangleleft\triangleright\; /P \vee Q/_s$

MT, NB and DS appear in the forms,

MT	$/P \supset Q/_s$	$\backslash\neg Q\backslash_s$	$/\neg P/_s$
NB	$/P \equiv Q/_s$	$\backslash\neg P\backslash_s$	$/\neg Q/_s$
	$/P \equiv Q/_s$	$\backslash\neg Q\backslash_s$	$/\neg P/_s$
DS	$/P \vee Q/_s$	$\backslash\neg P\backslash_s$	$/Q/_s$
	$/P \vee Q/_s$	$\backslash\neg Q\backslash_s$	$/P/_s$

As examples, here are a few cases where the logics do not all have the same results.

$P \to Q \vdash_{\text{N}\upsilon\text{X}_*} \neg Q \to \neg P$

1	$(P \to Q)_0$	P
2	$\neg Q_1$	A (g, $\to$I$*$)
3	$\overline{P}_1$	A (c, $\neg$I)
4	$\overline{Q}_1$	1,3 $\to$E$*$
5	$\neg Q_1$	2 R
6	$\neg P_1$	3-5 $\neg$I
7	$(\neg Q \to \neg P)_0$	2-6 $\to$I$*$

This derivation satisfies constraints (n) and (s), but does not go through in the 4-systems insofar as there is no "purchase" for application of $\rightarrow E_4$ with (1) and only $\overline{P}_1$, rather than P_1, at (3).

$P \wedge \neg Q \vdash_{N\upsilon X_4} \neg(P \rightarrow Q)$

1	$(P \wedge \neg Q)_0$	P
2	$\quad \overline{(P \rightarrow Q)}_0$	A (*c*, ¬I)
3	$\quad P_0$	1 ∧E
4	$\quad \overline{Q}_0$	2,3 $\rightarrow E_4$
5	$\quad \neg Q_0$	1 ∧E
6	$\neg(P \rightarrow Q)_0$	2-5 ¬I

This derivation satisfies constraint (n), though not (s). It is blocked in either star system insofar as the contradiction does not arise; by $\rightarrow E_*$, we might get Q_0 at (4), but this does not contradict $\neg Q_0$ for ¬I.

$\vdash_{N\upsilon K_x} [(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R)$

1	$\quad [(P \rightarrow Q) \wedge (Q \rightarrow R)]_1$	A (*g*, $\rightarrow I_x$)
2	$\quad\quad P_2$	A (*g*, $\rightarrow I_x$)
3	$\quad\quad (P \rightarrow Q)_1$	1 ∧E
4	$\quad\quad Q_2$	2,3 $\rightarrow E_x$
5	$\quad\quad (Q \rightarrow R)_1$	1 ∧E
6	$\quad\quad R_2$	4,5 $\rightarrow E_x$
7	$\quad (P \rightarrow R)_1$	2-6 $\rightarrow I_x$
8	$([(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R))_0$	1-7 $\rightarrow I_x$

This derivation works with either the star- or 4-rules. But it fails constraints (n) and (s) insofar as $s = 1$ for lines (4), (6) and (7).

7.3 SOUNDNESS AND COMPLETENESS

Preliminaries: Begin with generalized notions of validity. Given any model $\langle W, N, \overline{N}, h\rangle$, let m be a map from subscripts into W such that $m(0)$ is some member of N. Then say $\langle W, N, \overline{N}, h\rangle_m$ is $\langle W, N, \overline{N}, h\rangle$ ***with*** map m. Then, where Γ is a set of expressions of our language for derivations, $h_m(\Gamma) = 1$ iff for each $/A_s/ \in \Gamma$, $h_{m(s)}(/A/) = 1$. Now expand notions of validity for subscripts and overlines as follows,

VυX* $\Gamma \models^*_{\upsilon X} /A/_s$ iff there is no υX interpretation $\langle W, N, \overline{N}, h\rangle_m$ such that $h_m(\Gamma) = 1$ but $h_{m(s)}(/A/) = 0$.

NυX* $\Gamma \vdash^*_{N\upsilon X} /A/_s$ iff there is an $N\upsilon X$ derivation of $/A/_s$ from the members of Γ.

These notions reduce to the standard ones when all the members of Γ and A are without overlines and have subscript 0. As usual, for the following, cases omitted are like ones worked, and so left to the reader.

THEOREM 7.1 NυX *is sound: If* $\Gamma \vdash_{N\upsilon X} A$ *then* $\Gamma \models_{\upsilon X} A$.

For the $(\rightarrow)_*$ case, it will be useful to have a further preliminary.

L7.0 For an interpretation $\langle W, N, \overline{N}, h\rangle$, consider $\langle W', N', \overline{N}', h'\rangle$ such that corresponding to each $w \in W$ there are $w', w^* \in W'$ where, (i) $w' \in /N'/$ iff $w \in /N/$, and $w^* \in /N'/$ iff $w \in \backslash N\backslash$; (ii) $h'_{w'}(/p/) = 1$ iff $h_w(/p/) = 1$, and $h'_{w^*}(/p/) = 1$ iff $h_w(\backslash p\backslash) = 1$; (iii) for $w' \notin /N'/$, $h'_{w'}(/P \rightarrow Q/) = 1$ iff $h_w(/P \rightarrow Q/) = 1$, and for $w^* \notin /N'/$, $h'_{w^*}(/P \rightarrow Q/) = 1$ iff $h_w(\backslash P \rightarrow Q\backslash) = 1$. Then,

For the star systems and interpretations as above, for any $/A/$, we have (i) $h'_{w'}(/A/) = 1$ iff $h_w(/A/) = 1$ and (ii) $h'_{w^*}(/A/) = 1$ iff $h_w(\backslash A\backslash) = 1$.

Basis: $/A/$ is an atomic $/p/$. (i) By construction, $h'_{w'}(/p/) = 1$ iff $h_w(/p/) = 1$; so $h'_{w'}(/A/) = 1$ iff $h_w(/A/) = 1$. Similarly, (ii) by construction, $h'_{w^*}(/p/) = 1$ iff $h_w(\backslash p\backslash) = 1$; so $h'_{w^*}(/A/) = 1$ iff $h_w(\backslash A\backslash) = 1$.

Assp: For any i, $0 \leqslant i < k$, if $/A/$ has i operators, (i) $h'_{w'}(/A/) = 1$ iff $h_w(/A/) = 1$ and (ii) $h'_{w^*}(/A/) = 1$ iff $h_w(\backslash A\backslash) = 1$.

Show: If $/A/$ has k operators, then (i) $h'_{w'}(/A/) = 1$ iff $h_w(/A/) = 1$ and (ii) $h'_{w^*}(/A/) = 1$ iff $h_w(\backslash A\backslash) = 1$.

If $/A/$ has k operators, then it is of the form, $/\neg P/$, $/P \wedge Q/$, $/P \vee Q/$, or $/P \rightarrow Q/$, where P and Q have $< k$ operators.

($\neg$) $/A/$ is $/\neg P/$. (i) $h'_{w'}(/A/) = 1$ iff $h'_{w'}(/\neg P/) = 1$; by H$\upsilon(\neg)$, iff $h'_{w'}(\backslash P\backslash) = 0$; by assumption iff $h_w(\backslash P\backslash) = 0$; by H$\upsilon(\neg)$, iff $h_w(/\neg P/) = 1$; iff $h_w(/A/) = 1$. (ii) (i) $h'_{w^*}(/A/) = 1$ iff $h'_{w^*}(/\neg P/) = 1$; by H$\upsilon(\neg)$, iff $h'_{w^*}(\backslash P\backslash) = 0$; by assumption iff $h_w(/P/) = 0$; by H$\upsilon(\neg)$, iff $h_w(\backslash\neg P\backslash) = 1$; iff $h_w(\backslash A\backslash) = 1$.

($\wedge$) $/A/$ is $/P \wedge Q/$. (i) $h'_{w'}(/A/) = 1$ iff $h'_{w'}(/P \wedge Q/) = 1$; by H$\upsilon(\wedge)$, iff $h'_{w'}(/P/) = 1$ and $h'_{w'}(/Q/) = 1$; by assumption, iff $h_w(/P/) = 1$ and $h_w(/Q/) = 1$; by H$\upsilon(\wedge)$, iff $h_w(/P \wedge Q/) = 1$; iff $h_w(/A/) = 1$. (ii) $h'_{w^*}(/A/) = 1$ iff $h'_{w^*}(/P \wedge Q/) = 1$; by H$\upsilon(\wedge)$, iff $h'_{w^*}(/P/) = 1$ and $h'_{w^*}(/Q/) = 1$; by assumption, iff $h_w(\backslash P\backslash) = 1$ and $h_w(\backslash Q\backslash) = 1$; by H$\upsilon(\wedge)$, iff $h_w(\backslash P \wedge Q\backslash) = 1$; iff $h_w(\backslash A\backslash) = 1$.

($\vee$)

($\rightarrow$) A is $/P \rightarrow Q/$. (i) Suppose $w' \notin /N'/$; then by construction, $h'_{w'}(/P \rightarrow Q/) = 1$ iff $h_w(/P \rightarrow Q/) = 1$; so $h'_{w'}(/A/) = 1$ iff $h_w(/A/) = 1$. So suppose $w' \in /N'/$; then by construction, $w \in /N/$. $h'_{w'}(/A/) = 0$ iff $h'_{w'}(/A \rightarrow B/) = 0$; since $w' \in /N'/$, by H$\upsilon(\rightarrow)_*$ iff either there is an $x' \in W'$ such that $h'_{x'}(/\!/P/\!/) = 1$ and $h'_{x'}(/\!/Q/\!/) = 0$, or there is a $y^* \in W'$ such that $h'_{y^*}(/\!/P/\!/) = 1$ and $h'_{y^*}(/\!/Q/\!/) = 0$; by assumption, iff either $h_x(/\!/P/\!/) = 1$ and

$h_x(/\!\!/Q/\!\!/) = 0$, or $h_y(\backslash\!\!\backslash P\backslash\!\!\backslash) = 1$ and $h_y(\backslash\!\!\backslash Q\backslash\!\!\backslash) = 0$; given either of these, since $w \in /N/$, by $H\upsilon(\rightarrow)_*$, iff $h_w(/P \rightarrow Q/) = 0$; iff $h_w(/A/) = 0$.

(ii) Suppose $w^* \notin /N'/$; then by construction, $h'_{w^*}(/P \rightarrow Q/) = 1$ iff $h_w(\backslash P \rightarrow Q\backslash) = 1$; so $h'_{w^*}(/A/) = 1$ iff $h_w(\backslash A\backslash) = 1$. So suppose $w^* \in /N'/$; then $w \in \backslash N\backslash$. $h'_{w^*}(/A/) = 0$ iff $h'_{w^*}(/A \rightarrow B/) = 0$; since $w^* \in /N'/$, by $H\upsilon(\rightarrow)_*$ iff either there is an $x' \in W'$ such that $h'_{x'}(/\!\!/P/\!\!/) = 1$ and $h'_{x'}(/\!\!/Q/\!\!/) = 0$, or there is a $y^* \in W'$ such that $h'_{y^*}(/\!\!/P/\!\!/) = 1$ and $h'_{y^*}(/\!\!/Q/\!\!/) = 0$; by assumption, iff either $h_x(/\!\!/P/\!\!/) = 1$ and $h_x(/\!\!/Q/\!\!/) = 0$, or $h_y(\backslash\!\!\backslash P\backslash\!\!\backslash) = 1$ and $h_y(\backslash\!\!\backslash Q\backslash\!\!\backslash) = 0$; given either of these, since $w \in \backslash N\backslash$, by $H\upsilon(\rightarrow)_*$, iff $h_w(\backslash P \rightarrow Q\backslash) = 0$; iff $h_w(\backslash A\backslash) = 0$.

For any A, (i) $h'_{w'}(/A/) = 1$ iff $h_w(/A/) = 1$ and (ii) $h'_{w^*}(/A/) = 1$ iff $h_w(\backslash A\backslash) = 1$.

L7.1 If $\Gamma \subseteq \Gamma'$ and $\Gamma \vDash^*_{\upsilon X} /P/_s$ then $\Gamma' \vDash^*_{\upsilon X} /P/_s$.

Suppose $\Gamma \subseteq \Gamma'$ and $\Gamma \vDash^*_{\upsilon X} /P/_s$, but $\Gamma' \nvDash^*_{\upsilon X} /P/_s$. From the latter, by $V\upsilon X^*$, there is some υX interpretation $\langle W, N, \overline{N}, h\rangle_m$ such that $h_m(\Gamma') = 1$ but $h_{m(s)}(/P/) = 0$. But since $h_m(\Gamma') = 1$ and $\Gamma \subseteq \Gamma'$, $h_m(\Gamma) = 1$; so $h_m(\Gamma) = 1$ but $h_{m(s)}(/P/) = 0$; so by $V\upsilon X^*$, $\Gamma \nvDash^*_{\upsilon X} /P/_s$. This is impossible; reject the assumption: if $\Gamma \subseteq \Gamma'$ and $\Gamma \vDash^*_{\upsilon X} /P/_s$, then $\Gamma' \vDash^*_{\upsilon X} /P/_s$.

Main result: For each line in a derivation let $\mathcal{P}_i$ be the expression on line i and Γ_i be the set of all premises and assumptions whose scope includes line i. We set out to show "generalized" soundness: if $\Gamma \vdash^*_{N\upsilon X} \mathcal{P}$ then $\Gamma \vDash^*_{\upsilon X} \mathcal{P}$. As above, this reduces to the standard result when $\mathcal{P}$ and all the members of Γ are without overlines and have subscript 0. Suppose $\Gamma \vdash^*_{N\upsilon X} \mathcal{P}$. Then there is a derivation of $\mathcal{P}$ from premises in Γ where $\mathcal{P}$ appears under the scope of the premises alone. By induction on line number of this derivation, we show that for each line i of this derivation, $\Gamma_i \vDash^*_{\upsilon X} \mathcal{P}_i$. The case when $\mathcal{P}_i = \mathcal{P}$ is the desired result.

Basis: $\mathcal{P}_1$ is a premise or an assumption $/A/_s$. Then $\Gamma_1 = \{/A/_s\}$; it follows that for any $\langle W, N, \overline{N}, h\rangle_m$, $h_m(\Gamma_1) = 1$ iff $h_{m(s)}(/A/) = 1$; so there is no $\langle W, N, \overline{N}, h\rangle_m$ such that $h_m(\Gamma_1) = 1$ but $h_{m(s)}(/A/) = 0$. So by $V\upsilon X^*$, $\Gamma_1 \vDash^*_{\upsilon X} /A/_s$, where this is just to say, $\Gamma_1 \vDash^*_{\upsilon X} \mathcal{P}_1$.

Assp: For any $i, 1 \leqslant i < k, \Gamma_i \vDash^*_{\upsilon X} \mathcal{P}_i$.

Show: $\Gamma_k \vDash^*_{\upsilon X} \mathcal{P}_k$.

$\mathcal{P}_k$ is either a premise, an assumption, or arises from previous lines by R, $\wedge$I, $\wedge$E, $\vee$I, $\vee$E, $\neg$I, $\neg$E or, depending on the system, $\rightarrow I_4$, $\rightarrow E_4$, $\rightarrow I_*$, or $\rightarrow E_*$. If $\mathcal{P}_k$ is a premise or an assumption, then as in the basis, $\Gamma_k \vDash^*_{\upsilon X} \mathcal{P}_k$. So suppose $\mathcal{P}_k$ arises by one of the rules.

(R)

(∧I)

(∧E)

(∨I)

(∨E)

(¬I) If $\mathcal{P}_k$ arises by ¬I, then the picture is like this,

$$\begin{array}{r|l}
 & \quad /A/_s \\
 & \quad \vdots \\
i & \quad //B//_t \\
j & \quad \backslash\backslash\neg B\backslash\backslash_t \\
k & \backslash\neg A\backslash_s
\end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $\backslash\neg A\backslash_s$. By assumption, $\Gamma_i \models^*_{\upsilon X} //B//_t$ and $\Gamma_j \models^*_{\upsilon X} \backslash\backslash\neg B\backslash\backslash_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{/A/_s\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{/A/_s\}$; so by L7.1, $\Gamma_k \cup \{/A/_s\} \models^*_{\upsilon X} //B//_t$ and $\Gamma_k \cup \{/A/_s\} \models^*_{\upsilon X} \backslash\backslash\neg B\backslash\backslash_t$. Suppose $\Gamma_k \not\models^*_{\upsilon X} \backslash\neg A\backslash_s$; then by V$\upsilon$X*, there is an υX interpretation $\langle W, N, \overline{N}, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(\backslash\neg A\backslash) = 0$; so by H$\upsilon$(¬), $h_{m(s)}(/A/) = 1$; so $h_m(\Gamma_k) = 1$ and $h_{m(s)}(/A/) = 1$; so $h_m(\Gamma_k \cup \{/A/_s\}) = 1$; so by V$\upsilon$X*, $h_{m(t)}(//B//) = 1$ and $h_{m(t)}(\backslash\backslash\neg B\backslash\backslash) = 1$; from the latter, by H$\upsilon$(¬), $h_{m(t)}(//B//) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{\upsilon X} \backslash\neg A\backslash_s$, which is to say, $\Gamma_k \models^*_{\upsilon X} \mathcal{P}_k$.

(¬E)

(→I4) If $\mathcal{P}_k$ arises by →I4, then the picture is like this,

$$\begin{array}{r|l}
 & \quad A_t \\
 & \quad \vdots \\
i & \quad /B/_t \\
k & /A \to B/_s
\end{array}$$

where $i < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $/A \to B/_s$. For these systems, either by condition K, $W = N = \overline{N}$ or by constraint (n), $s = 0$; in the first case, $m(s) \in N$ and $m(s) \in \overline{N}$; so $m(s) \in /N/$; in the other case, by the construction of m, $m(s) \in N$; so with $N = \overline{N}$ by condition (4), $m(s) \in \overline{N}$; so $m(s) \in /N/$; in either case, $m(s) \in /N/$. By assumption, $\Gamma_i \models^*_{\upsilon X} /B/_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{A_t\}$; so by L7.1, $\Gamma_k \cup \{A_t\} \models^*_{\upsilon X} /B/_t$. Suppose $\Gamma_k \not\models^*_{\upsilon X} /A \to B/_s$; then by V$\upsilon$X*, there is an υX interpretation $\langle W, N, \overline{N}, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(/A \to B/) = 0$; so, since $m(s) \in /N/$, by Hυ(→)4, there is some $w \in W$ such that $h_w(A) = 1$ and $h_w(/B/) = 0$. Now consider a map m' like

m except that $m'(t) = w$, and consider $\langle W, N, \overline{N}, h\rangle_{m'}$; since t does not appear in Γ_k, it remains that $h_{m'}(\Gamma_k) = 1$; and since $m'(t) = w$, $h_{m'(t)}(A) = 1$; so $h_{m'}(\Gamma_k \cup \{A_t\}) = 1$; so by V$\upsilon$X*, $h_{m'(t)}(/B/) = 1$. But $m'(t) = w$; so $h_w(/B/) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{\upsilon X} /A \rightarrow B/_s$, which is to say, $\Gamma_k \models^*_{\upsilon X} \mathcal{P}_k$.

($\rightarrow$E4) If $\mathcal{P}_k$ arises by $\rightarrow$E4, then the picture is like this,

$$\begin{array}{r|l} i & /A \rightarrow B/_s \\ j & A_t \\ & \\ k & /B/_t \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $/B/_t$. For these systems, either by condition K, $W = N = \overline{N}$ or by constraint (n), $s = 0$; in the first case, $m(s) \in N$ and $m(s) \in \overline{N}$; so $m(s) \in /N/$; in the other case, by the construction of m, $m(s) \in N$; so with $N = \overline{N}$ by condition (4), $m(s) \in \overline{N}$; so $m(s) \in /N/$; in either case, $m(s) \in /N/$. By assumption, $\Gamma_i \models^*_{\upsilon X} /A \rightarrow B/_s$ and $\Gamma_j \models^*_{\upsilon X} A_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L7.1, $\Gamma_k \models^*_{\upsilon X} /A \rightarrow B/_s$ and $\Gamma_k \models^*_{\upsilon X} A_t$. Suppose $\Gamma_k \not\models^*_{\upsilon X} /B/_t$; then by V$\upsilon$X*, there is some υX interpretation $\langle W, N, \overline{N}, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(t)}(/B/) = 0$; since $h_m(\Gamma_k) = 1$, by VυX*, $h_{m(s)}(/A \rightarrow B/) = 1$ and $h_{m(t)}(A) = 1$; from the first of these, since $m(s) \in /N/$, by H$\upsilon(\rightarrow)_4$, there is no $w \in W$ such that $h_w(A) = 1$ and $h_w(/B/) = 0$; so it is not the case that $h_{m(t)}(A) = 1$ and $h_{m(t)}(/B/) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{\upsilon X} /B/_t$, which is to say, $\Gamma_k \models^*_{\upsilon X} \mathcal{P}_k$.

($\rightarrow$I*) If $\mathcal{P}_k$ arises by $\rightarrow$I*, then the picture is like this,

$$\begin{array}{r|l} & \quad \underline{/\!/A/\!/_t} \\ & \\ i & \quad /\!/B/\!/_t \\ k & /A \rightarrow B/_s \end{array}$$

where $i < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $/A \rightarrow B/_s$. For these systems, either by condition K, $W = N = \overline{N}$ or by constraint (s), $/A \rightarrow B/_s$ is of the sort, $(A \rightarrow B)_0$; in the first case, $m(s) \in N$ and $m(s) \in \overline{N}$; so $m(s) \in /N/$; in the other case, $s = 0$; so by the construction of m, $m(s) \in N$, which is to say $m(s) \in /N/$; so in either case, $m(s) \in /N/$. By assumption, $\Gamma_i \models^*_{\upsilon X} /\!/B/\!/_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{/\!/A/\!/_t\}$; so by L7.1, $\Gamma_k \cup \{/\!/A/\!/_t\} \models^*_{\upsilon X} /\!/B/\!/_t$. Suppose $\Gamma_k \not\models^*_{\upsilon X} /A \rightarrow B/_s$; then by V$\upsilon$X*, there is an υX interpretation $\langle W, N, \overline{N}, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(/A \rightarrow B/) = 0$; so by H$\upsilon(\rightarrow)_*$, there is some $x \in W$ such that $h_x(A) = 1$ and $h_x(B) = 0$, or $h_x(\overline{A}) = 1$ and $h_x(\overline{B}) = 0$. Without loss of generality, suppose $h_x(A) = 1$ and $h_x(B) = 0$; then by L7.0, there is an interpretation $\langle W', N', \overline{N}', h'\rangle$ where $h'_{w'}(/P/) = 1$ iff $h_w(/P/) = 1$

and $h'_{w^*}(/P/) = 1$ iff $h_w(\backslash P \backslash) = 1$. So with $m(s) = w$ iff $m'(s) = w'$, it remains that $h'_{m'}(\Gamma_k) = 1$; and we have that $x', x^* \in W'$ are such that $h'_{x'}(A) = 1$ and $h'_{x'}(B) = 0$, and $h'_{x^*}(\overline{A}) = 1$ and $h'_{x^*}(\overline{B}) = 0$; one of these is a y such that $h'_y(/\!/A/\!/) = 1$ and $h'_y(/\!/B/\!/) = 0$. Now consider a map n like m' except that $n(t) = y$, and consider $\langle W', N', \overline{N}', h'\rangle_n$; since t does not appear in Γ_k, it remains that $h'_n(\Gamma_k) = 1$; and since $n(t) = y$, $h'_{n(t)}(/\!/A/\!/) = 1$; so $h'_n(\Gamma_k \cup \{/\!/A/\!/_t\}) = 1$; so by V$\upsilon$X*, $h'_{n(t)}(/\!/B/\!/) = 1$. But $n(t) = y$; so $h'_y(/\!/B/\!/) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{\upsilon X} /A \to B/_s$, which is to say, $\Gamma_k \models^*_{\upsilon X} \mathcal{P}_k$.

($\to$E*) If $\mathcal{P}_k$ arises by $\to$E*, then the picture is like this,

$$\begin{array}{r|l} i & /A \to B/_s \\ j & /\!/A/\!/_t \\ & \\ k & /\!/B/\!/_t \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $/\!/B/\!/_t$. For these systems, either by condition K, $W = N = \overline{N}$ or by constraint (s), $/A \to B/_s$ is of the sort, $(A \to B)_0$; in the first case, $m(s) \in N$ and $m(s) \in \overline{N}$; so $m(s) \in /N/$; in the other case, $s = 0$; so by the construction of m, $m(s) \in N$, which is to say $m(s) \in /N/$; so in either case, $m(s) \in /N/$. By assumption, $\Gamma_i \models^*_{\upsilon X} /A \to B/_s$ and $\Gamma_j \models^*_{\upsilon X} /\!/A/\!/_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L7.1, $\Gamma_k \models^*_{\upsilon X} /A \to B/_s$ and $\Gamma_k \models^*_{\upsilon X} /\!/A/\!/_t$. Suppose $\Gamma_k \not\models^*_{\upsilon X} /\!/B/\!/_t$; then by V$\upsilon$X*, there is some υX interpretation $\langle W, N, \overline{N}, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(t)}(/\!/B/\!/) = 0$; since $h_m(\Gamma_k) = 1$, by VυX*, $h_{m(s)}(/A \to B/) = 1$ and $h_{m(t)}(/\!/A/\!/) = 1$; from the first of these, since $m(s) \in /N/$, by H$\upsilon(\to)_*$, there is no $w \in W$ such that $h_w(/\!/A/\!/) = 1$ and $h_w(/\!/B/\!/) = 0$; so it is not the case that $h_{m(t)}(/\!/A/\!/) = 1$ and $h_{m(t)}(/\!/B/\!/) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{\upsilon X} /\!/B/\!/_t$, which is to say, $\Gamma_k \models^*_{\upsilon X} \mathcal{P}_k$.

For any i, $\Gamma_i \models^*_{\upsilon X} \mathcal{P}_i$.

THEOREM 7.2 NυX *is complete: if* $\Gamma \models_{\upsilon X} A$ *then* $\Gamma \vdash_{N\upsilon X} A$.

Suppose $\Gamma \models_{\upsilon X} A$; then $\Gamma_0 \models^*_{\upsilon X} A_0$; we show that $\Gamma_0 \vdash^*_{N\upsilon X} A_0$. As usual, this reduces to the standard notion. For the following, fix on some particular υX. Then definitions of *consistency* etc. are relative to it.

CON Γ is CONSISTENT iff there is no A_s such that $\Gamma \vdash^*_{N\upsilon X} /A/_s$ and $\Gamma \vdash^*_{N\upsilon X} \backslash\neg A\backslash_s$.

L7.2 If s is 0 or appears in Γ, and $\Gamma \nvdash^*_{N\upsilon X} \backslash\neg P\backslash_s$, then $\Gamma \cup \{/P/_s\}$ is consistent.

Suppose s is 0 or appears in Γ and $\Gamma \nvdash^{*}_{N\upsilon X} \backslash\neg P\backslash_s$ but $\Gamma \cup \{/P/_s\}$ is inconsistent. Then there is some A_t such that $\Gamma \cup \{/P/_s\} \vdash^{*}_{N\upsilon X} /\!\!/A/\!\!/_t$ and $\Gamma \cup \{/P/_s\} \vdash^{*}_{N\upsilon X} \backslash\!\!\backslash\neg A\backslash\!\!\backslash_t$. But then we can argue,

1	Γ	
2	$\quad /P/_s$	A (c, $\neg$I)
3	$\quad /\!\!/A/\!\!/_t$	from $\Gamma \cup \{/P/_s\}$
4	$\quad \backslash\!\!\backslash\neg A\backslash\!\!\backslash_t$	from $\Gamma \cup \{/P/_s\}$
5	$\backslash\neg P\backslash_s$	2-4 $\neg$I

where the assumption is allowed insofar as s is either 0 or appears in Γ; so $\Gamma \vdash^{*}_{N\upsilon X} \backslash\neg P\backslash_s$. But this is impossible; reject the assumption: if s is 0 or appears in Γ and $\Gamma \nvdash^{*}_{N\upsilon X} \backslash\neg P\backslash_s$, then $\Gamma \cup \{/P/_s\}$ is consistent.

L7.3 There is an enumeration of all the subscripted formulas, $\mathcal{P}_1$ $\mathcal{P}_2 \ldots$

Proof by construction as usual.

MAX Γ is S-MAXIMAL iff for any A_s either $\Gamma \vdash^{*}_{N\upsilon X} /A/_s$ or $\Gamma \vdash^{*}_{N\upsilon X} \backslash\neg A\backslash_s$.

SGT Γ is a SCAPEGOAT set for $(\rightarrow)_{K_4}$ iff for every formula of the form $\neg(A \rightarrow B)$, if $\Gamma \vdash^{*}_{N\upsilon K_4} /\neg(A \rightarrow B)/_s$ then there is some t such that $\Gamma \vdash^{*}_{N\upsilon K_4} A_t$ and $\Gamma \vdash^{*}_{N\upsilon K_4} /\neg B/_t$.

Γ is a SCAPEGOAT set for $(\rightarrow)_{N_4}$ iff for every formula of the form $\neg(A \rightarrow B)$, if $\Gamma \vdash^{*}_{N\upsilon K_4} /\neg(A \rightarrow B)/_0$ then there is some t such that $\Gamma \vdash^{*}_{N\upsilon K_4} A_t$ and $\Gamma \vdash^{*}_{N\upsilon K_4} /\neg B/_t$.

Γ is a SCAPEGOAT set for $(\rightarrow)_{K_*}$ iff for every formula of the form $\neg(A \rightarrow B)$, if $\Gamma \vdash^{*}_{N\upsilon K_*} /\neg(A \rightarrow B)/_s$ then there is some t such that $\Gamma \vdash^{*}_{N\upsilon K_*} A_t$ and $\Gamma \vdash^{*}_{N\upsilon K_*} \overline{\neg B}_t$.

Γ is a SCAPEGOAT set for $(\rightarrow)_{N_*}$ iff for every formula of the form $\neg(A \rightarrow B)$, if $\Gamma \vdash^{*}_{N\upsilon K_*} \overline{\neg(A \rightarrow B)}_0$ then there is some t such that $\Gamma \vdash^{*}_{N\upsilon K_*} A_t$ and $\Gamma \vdash^{*}_{N\upsilon K_*} \overline{\neg B}_t$.

C(Γ') For Γ with unsubscripted formulas and the corresponding Γ_0, we construct Γ' as follows. Set $\Omega_0 = \Gamma_0$. By L7.3, there is an enumeration, $\mathcal{P}_1, \mathcal{P}_2 \ldots$ of all the formulas; let $\mathcal{E}_0$ be this enumeration. Then for the first $/A/_s$ in $\mathcal{E}_{i-1}$ such that s is 0 or included in Ω_{i-1}, let $\mathcal{E}_i$ be like $\mathcal{E}_{i-1}$ but without $/A/_s$, and set,

$$\begin{array}{lll} & \Omega_i = \Omega_{i-1} & \text{if } \Omega_{i-1} \vdash^*_{N\upsilon X} \backslash\neg A\backslash_s \\ & \Omega_{i^*} = \Omega_{i-1} \cup \{/A/_s\} & \text{if } \Omega_{i-1} \nvdash^*_{N\upsilon X} \backslash\neg A\backslash_s \end{array}$$

and

$$\begin{array}{lll} \upsilon K4: & \Omega_i = \Omega_{i^*} & \text{if } A_s \text{ is not of the form } /\neg(P \to Q)/_s \\ & \Omega_i = \Omega_{i^*} \cup \{P_t, /\neg Q/_t\} & \text{if } A_s \text{ is of the form } /\neg(P \to Q)/_s \\ \upsilon N4: & \Omega_i = \Omega_{i^*} & \text{if } A_s \text{ is not of the form } /\neg(P \to Q)/_0 \\ & \Omega_i = \Omega_{i^*} \cup \{P_t, /\neg Q/_t\} & \text{if } A_s \text{ is of the form } /\neg(P \to Q)/_0 \\ \upsilon K_*: & \Omega_i = \Omega_{i^*} & \text{if } A_s \text{ is not of the form } /\neg(P \to Q)/_s \\ & \Omega_i = \Omega_{i^*} \cup \{P_t, \overline{\neg Q}_t\} & \text{if } A_s \text{ is of the form } /\neg(P \to Q)/_s \\ \upsilon N_*: & \Omega_i = \Omega_{i^*} & \text{if } A_s \text{ is not of the form } \overline{\neg(P \to Q)}_0 \\ & \Omega_i = \Omega_{i^*} \cup \{P_t, \overline{\neg Q}_t\} & \text{if } A_s \text{ is of the form } \overline{\neg(P \to Q)}_0 \end{array}$$

-where t is the first subscript not included in Ω_{i^*}

then

$\Gamma' = \bigcup_{i \geqslant 0} \Omega_i$

Note that there is always sure to be a subscript t not in Ω_{i^*} insofar as there are infinitely many subscripts, and at any stage only finitely many formulas are added – the only subscripts in the initial Ω_0 being 0. Suppose s appears in Γ'; then there is some Ω_i in which it is first appears; and any formula $\mathcal{P}_j$ in the original enumeration that has subscript s is sure to be "considered" for inclusion at a subsequent stage.

L7.4 For any s included in Γ', Γ' is s-maximal.

Suppose s is included in Γ' but Γ' is not s-maximal. Then there is some A_s such that $\Gamma' \nvdash^*_{N\upsilon X} /A/_s$ and $\Gamma' \nvdash^*_{N\upsilon X} \backslash\neg A\backslash_s$. For any i, each member of Ω_{i-1} is in Γ'; so if $\Omega_{i-1} \vdash^*_{N\upsilon X} \backslash\neg A\backslash_s$ then $\Gamma' \vdash^*_{N\upsilon X} \backslash\neg A\backslash_s$; but $\Gamma' \nvdash^*_{N\upsilon X} \backslash\neg A\backslash_s$; so $\Omega_{i-1} \nvdash^*_{N\upsilon X} \backslash\neg A\backslash_s$; so since s is included in Γ', there is a stage in the construction that sets $\Omega_{i^*} = \Omega_{i-1} \cup \{/A/_s\}$; so by construction, $/A/_s \in \Gamma'$; so $\Gamma' \vdash^*_{N\upsilon X} /A/_s$. This is impossible; reject the assumption: Γ' is s-maximal.

L7.5 If Γ_0 is consistent, then each Ω_i is consistent.

Suppose Γ_0 is consistent.

Basis: $\Omega_0 = \Gamma_0$ and Γ_0 is consistent; so Ω_0 is consistent.

Assp: For any $i, 0 \leqslant i < k$, Ω_i is consistent.

Show: Ω_k is consistent.

Ω_k is either (i) Ω_{k-1}, (ii) $\Omega_{k^*} = \Omega_{k-1} \cup \{/A/_s\}$, (iii) $\Omega_{k^*} \cup \{P_t, /\neg Q/_t\}$ in $\upsilon K4$ or $\upsilon N4$, or (iv) $\Omega_{k^*} \cup \{P_t, \overline{\neg Q}_t\}$ in υK_* or υN_*.

(i) Suppose Ω_k is Ω_{k-1}. By assumption, Ω_{k-1} is consistent; so Ω_k is consistent.

(ii) Suppose Ω_k is $\Omega_{k^*} = \Omega_{k-1} \cup \{/A/_s\}$. Then by construction, s is 0 or in Ω_{k-1} and $\Omega_{k-1} \nvdash^*_{N\upsilon X} \backslash\neg A\backslash_s$; so by L7.2, $\Omega_{k-1} \cup \{/A/_s\}$ is consistent; so Ω_k is consistent.

(iii) Suppose Ω_k is $\Omega_{k^*} \cup \{P_t, /\neg Q/_t\}$ in υK_4 or υN_4. In this case, as above, Ω_{k*} is consistent and by construction, $/\neg(P \to Q)/_s \in \Omega_{k^*}$ (in υN_4, with $s = 0$). Suppose Ω_k is inconsistent. Then there is some A_u such that $\Omega_{k^*} \cup \{P_t, /\neg Q/_t\} \vdash^*_{N\upsilon X} //A//_u$ and $\Omega_{k^*} \cup \{P_t, /\neg Q/_t\} \vdash^*_{N\upsilon X} \backslash\backslash\neg A\backslash\backslash_u$. So reason as follows,

1	Ω_{k^*}	
2	$\quad P_t$	A (g, $\to$I$_4$)
3	$\quad\quad /\neg Q/_t$	A (c, $\neg$E)
4	$\quad\quad //A//_u$	from $\Omega_{k^*} \cup \{P_t, /\neg Q/_t\}$
5	$\quad\quad \backslash\backslash\neg A\backslash\backslash_u$	from $\Omega_{k^*} \cup \{P_t, /\neg Q/_t\}$
6	$\quad \backslash Q\backslash_t$	3-5 $\neg$E
7	$\backslash P \to Q\backslash_s$	2-6 $\to$I$_4$

where, by construction, t is not in Ω_{k^*} and for υN_4, $s = 0$. So $\Omega_{k^*} \vdash^*_{N\upsilon X} \backslash P \to Q\backslash_s$; but $/\neg(P \to Q)/_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{N\upsilon X} /\neg(P \to Q)/_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

(iv) Suppose Ω_k is $\Omega_{k^*} \cup \{P_t, \overline{\neg Q_t}\}$ in υK_* or υN_*. In this case, as above, Ω_{k*} is consistent and by construction, $/\neg(P \to Q)/_s \in \Omega_{k^*}$ (in υK_*, with overline and $s = 0$). Suppose Ω_k is inconsistent. Then there is some A_u such that $\Omega_{k^*} \cup \{P_t, \overline{\neg Q_t}\} \vdash^*_{N\upsilon X} //A//_u$ and $\Omega_{k^*} \cup \{P_t, \overline{\neg Q_t}\} \vdash^*_{N\upsilon X} \backslash\backslash\neg A\backslash\backslash_u$. So reason as follows,

1	Ω_{k^*}	
2	$\quad P_t$	A (g, $\to$I$*$)
3	$\quad\quad \overline{\neg Q_t}$	A (c, $\neg$E)
4	$\quad\quad //A//_u$	from $\Omega_{k^*} \cup \{P_t, \overline{\neg Q_t}\}$
5	$\quad\quad \backslash\backslash\neg A\backslash\backslash_u$	from $\Omega_{k^*} \cup \{P_t, \overline{\neg Q_t}\}$
6	$\quad Q_t$	3-5 $\neg$E
7	$\backslash P \to Q\backslash_s$	2-6 $\to$I$*$

where, by construction, t is not in Ω_{k^*} and for υN_*, $\backslash P \to Q\backslash_s$ is without overline and $s = 0$. So $\Omega_{k^*} \vdash^*_{N\upsilon X} \backslash P \to Q\backslash_s$; but $/\neg(P \to Q)/_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{N\upsilon X} /\neg(P \to Q)/_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

For any i, Ω_i is consistent.

L7.6 If Γ_0 is consistent, then Γ' is consistent.

Reasoning parallel to L2.6 and L6.6.

L7.7 If Γ_0 is consistent, then Γ' is a scapegoat set for $(\to)_{K_4}$, $(\to)_{N_4}$, $(\to)_{K_*}$, and $(\to)_{N_*}$.

For $(\to)_{K_4}$ and $(\to)_{N_4}$. Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{N\upsilon X} /\neg(P \to Q)/_s$. By L7.6, Γ' is consistent; and by the constraints on subscripts, s is included in Γ'. Since Γ' is consistent, $\Gamma' \nvdash^{*}_{N\upsilon X} \backslash\neg\neg(P \rightarrow Q)\backslash_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{/\neg(P \rightarrow Q)/_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{P_t, /\neg Q/_t\}$; so by construction, $P_t \in \Gamma'$ and $/\neg Q/_t \in \Gamma'$; so $\Gamma' \vdash^{*}_{N\upsilon X} P_t$ and $\Gamma' \vdash^{*}_{N\upsilon X} /\neg Q/_t$. So Γ' is a scapegoat set for $(\rightarrow)_{K_4}$ and $(\rightarrow)_{N_4}$ – where the argument for $(\rightarrow)_{N_4}$ assumes $s = 0$.

For $(\rightarrow)_{K_*}$ and $(\rightarrow)_{N_*}$. Suppose Γ_0 is consistent and $\Gamma' \vdash^{*}_{N\upsilon X} /\neg(P \rightarrow Q)/_s$. By L7.6, Γ' is consistent; and by the constraints on subscripts, s is included in Γ'. Since Γ' is consistent, $\Gamma' \nvdash^{*}_{N\upsilon X} \backslash\neg\neg(P \rightarrow Q)\backslash_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{/\neg(P \rightarrow Q)/_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{P_t, \overline{\neg Q}_t\}$; so by construction, $P_t \in \Gamma'$ and $\overline{\neg Q}_t \in \Gamma'$; so $\Gamma' \vdash^{*}_{N\upsilon X} P_t$ and $\Gamma' \vdash^{*}_{N\upsilon X} \overline{\neg Q}_t$. So Γ' is a scapegoat set for $(\rightarrow)_{K_*}$ and $(\rightarrow)_{N_*}$ – where the argument for $(\rightarrow)_{N_*}$ assumes $/\neg(P \rightarrow Q)/_s$ is with overline and $s = 0$.

C(I) We construct an interpretation $I = \langle W, N, \overline{N}, h\rangle$ based on Γ' as follows.

υK_x: For the K systems, let W have a member w_s corresponding to each subscript s included in Γ'. Then set $N = \overline{N} = W$ and $h_{w_s}(/p/) = 1$ iff $\Gamma' \vdash^{*}_{N\upsilon X} /p/_s$.

υN_4: Let W have a member w_s corresponding to each subscript s included in Γ'. Then set $N = \overline{N} = \{w_0\}$; $h_{w_s}(/p/) = 1$ iff $\Gamma' \vdash^{*}_{N\upsilon X} /p/_s$; and for $s \neq 0$, $h_{w_s}(/A \rightarrow B/) = 1$ iff $\Gamma' \vdash^{*}_{N\upsilon X} /A \rightarrow B/_s$.

υN_*: Let W have a member w_s corresponding to each subscript s included in Γ'. Then set $N = \{w_0\}$ and $\overline{N} = \phi$; $h_{w_s}(/p/) = 1$ iff $\Gamma' \vdash^{*}_{N\upsilon X} /p/_s$; $h_{w_s}(\overline{P \rightarrow Q}) = 1$ iff $\Gamma' \vdash^{*}_{N\upsilon X} \overline{(P \rightarrow Q)}_s$; and for $s \neq 0$, $h_{w_s}(A \rightarrow B) = 1$ iff $\Gamma' \vdash^{*}_{N\upsilon X} (A \rightarrow B)_s$.

L7.8 If Γ_0 is consistent then for $\langle W, N, \overline{N}, h\rangle$ constructed as above, and for any s included in Γ', $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^{*}_{N\upsilon X} /A/_s$.

Suppose Γ_0 is consistent and s is included in Γ'. By L7.4, Γ' is s-maximal. By L7.6 and L7.7, Γ' is consistent and a scapegoat set for the different conditionals. Now by induction on the number of operators in $/A/_s$,

Basis: If $/A/_s$ has no operators, then it is a parameter $/p/_s$ and by construction, $h_{w_s}(/p/) = 1$ iff $\Gamma' \vdash^{*}_{N\upsilon X} /p/_s$. So $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^{*}_{N\upsilon X} /A/_s$.

Assp: For any i, $0 \leqslant i < k$, if $/A/_s$ has i operators, then $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^{*}_{N\upsilon X} /A/_s$.

Show: If $/A/_s$ has k operators, then $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^{*}_{N\upsilon X} /A/_s$.

If $/A/_s$ has k operators, then it is of the form $/\neg P/_s$, $/P \wedge Q/_s$, $/P \vee Q/_s$ or $/P \rightarrow Q/_s$, where P and Q have $< k$ operators.

$(\neg)$ $/A/_s$ is $/\neg P/_s$. (i) Suppose $h_{w_s}(/A/) = 1$; then $h_{w_s}(/\neg P/) = 1$; so by $H\upsilon(\neg)$, $h_{w_s}(\backslash P\backslash) = 0$; so by assumption, $\Gamma' \nvdash^{*}_{N\upsilon X} \backslash P\backslash_s$;

so by s-maximality, $\Gamma' \vdash^*_{N\upsilon X} /\neg P/_s$, where this is to say, $\Gamma' \vdash^*_{N\upsilon X} /A/_s$. (ii) Suppose $\Gamma' \vdash^*_{N\upsilon X} /A/_s$; then $\Gamma' \vdash^*_{N\upsilon X} /\neg P/_s$; so by consistency, $\Gamma' \nvdash^*_{N\upsilon X} \backslash P\backslash_s$; so by assumption, $h_{w_s}(\backslash P\backslash) = 0$; so by $H\upsilon(\neg)$, $h_{w_s}(/\neg P/) = 1$, where this is to say, $h_{w_s}(/A/) = 1$. So $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{N\upsilon X} /A/_s$.

($\wedge$)

($\vee$)

($\rightarrow$) $/A/_s$ is $/P \rightarrow Q/_s$. (i) Suppose $h_{w_s}(/A/) = 1$ but $\Gamma' \nvdash^*_{N\upsilon X} /A/_s$; then $h_{w_s}(/P \rightarrow Q/) = 1$, but $\Gamma' \nvdash^*_{N\upsilon X} /P \rightarrow Q/_s$; from the latter, by s-maximality, $\Gamma' \vdash^*_{N\upsilon X} \backslash\neg(P \rightarrow Q)\backslash_s$.

υK_4: In this case, $N = \overline{N} = K$; so $w_s \in /N/$. Since Γ' is a scapegoat set for $(\rightarrow)_{K_4}$, there is some t such that $\Gamma' \vdash^*_{N\upsilon K_4} P_t$ and $\Gamma' \vdash^*_{N\upsilon K_4} \backslash\neg Q\backslash_t$; from the latter, by consistency, $\Gamma' \nvdash^*_{N\upsilon K_4} /Q/_t$; so by our assumption, $h_{w_t}(P) = 1$ and $h_{w_t}(/Q/) = 0$; so since $w_s \in /N/$, by $H\upsilon(\rightarrow)_4$, $h_{w_s}(/P \rightarrow Q/) = 0$. This is impossible; reject the assumption: if $h_{w_s}(/A/) = 1$, then $\Gamma' \vdash^*_{N\upsilon X} /A/_s$.

υN_4: In this case, when $s = 0$, $w_s \in /N/$ and reasoning is as above. Otherwise, by construction, if $h_{w_s}(/A/) = 1$ then $\Gamma' \vdash^*_{N\upsilon X} /A/_s$.

υK_*: In this case, $N = \overline{N} = K$; so $w_s \in /N/$. Since Γ' is a scapegoat set for $\overline{(\rightarrow)}_{K_*}$, there is some t such that $\Gamma' \vdash^*_{N\upsilon K_*} P_t$ and $\Gamma' \vdash^*_{N\upsilon K_*} \overline{\neg Q}_t$; from the latter, by consistency, $\Gamma' \nvdash^*_{N\upsilon K_*} Q_t$; so by assumption, $h_{w_t}(P) = 1$ and $h_{w_t}(Q) = 0$; so since $w_s \in /N/$, by $H\upsilon(\rightarrow)_*$, $h_{w_s}(/P \rightarrow Q/) = 0$. This is impossible; reject the assumption: if $h_{w_s}(/A/) = 1$, then $\Gamma' \vdash^*_{N\upsilon X} /A/_s$.

υN_*: In this case, when $s = 0$ and $/P \rightarrow Q/$ is without overline – so that $\backslash\neg(P \rightarrow Q)\backslash$ is $\overline{\neg(P \rightarrow Q)}$ – $w_s \in /N/$ and reasoning is as immediately above. Otherwise, by construction, if $h_{w_s}(/A/) = 1$ then $\Gamma' \vdash^*_{N\upsilon X} /A/_s$.

So in any of these cases, if $h_{w_s}(/A/) = 1$ then $\Gamma' \vdash^*_{N\upsilon X} /A/_s$.

(ii) Suppose $\Gamma' \vdash^*_{N\upsilon X} /A/_s$ but $h_{w_s}(/A/) = 0$; then $\Gamma' \vdash^*_{N\upsilon X} /P \rightarrow Q/_s$ but $h_{w_s}(/P \rightarrow Q/) = 0$.

υK_4: From the latter, by $H\upsilon(\rightarrow)_4$, there is some $w_t \in W$ such that $h_{w_t}(P) = 1$ and $h_{w_t}(/Q/) = 0$; so by assumption, $\Gamma' \vdash^*_{N\upsilon K_4} P_t$ and $\Gamma' \nvdash^*_{N\upsilon K_4} /Q/_t$; so by s-maximality, $\Gamma' \vdash^*_{N\upsilon K_4} \backslash\neg Q\backslash_t$. So by reasoning as follows,

1	Γ'		
2		$/P \to Q/_s$	A (*c*, ¬I)
3		P_t	from Γ'
4		$/Q/_t$	2,3 →E4
5		$\backslash\neg Q\backslash_t$	from Γ'
6	$\backslash\neg(P \to Q)\backslash_s$		2-5 ¬I

$\Gamma' \vdash^*_{N\upsilon K_4} \backslash\neg(P \to Q)\backslash_s$; so by consistency, $\Gamma' \nvdash^*_{N\upsilon K_4} /P \to Q/_s$. This is impossible; reject the assumption: if $\Gamma' \vdash^*_{N\upsilon X} /A/_s$ then $h_{w_s}(/A/) = 1$.

υN_4: When $s = 0$, the reasoning is as above. Otherwise, by construction, if $\Gamma' \vdash^*_{N\upsilon X} /A/_s$, then $h_{w_s}(/A/) = 1$.

υK_*: From the latter, by $H\upsilon(\to)_*$, there is some $w_t \in W$ such that $h_{w_t}(/\!/P/\!/) = 1$ and $h_{w_t}(/\!/Q/\!/) = 0$; so by assumption, $\Gamma' \vdash^*_{N\upsilon K_4} /\!/P/\!/_t$ and $\Gamma' \nvdash^*_{N\upsilon K_4} /\!/Q/\!/_t$; so by s-maximality, $\Gamma' \vdash^*_{N\upsilon K_4} \backslash\!\backslash\neg Q\backslash\!\backslash_t$. So by reasoning as follows,

1	Γ'		
2		$/P \to Q/_s$	A (*c*, ¬I)
3		$/\!/P/\!/_t$	from Γ'
4		$/\!/Q/\!/_t$	2,3 →E*
5		$\backslash\!\backslash\neg Q\backslash\!\backslash_t$	from Γ'
6	$\backslash\neg(P \to Q)\backslash_s$		2-5 ¬I

$\Gamma' \vdash^*_{N\upsilon K_*} \backslash\neg(P \to Q)\backslash_s$; so by consistency, $\Gamma' \nvdash^*_{N\upsilon K_*} /P \to Q/_s$. This is impossible; reject the assumption: if $\Gamma' \vdash^*_{N\upsilon X} /A/_s$ then $h_{w_s}(/A/) = 1$.

υN_*: When $s = 0$ and $/P \to Q/$ is without overline, the reasoning is as immediately above. Otherwise, by construction, if $\Gamma' \vdash^*_{N\upsilon X} /A/_s$ then $h_{w_s}(/A/) = 1$.

So in any of these cases, if $\Gamma' \vdash^*_{N\upsilon X} /A/_s$ then $h_{w_s}(/A/) = 1$. So $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{N\upsilon X} /A/_s$.

For any A_s, $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{N\upsilon X} /A/_s$.

L7.9 If Γ_0 is consistent, then $\langle W, N, \overline{N}, h\rangle$ constructed as above is an υX interpretation.

This is immediate, by construction.

MAP For any $w_s \in W$, set $m(s) = w_s$; otherwise $m(s)$ is arbitrary.

L7.10 If Γ_0 is consistent, then $h_m(\Gamma_0) = 1$.

Reasoning parallel to L2.10 and L6.9.

Main result: Suppose $\Gamma \models_{\upsilon X} A$ but $\Gamma \nvdash_{N\upsilon X} A$. Then $\Gamma_0 \models^*_{\upsilon X} A_0$ but $\Gamma_0 \nvdash^*_{N\upsilon X} A_0$. By (DN), if $\Gamma_0 \vdash^*_{N\upsilon X} \neg\neg A_0$, then $\Gamma_0 \vdash^*_{N\upsilon X} A_0$; so $\Gamma_0 \nvdash^*_{N\upsilon X} \neg\neg A_0$; so by L7.2, $\Gamma_0 \cup \{\overline{\neg A_0}\}$

is consistent; so by L7.9 and L7.10, there is an υX interpretation $\langle W, N, \overline{N}, h\rangle_m$ constructed as above such that $h_m(\Gamma_0 \cup \{\overline{\neg A_0}\}) = 1$; so $h_{m(0)}(\overline{\neg A}) = 1$; so by $H\upsilon(\neg)$, $h_{m(0)}(A) = 0$; so $h_m(\Gamma_0) = 1$ and $h_{m(0)}(A) = 0$; so by $V\upsilon X^*$, $\Gamma_0 \not\models^*_{\upsilon X} A_0$. This is impossible; reject the assumption: if $\Gamma \models_{\upsilon X} A$, then $\Gamma \vdash_{N\upsilon X} A$.

8 MAINSTREAM RELEVANT LOGICS: *Bx* (CH. 10,11)

The treatment here for Priest's chapter 11 is minimal: there are only resources for *CK* first introduced in chapter 11, not chapter 10. I abandon the four-valued approach from previous sections, and follow Priest in developing the star-semantics on its own terms.[7]

8.1 LANGUAGE / SEMANTIC NOTIONS

LBx The VOCABULARY consists of propositional parameters $p_0, p_1 \ldots$ with the operators, $\neg$, $\wedge$, $\vee$, $\rightarrow$, (and $>$). Each propositional parameter is a FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A > B)$. $A \supset B$ abbreviates $\neg A \vee B$, and $A \equiv B$ abbreviates $(A \supset B) \wedge (B \supset A)$.

IBRX Without '>' in the language, an INTERPRETATION is $\langle W, N, R, *, \nu\rangle$ where W is a set of worlds; N is a subset of W; R is a subset of $W^3 = W \times W \times W$; $*$ is a function from worlds to worlds such that $w^{**} = w$; and ν is a function such that for any $w \in W$ and p, $\nu_w(p) = 1$ or $\nu_w(p) = 0$. As a constraint on interpretations, we require also,

NC For any $w \in N$, $Rwxy$ iff $x = y$

Where x is empty or indicates some combination of the following constraints,

(C8) If $Rabc$, then Rac^*b^*

(C9) If there is an x such that $Rabx$ and $Rxcd$ then there is a y such that $Racy$ and $Rbyd$

(C10) If there is an x such that $Rabx$ and $Rxcd$ then there is a y such that $Rbcy$ and $Rayd$

(C11) If $Rabc$ then $Rbac$

(C12) If $Rabc$ then there is an x such that $Rabx$ and $Rxbc$

(C13) If $Rabx$ and $Rxcd$ then $Racd$

[7]The four-valued approach does apply to some of these logics. But it is complicated considerably (as we have already begun to see with the double normal worlds for υX_* of the previous section), and the approach does not apply to all the logics. For details, see [9, 8], and for related derivations along the lines of the four-valued approach from this paper my [11]. As I suggest, this incapacity may be related to motivations for systems like *DW* which do not transfer naturally to stronger systems like *TW*, *RW* and especially *R*.

($\preceq$) If $Rabc$ then (i) if $v_a(p) = 1$, then $v_c(p) = 1$, and (ii) if $v_{c^*}(p) = 1$, then $v_{a^*}(p) = 1$.

$\langle W, N, R, *, v\rangle$ is a *Bx* interpretation when it meets the constraints from x. System B has none of the extra constraints; B_{DW} is B_{C8}; B_{TW} is B_{C8-C10}; B_{RW} is B_{C8-C11}; B_R is B_{C8-C12}; and B_{CK} is $B_{C8-C11,C13,\preceq}$.

IBcx When '>' is in the language, an interpretation is $\langle W, N, R, \{R_A \mid A \in \mathfrak{I}\}, *, v\rangle$, where $\mathfrak{I}$ is the set of all formulas and R_A is a subset of W^2. Condition NC remains in place, but none of C8 - C13 or ($\preceq$). That is all for B_C (what Priest calls C_B). Where $f_A(w) = \{x \in W \mid wR_Ax\}$, and $[A] = \{x \in W \mid v_w(A) = 1\}$, B_{C+} adds the constraints,

(1) For any $w \in N$, $f_A(w) \subseteq [A]$

(2) For any $w \in N$, if $w \in [A]$, then $w \in f_A(w)$

TB For complex expressions,

($\neg$) $v_w(\neg A) = 1$ if $v_{w^*}(A) = 0$, and 0 otherwise.

($\wedge$) $v_w(A \wedge B) = 1$ if $v_w(A) = 1$ and $v_w(B) = 1$, and 0 otherwise.

($\vee$) $v_w(A \vee B) = 1$ if $v_w(A) = 1$ or $v_w(B) = 1$, and 0 otherwise.

($\rightarrow$) $v_w(A \rightarrow B) = 1$ iff there are no $x, y \in W$ such that $Rwxy$ and $v_x(A) = 1$ but $v_y(B) = 0$.

($>$) $v_w(A > B) = 1$ iff there is no $x \in W$ such that wR_Ax and $v_x(B) = 0$.

For a set Γ of formulas, $v_w(\Gamma) = 1$ iff $v_w(A) = 1$ for each $A \in \Gamma$; then,

VBx $\Gamma \models_{Bx} A$ iff there is no *Bx* interpretation $\langle W, N, R, *, v\rangle$ / $\langle W, N, R, \{R_A \mid A \in \mathfrak{I}\}, *, v\rangle$ and $w \in N$ such that $v_w(\Gamma) = 1$ and $v_w(A) = 0$.

8.2 NATURAL DERIVATIONS: *NBx*

Allow subscripts of the sort i and $i^{\#}$. Where s is a subscript i or $i^{\#}$, $\bar{s}$ is the other. Say s is "introduced" as a subscript when either s or $\bar{s}$ is a subscript. For subscripts s, t, u allow also expressions of the sort $s \simeq t$, $s.t.u$ and $A_{s/t}$. Let $\mathcal{P}(s)$ be any expression in which s appears, and $\mathcal{P}(t)$ the same expression with one or more instances of s replaced by t.

R | P_s
  | P_s

$\neg$I | | $P_{\bar{s}}$
  | | Q_t
  | | $\neg Q_{\bar{t}}$
  | $\neg P_s$

$\neg$E | | $\neg P_{\bar{s}}$
  | | Q_t
  | | $\neg Q_{\bar{t}}$
  | P_s

$\wedge$I
P_s
Q_s
$(P \wedge Q)_s$

$\wedge$E
$(P \wedge Q)_s$
P_s

$\wedge$E
$(P \wedge Q)_s$
Q_s

$\vee$I
P_s
$(P \vee Q)_s$

$\vee$I
P_s
$(Q \vee P)_s$

$\vee$E
$(P \vee Q)_s$
| P_s
| R_t
| Q_s
| R_t
R_t

$\supset$I
| $P_{\overline{s}}$
| Q_s
$(P \supset Q)_s$

$\supset$E
$(P \supset Q)_s$
$P_{\overline{s}}$
Q_s

$\equiv$I
| $P_{\overline{s}}$
| Q_s
| $Q_{\overline{s}}$
| P_s
$(P \equiv Q)_s$

$\equiv$E
$(P \equiv Q)_s$
$P_{\overline{s}}$
Q_s

$\equiv$E
$(P \equiv Q)_s$
$Q_{\overline{s}}$
P_s

$\rightarrow$I
| $s.t.u$
| P_t
| Q_u
$(P \rightarrow Q)_s$
where t and u are not introduced in any undischarged premise or assumption

$\rightarrow$E
$s.t.u$
$(P \rightarrow Q)_s$
P_t
Q_u

$\not\rightarrow$I
$\overline{s}.t.u$
P_t
$\neg Q_{\overline{u}}$
$\neg(P \rightarrow Q)_s$

$\not\rightarrow$E
$\neg(P \rightarrow Q)_s$
| $\overline{s}.t.u$
| P_t
| $\neg Q_{\overline{u}}$
| R_v
R_v
where t and u are not introduced in any undischarged premise or assumption or by v

0I
$s \simeq t$
$0.s.t$

0E
$0.s.t$
$s \simeq t$

$\simeq$I
$s \simeq s$

$\simeq$E
$s \simeq t$
$\mathcal{P}(s)$
$\mathcal{P}(t)$

$s \simeq t$
$\mathcal{P}(\overline{s})$
$\mathcal{P}(\overline{t})$

These are the rules of *NB*, where $\supset$I, $\supset$E, $\equiv$I, $\equiv$E and, as we shall see, $\not\rightarrow$I and $\not\rightarrow$E are derived. With $s \simeq t$, we can introduce $s \simeq s$ by $\simeq$I, and then get $t \simeq s$ by $\simeq$E; so informally, we let $\simeq$E include also a derived rule that reverses order around '$\simeq$' – using $s \simeq t$ to replace some instance(s) of t ($\overline{t}$) with s ($\overline{s}$). As usual, subscripts are 0 or introduced in an assumption that requires new subscripts (and similarly for the following). To make things easier to follow, cite lines for $\rightarrow$E only in the order listed above: first access, then the conditional, then the antecedent.

For relevant systems NB_x, include rules from the following as appropriate.

$$\text{AM8}\ \begin{array}{|l} s.t.u \\ \\ s.\overline{u}.\overline{t} \end{array} \qquad \text{AM9}\ \begin{array}{|l} s.t.x \\ x.u.v \\ \quad\begin{array}{|l} s.u.y \\ \underline{t.y.v} \\ \\ P_w \end{array} \\ P_w \end{array} \qquad \text{AM10}\ \begin{array}{|l} s.t.x \\ x.u.v \\ \quad\begin{array}{|l} t.u.y \\ \underline{s.y.v} \\ \\ P_w \end{array} \\ P_w \end{array} \qquad \text{AM11}\ \begin{array}{|l} s.t.u \\ \\ t.s.u \end{array}$$

$$\text{AM12}\ \begin{array}{|l} s.t.u \\ \quad\begin{array}{|l} s.t.y \\ \underline{y.t.u} \\ \\ P_w \end{array} \\ P_w \end{array} \qquad \text{AM13}\ \begin{array}{|l} s.t.u \\ u.v.w \\ \\ s.v.w \end{array} \qquad \text{AM}\preceq\ \begin{array}{|l} s.t.u \\ P_s \\ \\ P_u \end{array} \qquad \text{AM}\preceq\ \begin{array}{|l} s.t.u \\ P_{\overline{u}} \\ \\ P_{\overline{s}} \end{array}$$

For AM9, AM10 and AM12, y is not introduced in any undischarged premise or assumption, or by w. Note that the right-hand version of AM$\preceq$ is a derived rule in NB_{CK}: from $s.t.u$ it follows by AM11 that $t.s.u$; and from AM8 that $t.\overline{u}.\overline{s}$; so from AM11 that $\overline{u}.t.\overline{s}$; so from $P_{\overline{u}}$ by the left-hand version that $P_{\overline{s}}$.

For the systems NB_{C_x} revert to the rules of *NB*. Then add >I and >E. As we show just below, $\not>$I and $\not>$E are derived.

$$>\text{I}\ \begin{array}{|l} \quad\begin{array}{|l} \underline{P_{s/t}} \\ \\ Q_t \end{array} \\ (P > Q)_s \end{array} \qquad >\text{E}\ \begin{array}{|l} (P > Q)_s \\ P_{s/t} \\ \\ Q_t \end{array} \qquad \not>\text{I}\ \begin{array}{|l} P_{\overline{s}/t} \\ \neg Q_{\overline{t}} \\ \\ \neg(P > Q)_s \end{array} \qquad \not>\text{E}\ \begin{array}{|l} \neg(P > Q)_s \\ \quad\begin{array}{|l} P_{\overline{s}/t} \\ \underline{\neg Q_{\overline{t}}} \\ \\ R_u \end{array} \\ R_u \end{array}$$

>I: where t is not introduced in any undischarged premise or assumption

$\not>$E: where t is not introduced in any undischarged premise or assumption, or by u

As before, corresponding to constraints (1) and (2) for the C^+ system, are AMP1 and AMP2, now restricted to apply just at the normal world 0.

$$\text{AMP1}\ \begin{array}{|l} P_{0/t} \\ \\ P_t \end{array} \qquad \text{AMP2}\ \begin{array}{|l} P_0 \\ \\ P_{0/0} \end{array}$$

Where Γ is a set of unsubscripted formulas, let Γ_0 be those same formulas, each with subscript 0. Then,

NBx $\Gamma \vdash_{NBx} A$ iff there is an *NBx* derivation of A_0 from the members of Γ_0.

Derived rules carry over much as one would expect. Thus, e.g.,

$$\text{MT}\ \begin{array}{|l} (P \supset Q)_s \\ \neg Q_{\overline{s}} \\ \\ \neg P_s \end{array} \qquad \text{NB}\ \begin{array}{|l} (P \equiv Q)_s \\ \neg P_{\overline{s}} \\ \\ \neg Q_s \end{array} \quad \begin{array}{|l} (P \equiv Q)_s \\ \neg Q_{\overline{s}} \\ \\ \neg P_s \end{array} \qquad \text{DS}\ \begin{array}{|l} (P \vee Q)_s \\ \neg P_{\overline{s}} \\ \\ Q_s \end{array} \quad \begin{array}{|l} (P \vee Q)_s \\ \neg Q_{\overline{s}} \\ \\ P_s \end{array}$$

Impl $(P \supset Q)_s \ \triangleleft\triangleright\ (\neg P \vee Q)_s$
$(\neg P \supset Q)_s \ \triangleleft\triangleright\ (P \vee Q)_s$

As examples, $\not\to$I, $\not\to$E, $\not>$I and $\not>$E are derived rules in NB_x and NB_{Cx}.

$\not\to$I

1	$\overline{s}.t.u$	P
2	P_t	P
3	$\neg Q_{\overline{u}}$	P
4	$(P \to Q)_{\overline{s}}$	A (c, ¬I)
5	Q_u	1,4,2 →E
6	$\neg Q_{\overline{u}}$	3 R
7	$\neg(P \to Q)_s$	4-6 ¬I

$\not\to$E

1	$\neg(P \to Q)_s$	P
2	$\neg R_{\overline{v}}$	A (c, ¬E)
3	$\overline{s}.t.u$	A (g, →I)
4	P_t	
5	$\neg Q_{\overline{u}}$	A (c, ¬E)
	⋮	with 1,3,4,5
6	R_v	as for $\not\to$E
7	$\neg R_{\overline{v}}$	2 R
8	Q_u	5-7 ¬E
9	$(P \to Q)_{\overline{s}}$	3-8 →I
10	$\neg(P \to Q)_s$	1 R
11	R_v	2-10 ¬E

$\not>$I

1	$P_{\overline{s}/t}$	P
2	$\neg Q_{\overline{t}}$	P
3	$(P > Q)_{\overline{s}}$	A (c, ¬I)
4	Q_t	1,3 >E
5	$\neg Q_{\overline{t}}$	2 R
6	$\neg(P > Q)_s$	3-5 ¬I

$\not>$E

1	$\neg(P > Q)_s$	P
2	$\neg R_{\overline{u}}$	A (c, ¬E)
3	$P_{\overline{s}/t}$	A (g, >I)
4	$\neg Q_{\overline{t}}$	A (c, ¬E)
	⋮	with 1,3,4
5	R_u	as for $\not>$E
6	$\neg R_{\overline{u}}$	2 R
7	Q_t	4-6 ¬E
8	$(P > Q)_{\overline{s}}$	3-7 >I
9	$\neg(P > Q)_s$	1 R
10	R_u	2-9 ¬E

Note the way overlines work (much the way slashes worked before). For $\not\to$E, note that the application of →I depends on the restriction that t and u are not introduced by v; and similarly, for $\not>$E the application of >I depends on the restriction that t is not introduced by u.

As further examples, here are a few key results that parallel ones from Priest's text.

A3 $\vdash_{NBx} (A \wedge B) \rightarrow A$

1	0.1.2	A (g, →I)
2	$(A \wedge B)_1$	
3	A_1	2 ∧E
4	$1 \simeq 2$	1 0E
5	A_2	3,4 ≃E
6	$[(A \wedge B) \rightarrow A]_0$	1-5 →I

A5 $\vdash_{NBx} [(A \rightarrow B) \wedge (A \rightarrow C)] \rightarrow [A \rightarrow (B \wedge C)]$

1	0.1.2	A (g, →I)
2	$[(A \rightarrow B) \wedge (A \rightarrow C)]_1$	
3	2.3.4	A (g, →I)
4	A_3	
5	$1 \simeq 2$	1 0E
6	1.3.4	3,5 ≃E
7	$(A \rightarrow B)_1$	2 ∧E
8	$(A \rightarrow C)_1$	2 ∧E
9	B_4	6,7,4 →E
10	C_4	6,8,4 →E
11	$(B \wedge C)_4$	9,10 ∧I
12	$[A \rightarrow (B \wedge C)]_2$	3-11 →I
13	$([(A \rightarrow B) \wedge (A \rightarrow C)] \rightarrow [A \rightarrow (B \wedge C)])_0$	1-12 →I

R5 $(A \rightarrow \neg B) \vdash_{NBx} (B \rightarrow \neg A)$

1	$(A \rightarrow \neg B)_0$	P
2	0.1.2	A (g, →I)
3	B_1	
4	$A_{2^\#}$	A (c, ¬I)
5	$2^\# \simeq 2^\#$	≃I
6	$0.2^\#.2^\#$	5 0I
7	$\neg B_{2^\#}$	6,1,4 →E
8	$1 \simeq 2$	2 0E
9	B_2	3,8 ≃E
10	$\neg A_2$	4-9 ¬I
11	$(B \rightarrow \neg A)_0$	2-10 →I

A9 $\vdash_{NB_{TW}} (A \to B) \to [(B \to C) \to (A \to C)]$

1	$0.1.2$	A (g, $\to$I)
2	$(A \to B)_1$	
3	$2.3.4$	A (g, $\to$I)
4	$(B \to C)_3$	
5	$4.5.6$	A (g, $\to$I)
6	A_5	
7	$1 \simeq 2$	1 oE
8	$(A \to B)_2$	2,7 $\simeq$E
9	$2.5.7$	A (g, 3,5 AM9)
10	$3.7.6$	
11	B_7	9,8,6 $\to$E
12	C_6	10,4,11 $\to$E
13	C_6	3,5,9-12 AM9
14	$(A \to C)_4$	5-13 $\to$I
15	$[(B \to C) \to (A \to C)]_2$	3-14 $\to$I
16	$((A \to B) \to [(B \to C) \to (A \to C)])_0$	1-15 $\to$I

$\vdash_{NB_R} (\neg A \to A) \to A$

1	$0.1.2$	A (g, $\to$I)
2	$(\neg A \to A)_1$	
3	$\neg A_{2^\#}$	A (c, $\neg$E)
4	$0.2^\#.1^\#$	1 AM8
5	$0.2^\#.3$	A (g, 4 AM12)
6	$3.2^\#.1^\#$	
7	$3.1.2$	6 AM8
8	$1.3.2$	7 AM11
9	$1.2^\#.3^\#$	8 AM8
10	$A_{3^\#}$	9,2,3 $\to$E
11	$2^\# \simeq 3$	5 oE
12	A_2	10,11 $\simeq$E
13	A_2	4,5-12 AM12
14	$\neg A_{2^\#}$	3 R
15	A_2	3-14 $\neg$E
16	$[(\neg A \to A) \to A]_0$	1-15 $\to$I

A13 $\vdash_{NB_{CK}} A \to (B \to A)$

1	$\vert\ \vert\ 0.1.2$	A (g, →I)
2	$\vert\ \vert\ \underline{A_1}$	
3	$\vert\ \vert\ \vert\ 2.3.4$	A (g, →I)
4	$\vert\ \vert\ \vert\ \underline{B_3}$	
5	$\vert\ \vert\ \vert\ 1.0.2$	1 AM11
6	$\vert\ \vert\ \vert\ 1.3.4$	5,3 AM13
7	$\vert\ \vert\ \vert\ A_4$	2,6 AM⪯
8	$\vert\ \vert\ (B \to A)_2$	3-7 →I
9	$\vert\ [A \to (B \to A)]_0$	1-8 →I

$\vdash_{NB_{CK}} (A \vee B) \to ((A \to B) \to B)$

1	$\vert\ \vert\ 0.1.2$	A (g, →I)
2	$\vert\ \vert\ \underline{(A \vee B)_1}$	
3	$\vert\ \vert\ \vert\ 2.3.4$	A (g, →I)
4	$\vert\ \vert\ \vert\ \underline{(A \to B)_3}$	
5	$\vert\ \vert\ \vert\ \vert\ \underline{A_1}$	A (g, 2 ∨E)
6	$\vert\ \vert\ \vert\ \vert\ 1 \simeq 2$	1 0E
7	$\vert\ \vert\ \vert\ \vert\ A_2$	5,6 ≃E
8	$\vert\ \vert\ \vert\ \vert\ 3.2.4$	3 AM11
9	$\vert\ \vert\ \vert\ \vert\ B_4$	8,4,7 →E
10	$\vert\ \vert\ \vert\ \vert\ \underline{B_1}$	A (g, 2 ∨E)
11	$\vert\ \vert\ \vert\ \vert\ 1.0.2$	1 AM11
12	$\vert\ \vert\ \vert\ \vert\ 1.3.4$	11,3 AM13
13	$\vert\ \vert\ \vert\ \vert\ B_4$	10,12 AM⪯
14	$\vert\ \vert\ \vert\ B_4$	2,5-9,10-13 ∨E
15	$\vert\ \vert\ ((A \to B) \to B)_2$	3-14 →I
16	$\vert\ [(A \vee B) \to ((A \to B) \to B)]_0$	1-15 →I

8.3 SOUNDNESS AND COMPLETENESS

Preliminaries: Begin with generalized notions of validity. For a model $\langle W, N, R, *, v\rangle$ or $\langle W, N, R, \{R_A \mid A \in \mathfrak{I}\}, *, v\rangle$, let m be a map from subscripts into W such that $m(0) \in N$ and $m(\bar{s}) = m(s)^*$. Say $\langle W, N, R, *, v\rangle_m$ and $\langle W, N, R, \{R_A \mid A \in \mathfrak{I}\}, *, v\rangle_m$ are$\langle W, N, R, *, v\rangle$ and $\langle W, N, R, \{R_A \mid A \in \mathfrak{I}\}, *, v\rangle$ ***with*** map m. Then, where Γ is a set of expressions of our language for derivations, $v_m(\Gamma) = 1$ iff for each $A_s \in \Gamma$, $v_{m(s)}(A) = 1$, for each $s \simeq t \in \Gamma$, $m(s) = m(t)$, for each $s.t.u \in \Gamma$, $\langle m(s), m(t), m(u)\rangle \in R$, and for each $A_{s/t} \in \Gamma$, $\langle m(s), m(t)\rangle \in R_A$. Unless otherwise noted, reasoning is meant to be neutral between interpretations of the different types. Now expand notions of validity to include subscripted formulas, and alternate expressions as indicated in double brackets.

VBx* $\Gamma \models^*_{Bx} A_s$ $[\![s \simeq t/s.t.u/A_{s/t}]\!]$ iff there is no *Bx* interpretation with map m where $v_m(\Gamma) = 1$ but $v_{m(s)}(A) = 0$ $[\![m(s) \neq m(t)\ /\ \langle m(s), m(t), m(u)\rangle \notin$

$R \,/\, \langle m(s), m(t)\rangle \notin R_A]\!]$.

NBx* $\Gamma \vdash^{*}_{NBx} A_s$ $[\![s \simeq t \,/\, s.t.u \,/\, A_{s/t}]\!]$ iff there is an *NBx* derivation of A_s $[\![s \simeq t \,/\, s.t.u \,/\, A_{s/t}]\!]$ from the members of Γ.

These notions reduce to the standard ones when all the members of Γ and A have subscript 0 (and so are not of the sort $s \simeq t$, $s.t.u$ or $A_{s/t}$). For the following, cases omitted are like ones worked, and so left to the reader.

THEOREM 8.1 *NBx is sound: If* $\Gamma \vdash_{NBx} A$ *then* $\Gamma \vDash_{Bx} A$.

L8.1 If $\Gamma \subseteq \Gamma'$ and $\Gamma \vDash^{*}_{Bx} P_s$ $[\![s \simeq t \,/\, s.t.u \,/\, A_{s/t}]\!]$, then $\Gamma' \vDash^{*}_{Bx} P_s$ $[\![s \simeq t \,/\, s.t.u \,/\, A_{s/t}]\!]$.

Suppose $\Gamma \subseteq \Gamma'$ and $\Gamma \vDash^{*}_{Bx} P_s$ $[\![s \simeq t \,/\, s.t.u \,/\, A_{s/t}]\!]$, but $\Gamma' \nvDash^{*}_{Bx} P_s$ $[\![s \simeq t \,/\, s.t.u \,/\, A_{s/t}]\!]$. From the latter, by VBx*, there is some *Bx* interpretation with ν and m such that $\nu_m(\Gamma') = 1$ but $\nu_{m(s)}(P) = 0$ $[\![m(s) \neq m(t) \,/\, \langle m(s), m(t), m(u)\rangle \notin R \,/\, \langle m(s), m(t)\rangle \notin R_A]\!]$. But since $\nu_m(\Gamma') = 1$ and $\Gamma \subseteq \Gamma'$, $\nu_m(\Gamma) = 1$; so $\nu_m(\Gamma) = 1$ but $\nu_{m(s)}(P) = 0$ $[\![m(s) \neq m(t) \,/\, \langle m(s), m(t), m(u)\rangle \notin R \,/\, \langle m(s), m(t)\rangle \notin R_A]\!]$; so by VBx*, $\Gamma \nvDash^{*}_{Bx} P_s$ $[\![s \simeq t \,/\, s.t.u \,/\, A_{s/t}]\!]$. This is impossible; reject the assumption: if $\Gamma \subseteq \Gamma'$ and $\Gamma \vDash^{*}_{Bx} P_s$ $[\![s \simeq t \,/\, s.t.u \,/\, A_{s/t}]\!]$, then $\Gamma' \vDash^{*}_{Bx} P_s$ $[\![s \simeq t \,/\, s.t.u \,/\, A_{s/t}]\!]$.

Main result: For each line in a derivation let $\mathcal{P}_i$ be the expression on line i and Γ_i be the set of all premises and assumptions whose scope includes line i. We set out to show "generalized" soundness: if $\Gamma \vdash^{*}_{NBx} \mathcal{P}$ then $\Gamma \vDash^{*}_{Bx} \mathcal{P}$. As above, this reduces to the standard result when $\mathcal{P}$ and all the members of Γ are formulas with subscript 0. Suppose $\Gamma \vdash^{*}_{NBx} \mathcal{P}$. Then there is a derivation of $\mathcal{P}$ from premises in Γ where $\mathcal{P}$ appears under the scope of the premises alone. By induction on line number of this derivation, we show that for each line i of this derivation, $\Gamma_i \vDash^{*}_{Bx} \mathcal{P}_i$. The case when $\mathcal{P}_i = \mathcal{P}$ is the desired result.

Basis: $\mathcal{P}_1$ is a premise or an assumption A_s $[\![s \simeq t/s.t.u/A_{s/t}]\!]$. Then $\Gamma_1 = \{A_s\}$ $[\![\{s \simeq t\} \,/\, \{s.t.u\} \,/\, \{A_{s/t}\}]\!]$; so for any *Bx* interpretation with its ν and m, $\nu_m(\Gamma_1) = 1$ iff $\nu_{m(s)}(A) = 1$ $[\![m(s) = m(t) \,/\, \langle m(s), m(t), m(u)\rangle \in R \,/\, \langle m(s), m(t)\rangle \in R_A]\!]$; so there is no *Bx* interpretation with ν and m such that $\nu_m(\Gamma_1) = 1$ but $\nu_{m(s)}(A) = 0$ $[\![m(s) \neq m(t) \,/\, \langle m(s), m(t), m(u)\rangle \notin R \,/\, \langle m(s), m(t)\rangle \notin R_A]\!]$. So by VBx*, $\Gamma_1 \vDash^{*}_{Bx} A_s$ $[\![s \simeq t \,/\, s.t.u \,/\, A_{s/t}]\!]$, where this is just to say, $\Gamma_1 \vDash^{*}_{Bx} \mathcal{P}_1$.

Assp: For any $i, 1 \leqslant i < k, \Gamma_i \vDash^{*}_{Bx} \mathcal{P}_i$.

Show: $\Gamma_k \vDash^{*}_{Bx} \mathcal{P}_k$.

$\mathcal{P}_k$ is either a premise, an assumption, or arises from previous lines by R, $\wedge$I, $\wedge$E, $\vee$I, $\vee$E, $\neg$I, $\neg$E, $\rightarrow$I, $\rightarrow$E, $\simeq$I, $\simeq$E, $\circ$I, $\circ$E or, depending on the system, AM8, AM9, AM10, AM11, AM12, AM13, AM$\preceq$, $>$I, $>$E, AMP1, or AMP2. If $\mathcal{P}_k$ is a premise or an assumption, then as in the basis, $\Gamma_k \vDash^{*}_{Bx} \mathcal{P}_k$. So suppose $\mathcal{P}_k$ arises by one of the rules.

(R)

(∧I)

(∧E)

(∨I)

(∨E)

(¬I) If $\mathcal{P}_k$ arises by ¬I, then the picture is like this,

$$\begin{array}{r|l}
 & \quad \underline{A_{\bar{s}}} \\
 & \quad \\
i & \quad B_t \\
j & \quad \neg B_{\bar{t}} \\
k & \neg A_s
\end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $\neg A_s$. By assumption, $\Gamma_i \models^*_{Bx} B_t$ and $\Gamma_j \models^*_{Bx} \neg B_{\bar{t}}$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{A_{\bar{s}}\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{A_{\bar{s}}\}$; so by L8.1, $\Gamma_k \cup \{A_{\bar{s}}\} \models^*_{Bx} B_t$ and $\Gamma_k \cup \{A_{\bar{s}}\} \models^*_{Bx} \neg B_{\bar{t}}$. Suppose $\Gamma_k \not\models^*_{Bx} \neg A_s$; then by VBx*, there is a *Bx* interpretation with ν and m such that $\nu_m(\Gamma_k) = 1$ but $\nu_{m(s)}(\neg A) = 0$; so by TB(¬), $\nu_{m(s)^*}(A) = 1$; so by the construction of m, $\nu_{m(\bar{s})}(A) = 1$; so $\nu_m(\Gamma_k) = 1$ and $\nu_{m(\bar{s})}(A) = 1$; so $\nu_m(\Gamma_k \cup \{A_{\bar{s}}\}) = 1$; so by VBx*, $\nu_{m(t)}(B) = 1$ and $\nu_{m(\bar{t})}(\neg B) = 1$; from the latter, by TB(¬), $\nu_{m(\bar{t})^*}(B) = 0$; so by the construction of m, $\nu_{m(t)}(B) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} \neg A_s$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

(¬E)

(→I) If $\mathcal{P}_k$ arises by →I, then the picture is like this,

$$\begin{array}{r|l}
 & \quad s.t.u \\
 & \quad \underline{A_t} \\
 & \quad \\
i & \quad B_u \\
k & (A \rightarrow B)_s
\end{array}$$

where $i < k$, t, u are not introduced in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $(A \rightarrow B)_s$. By assumption, $\Gamma_i \models^*_{Bx} B_u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{s.t.u, A_t\}$; so by L8.1, $\Gamma_k \cup \{s.t.u, A_t\} \models^*_{Bx} B_u$. Suppose $\Gamma_k \not\models^*_{Bx} (A \rightarrow B)_s$; then by VBx*, there is a *Bx* interpretation with $\mathcal{W}$, R, ν and m such that $\nu_m(\Gamma_k) = 1$ but $\nu_{m(s)}(A \rightarrow B) = 0$; so by TB(→), there are $x, y \in \mathcal{W}$ such that $Rm(s)xy$ and $\nu_x(A) = 1$ but $\nu_y(B) = 0$. Now consider a map m' like m except that $m'(t) = x$, $m'(\bar{t}) = x^*$, $m'(u) = y$, and $m'(\bar{u}) = y^*$; since t and u (along with $\bar{t}$ and $\bar{u}$) do not appear in Γ_k, it remains that $\nu_{m'}(\Gamma_k) = 1$;

since $v_x(A) = 1$, $v_{m'(t)}(A) = 1$; and since $Rm(s)xy$, with $m(s) = m'(s)$, we have $\langle m'(s), m'(t), m'(u)\rangle \in R$; so $v_{m'}(\Gamma_k \cup \{s.t.u, A_t\}) = 1$; so by VBx*, $v_{m'(u)}(B) = 1$. But $m'(u) = y$; so $v_y(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} (A \to B)_s$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

(→E) If $\mathcal{P}_k$ arises by →E, then the picture is like this,

$$\begin{array}{r|l} h & s.t.u \\ i & (A \to B)_s \\ j & A_t \\ & \\ k & B_u \end{array}$$

where $h, i, j < k$ and $\mathcal{P}_k$ is B_u. By assumption, $\Gamma_h \models^*_{Bx} s.t.u$, $\Gamma_i \models^*_{Bx} (A \to B)_s$ and $\Gamma_j \models^*_{Bx} A_t$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L8.1, $\Gamma_k \models^*_{Bx} s.t.u$, $\Gamma_k \models^*_{Bx} (A \to B)_s$ and $\Gamma_k \models^*_{Bx} A_t$. Suppose $\Gamma_k \not\models^*_{Bx} B_u$; then by VBx*, there is some *Bx* interpretation with $\mathcal{W}$, R, v and m such that $v_m(\Gamma_k) = 1$ but $v_{m(u)}(B) = 0$; since $v_m(\Gamma_k) = 1$, by VBx*, $\langle m(s), m(t), m(u)\rangle \in R$, $v_{m(s)}(A \to B) = 1$ and $v_{m(t)}(A) = 1$; since $v_{m(s)}(A \to B) = 1$, by TB(→), there are no $x, y \in \mathcal{W}$ such that $Rm(s)xy$ and $v_x(A) = 1$ but $v_y(B) = 0$; so since $\langle m(s), m(t), m(u)\rangle \in R$, it is not the case that $v_{m(t)}(A) = 1$ and $v_{m(u)}(B) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} B_u$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

(≃I) If $\mathcal{P}_k$ arises by ≃I, then the picture is like this,

$$\begin{array}{r|l} & \\ k & s \simeq s \end{array}$$

where $\mathcal{P}_k$ is $s \simeq s$. Suppose $\Gamma_k \not\models^*_{Bx} s \simeq s$; then by VBx*, there is a *Bx* interpretation with v, and m such that $v_m(\Gamma_k) = 1$ but $m(s) \neq m(s)$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} s \simeq s$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

(≃E) If A_k arises by ≃E, then the picture is like this,

$$\begin{array}{r|l} i & s \simeq t \\ j & \mathcal{A}(s) \\ & \\ k & \mathcal{A}(t) \end{array} \quad \text{or} \quad \begin{array}{r|l} i & s \simeq t \\ k & \mathcal{A}(\bar{s}) \\ & \\ k & \mathcal{A}(\bar{t}) \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $\mathcal{A}(t)$ or $\mathcal{A}(\bar{t})$. By assumption, $\Gamma_i \models^*_{Bx} s \simeq t$ and $\Gamma_j \models^*_{Bx} \mathcal{A}(s)$ / $\mathcal{A}(\bar{s})$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L8.1, $\Gamma_k \models^*_{Bx} s \simeq t$ and $\Gamma_k \models^*_{Bx} \mathcal{A}(s)$ / $\mathcal{A}(\bar{s})$. In the right-hand case, $\mathcal{A}(\bar{s})$ is of the sort, A_u, $u \simeq v$, $u.v.w$ or $A_{u/v}$ where one of u, v, or w is $\bar{s}$. Suppose $\mathcal{A}(\bar{s})$ is $A_{\bar{s}}$ and $\Gamma_k \not\models^*_{Bx} A_{\bar{t}}$. Then by VBx*, there is some *Bx* interpretation with v and m such that $v_m(\Gamma_k) = 1$ but $v_{m(\bar{t})}(A) = 0$. Since $v_m(\Gamma_k) = 1$, by VBx*, $m(s) = m(t)$ and $v_{m(\bar{s})}(A) = 1$; since $m(s) = m(t)$, $m(s)^* = m(t)^*$; but by the construction of m, $m(s)^* =$

$m(\bar{s})$ and $m(t)^* = m(\bar{t})$; so $m(\bar{s}) = m(\bar{t})$; so $v_{m(\bar{t})}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} A_{\bar{t}}$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$. And similarly in the other cases.

(oI) If $\mathcal{P}_k$ arises by oI, then the picture is like this,

$$\begin{array}{r|l} i & s \simeq t \\ & \\ k & 0.s.t \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is 0.s.t. By assumption, $\Gamma_i \models^*_{Bx} s \simeq t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L8.1, $\Gamma_k \models^*_{Bx} s \simeq t$. Suppose $\Gamma_k \not\models^*_{Bx}$ 0.s.t; then by VBx*, there is a *Bx* interpretation with *W*, N, R, v and m such that $v_m(\Gamma_k) = 1$ but $\langle m(0), m(s), m(t)\rangle \notin R$; since $v_m(\Gamma_k) = 1$, by VBx*, $m(s) = m(t)$; and by the construction of m, $m(0) \in N$; so by NC, $\langle m(0), m(s), m(t)\rangle \in R$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx}$ 0.s.t, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

(oE) If $\mathcal{P}_k$ arises by oE, then the picture is like this,

$$\begin{array}{r|l} i & 0.s.t \\ & \\ k & s \simeq t \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is $s \simeq t$. By assumption, $\Gamma_i \models^*_{Bx}$ 0.s.t; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L8.1, $\Gamma_k \models^*_{Bx}$ 0.s.t. Suppose $\Gamma_k \not\models^*_{Bx} s \simeq t$; then by VBx*, there is a *Bx* interpretation with *W*, N, R, v and m such that $v_m(\Gamma_k) = 1$ but $m(s) \neq m(t)$; since $v_m(\Gamma_k) = 1$, by VBx*, $\langle m(0), m(s), m(t)\rangle \in R$; and by the construction of m, $m(0) \in N$; so by NC, $m(s) = m(t)$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} s \simeq t$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

(AM8) If $\mathcal{P}_k$ arises by AM8, then the picture is like this,

$$\begin{array}{r|l} i & s.t.u \\ & \\ k & s.\bar{u}.\bar{t} \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is $s.\bar{u}.\bar{t}$. Where this rule is included in *NBx*, *Bx* includes condition C8. By assumption, $\Gamma_i \models^*_{Bx}$ s.t.u; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L8.1, $\Gamma_k \models^*_{Bx}$ s.t.u. Suppose $\Gamma_k \not\models^*_{Bx} s.\bar{u}.\bar{t}$; then by VBx*, there is a *Bx* interpretation with R, v and m such that $v_m(\Gamma_k) = 1$ but $\langle m(s), m(\bar{u}), m(\bar{t})\rangle \notin R$; since $v_m(\Gamma_k) = 1$, by VBx*, $\langle m(s), m(t), m(u)\rangle \in R$; so by C8, $\langle m(s), m(u)^*, m(t)^*\rangle \in R$; so by the construction of m, $\langle m(s), m(\bar{u}), m(\bar{t})\rangle \in R$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} s.\bar{u}.\bar{t}$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

(AM9) If $\mathcal{P}_k$ arises by AM9, then the picture is like this,

$$
\begin{array}{r|l}
h & s.t.x \\
i & x.u.v \\
 & \quad \begin{array}{|l} s.u.y \\ \underline{t.y.v} \\ \\ A_w \end{array} \\
j & \\
k & A_w
\end{array}
$$

where $h, i, j < k$, y is not introduced in any member of Γ_k (in any undischarged premise or assumption) or by w, and $\mathcal{P}_k$ is A_w. Where this rule is included in *NBx*, *Bx* includes condition C9. By assumption, $\Gamma_h \models^*_{Bx}$ s.t.x, $\Gamma_i \models^*_{Bx}$ x.u.v and $\Gamma_j \models^*_{Bx} A_w$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k \cup \{s.u.y, t.y.v\}$; so by L8.1, $\Gamma_k \models^*_{Bx}$ s.t.x, $\Gamma_k \models^*_{Bx}$ x.u.v and $\Gamma_k \cup \{s.u.y, t.y.v\} \models^*_{Bx} A_w$. Suppose $\Gamma_k \not\models^*_{Bx} A_w$; then by VBx*, there is a *Bx* interpretation with *W*, R, v and m such that $v_m(\Gamma_k) = 1$ but $v_{m(w)}(A) = 0$; since $v_m(\Gamma_k) = 1$, by VBx*, $\langle m(s), m(t), m(x)\rangle \in R$ and $\langle m(x), m(u), m(v)\rangle \in R$; so by C9, there is some $z \in W$ such that $\langle m(s), m(u), z\rangle \in R$ and $\langle m(t), z, m(v)\rangle \in R$. Now consider a map m' like m except that $m'(y) = z$ and $m'(\overline{y}) = z^*$; since y (along with $\overline{y}$) does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m(s) = m'(s)$, and similarly for t, u and v, $\langle m'(s), m'(u), m'(y)\rangle \in R$ and $\langle m'(t), m'(y), m'(v)\rangle \in R$; so $v_{m'}(\Gamma_k \cup \{s.u.y, t.y.v\}) = 1$; so by VBx*, $v_{m'(w)}(A) = 1$. But since y is not introduced by w, $m'(w) = m(w)$; so $v_{m(w)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} A_w$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

(AM10)

(AM11)

(AM12)

(AM13) If $\mathcal{P}_k$ arises by AM13, then the picture is like this,

$$
\begin{array}{r|l}
i & s.t.u \\
j & u.v.w \\
 & \\
k & s.v.w
\end{array}
$$

where $i, j < k$ and $\mathcal{P}_k$ is s.v.w. Where this rule is included in *NBx*, *Bx* includes condition C13. By assumption, $\Gamma_i \models^*_{Bx}$ s.t.u and $\Gamma_j \models^*_{Bx}$ u.v.w; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L8.1, $\Gamma_k \models^*_{Bx}$ s.t.u and $\Gamma_j \models^*_{Bx}$ u.v.w. Suppose $\Gamma_k \not\models^*_{Bx}$ s.v.w; then by VBx*, there is a *Bx* interpretation with R, v and m such that $v_m(\Gamma_k) = 1$ but $\langle m(s), m(v), m(w)\rangle \notin R$; since $v_m(\Gamma_k) = 1$, by VBx*, we have $\langle m(s), m(t), m(u)\rangle \in R$ and $\langle m(u), m(v), m(w)\rangle \in R$; and so by C13, $\langle m(s), m(v), m(w)\rangle \in R$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx}$ s.v.w, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

(AM$\preceq$) If $\mathcal{P}_k$ arises by AM$\preceq$, then the picture is like this,

$$\begin{array}{l|l} i & s.t.u \\ j & A_s \\ & \\ k & A_u \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is A_u. In *BCK*, where this rule is included in *NBx*, *Bx* includes condition ($\preceq$) along with C8, C11 and C13. By assumption, $\Gamma_i \models^*_{Bx}$ s.t.u and $\Gamma_j \models^*_{Bx} A_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L8.1, $\Gamma_k \models^*_{Bx}$ s.t.u and $\Gamma_k \models^*_{Bx} A_s$. Suppose $\Gamma_k \not\models^*_{Bx} A_u$; then by VBx*, there is a *Bx* interpretation with v and m such that $v_m(\Gamma_k) = 1$ but $v_{m(u)}(A) = 0$; since $v_m(\Gamma_k) = 1$, by VBx*, $\langle m(s), m(t), m(u)\rangle \in R$ and $v_{m(s)}(A) = 1$; But given $Rabc$, under current constraints, (i) if $v_a(A) = 1$ then $v_c(A) = 1$ and (ii) if $v_{c^*}(A) = 1$ then $v_{a^*}(A) = 1$. Suppose $Rabc$.

Basis: A is a parameter p. (i) Suppose $v_a(A) = 1$; then $v_a(p) = 1$; so by ($\preceq$), $v_c(p) = 1$; so $v_c(A) = 1$. (ii) Suppose $v_{c^*}(A) = 1$; then $v_{c^*}(p) = 1$; so by ($\preceq$), $v_{a^*}(p) = 1$; so $v_{a^*}(A) = 1$.

Assp: For any i, $0 \leqslant i < k$, if A has i operator symbols then (i) if $v_a(A) = 1$ then $v_c(A) = 1$ and (ii) if $v_{c^*}(A) = 1$ then $v_{a^*}(A) = 1$.

Show: If A has k operator symbols then (i) if $v_a(A) = 1$ then $v_c(A) = 1$ and (ii) if $v_{c^*}(A) = 1$ then $v_{a^*}(A) = 1$.

In this system we do not have $>$ in the language. So if A has k operator symbols, it is of the form $\neg P$, $P \wedge Q$, $P \vee Q$, or $P \to Q$ where P and Q have $< k$ operator symbols.

($\neg$) A is $\neg P$. (i) Suppose $v_a(A) = 1$; then $v_a(\neg P) = 1$; so by TB($\neg$), $v_{a^*}(P) = 0$; so by assumption, $v_{c^*}(P) = 0$; so by TB($\neg$), $v_c(\neg P) = 1$; so $v_c(A) = 1$. (ii) Suppose $v_{c^*}(A) = 1$; then $v_{c^*}(\neg P) = 1$; so by TB($\neg$), $v_c(P) = 0$; so by assumption, $v_a(P) = 0$; so by TB($\neg$), $v_{a^*}(\neg P) = 1$; so $v_{a^*}(A) = 1$.

($\wedge$) A is $P \wedge Q$. (i) Suppose $v_a(A) = 1$; then $v_a(P \wedge Q) = 1$; so by TB($\wedge$), $v_a(P) = 1$ and $v_a(Q) = 1$; so by assumption, $v_c(P) = 1$ and $v_c(Q) = 1$; so by TB($\wedge$), $v_c(P \wedge Q) = 1$; so $v_c(A) = 1$. (ii) Suppose $v_{c^*}(A) = 1$; then $v_{c^*}(P \wedge Q) = 1$; so by TB($\wedge$), $v_{c^*}(P) = 1$ and $v_{c^*}(Q) = 1$; so by assumption, $v_{a^*}(P) = 1$ and $v_{a^*}(Q) = 1$; so by TB($\wedge$), $v_{a^*}(P \wedge Q) = 1$; so $v_{a^*}(A) = 1$.

($\vee$)

($\to$) A is $P \to Q$. (i) Suppose $v_a(A) = 1$ but $v_c(A) = 0$; then $v_a(P \to Q) = 1$ but $v_c(P \to Q) = 0$. From the latter, by TB($\to$), there are $w, x \in W$ such that $Rcwx$ and $v_w(P) = 1$ but $v_x(Q) = 0$. From the former, by TB($\to$), there are no $y, z \in W$ such that $Rayz$ and $v_y(P) = 1$ but $v_z(Q) = 0$. But since $Rabc$ and $Rcwx$, by C13,

$Rawx$; so it is not the case that $v_w(P) = 1$ and $v_x(Q) = 0$. This is impossible; reject the assumption: if $v_a(A) = 1$, then $v_c(A) = 1$. (ii) Suppose $v_{c^*}(A) = 1$ but $v_{a^*}(A) = 0$; then $v_{c^*}(P \rightarrow Q) = 1$ but $v_{a^*}(P \rightarrow Q) = 0$. From the latter, by TB($\rightarrow$), there are $w, x \in W$ such that Ra^*wx and $v_w(P) = 1$ but $v_x(Q) = 0$. From the former, by TB($\rightarrow$), there are no $y, z \in W$ such that Rc^*yz and $v_y(P) = 1$ but $v_z(Q) = 0$. But since $Rabc$, by C11, $Rbac$; so by C8, Rbc^*a^*; so by C11, Rc^*ba^*; so with Ra^*wx, by C13, Rc^*wx; so it is not the case that $v_w(P) = 1$ and $v_x(Q) = 0$. This is impossible; reject the assumption: if $v_{c^*}(A) = 1$, then $v_{a^*}(A) = 1$.

For any A, (i) if $v_a(A) = 1$ then $v_c(A) = 1$ and (ii) if $v_{c^*}(A) = 1$ then $v_{a^*}(A) = 1$.

So, returning to the main case, $v_{m(u)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} A_u$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

(>I)

(>E) If $\mathcal{P}_k$ arises by >E, then the picture is like this,

$$\begin{array}{r|l} i & (A > B)_s \\ j & A_{s/t} \\ & \\ k & B_t \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is B_t. By assumption, $\Gamma_i \models^*_{Bx} (A > B)_s$ and $\Gamma_j \models^*_{Bx} A_{s/t}$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L8.1, $\Gamma_k \models^*_{Bx} (A > B)_s$ and $\Gamma_k \models^*_{Bx} A_{s/t}$. Suppose $\Gamma_k \not\models^*_{Bx} B_t$; then by VBx*, there is some *Bx* interpretation with W, $\{R_A \mid A \in \mathfrak{I}\}$, v and m such that $v_m(\Gamma_k) = 1$ but $v_{m(t)}(B) = 0$; since $v_m(\Gamma_k) = 1$, by VBx*, $v_{m(s)}(A > B) = 1$ and $\langle m(s), m(t) \rangle \in R_A$; from the former, by TB(>), any $w \in W$ such that $m(s)R_A w$ has $v_w(B) = 1$; so $v_{m(t)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} B_t$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

(AMPi) If $\mathcal{P}_k$ arises by AMPi, then the picture is like this,

$$\begin{array}{r|l} i & A_{0/t} \\ & \\ k & A_t \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is A_t. Where this rule is in *NBx*, *Bx* includes condition (1). By assumption, $\Gamma_i \models^*_{Bx} A_{0/t}$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L8.1, $\Gamma_k \models^*_{Bx} A_{0/t}$. Suppose $\Gamma_k \not\models^*_{Bx} A_t$; then by VBx*, there is some *Bx* interpretation with N, $\{R_A \mid A \in \mathfrak{I}\}$, v and m such that $v_m(\Gamma_k) = 1$ but $v_{m(t)}(A) = 0$; since $v_m(\Gamma_k) = 1$, by VBx*, $m(t) \in f_A(m(0))$; but by the construction of m, $m(0) \in N$; so by condition (1),

$m(t) \in [A]$; so $v_{m(t)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} A_t$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

(AMP2) If $\mathcal{P}_k$ arises by AMP2, then the picture is like this,

i	A_0
k	$A_{0/0}$

where $i < k$ and $\mathcal{P}_k$ is $A_{0/0}$. Where this rule is in *NBx*, *Bx* includes condition (2). By assumption, $\Gamma_i \models^*_{Bx} A_0$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L8.1, $\Gamma_k \models^*_{Bx} A_0$. Suppose $\Gamma_k \not\models^*_{Bx} A_{0/0}$; then by VBx*, there is some *Bx* interpretation with N, $\{R_A \mid A \in \mathfrak{I}\}$, v and m such that $v_m(\Gamma_k) = 1$ but $m(0) \notin f_A(m(0))$; since $v_m(\Gamma_k) = 1$, by VCx*, $v_{m(0)}(A) = 1$; so $m(0) \in [A]$; and by the construction of m, $m(0) \in N$; so by condition (2), $m(0) \in f_A(m(0))$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} A_{0/0}$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

For any i, $\Gamma_i \models^*_{Bx} \mathcal{P}_i$.

THEOREM 8.2 *NB*x *is complete: if* $\Gamma \models_{Bx} A$ *then* $\Gamma \vdash_{NBx} A$.

Suppose $\Gamma \models_{Bx} A$; then $\Gamma_0 \models^*_{Bx} A_0$; we show that $\Gamma_0 \vdash^*_{NBx} A_0$. As usual, this reduces to the standard notion. For the following, fix on some particular constraint(s) x. Then definitions of *consistency* etc. are relative to it.

CON Γ is CONSISTENT iff there is no A_s such that $\Gamma \vdash^*_{NBx} A_s$ and $\Gamma \vdash^*_{NBx} \neg A_{\overline{s}}$.

L8.2 If s is 0 or introduced in Γ, and $\Gamma \nvdash^*_{NBx} \neg P_{\overline{s}}$, then $\Gamma \cup \{P_s\}$ is consistent.

Suppose s is 0 or introduced in Γ and $\Gamma \nvdash^*_{NBx} \neg P_{\overline{s}}$ but $\Gamma \cup \{P_s\}$ is inconsistent. Then there is some A_t such that $\Gamma \cup \{P_s\} \vdash^*_{NBx} A_t$ and $\Gamma \cup \{P_s\} \vdash^*_{NBx} \neg A_{\overline{t}}$. But then we can argue,

1	Γ	
2	P_s	A (c, ¬I)
3	A_t	from $\Gamma \cup \{P_s\}$
4	$\neg A_{\overline{t}}$	from $\Gamma \cup \{P_s\}$
5	$\neg P_{\overline{s}}$	2-4 ¬I

where the assumption is allowed insofar as s is either 0 or introduced in Γ; so $\Gamma \vdash^*_{NBx} \neg P_{\overline{s}}$. But this is impossible; reject the assumption: if s is 0 or introduced in Γ and $\Gamma \nvdash^*_{NBx} \neg P_{\overline{s}}$, then $\Gamma \cup \{P_s\}$ is consistent.

L8.3 There is an enumeration of all the subscripted formulas, $\mathcal{P}_1$ $\mathcal{P}_2 \ldots$ In addition, there is an enumeration of these formulas with access relations s.t.u and with pairs of the sort s.t.u / u.v.w.

Proof by construction.

MAX Γ is S-MAXIMAL iff for any A_s either $\Gamma \vdash^*_{NBx} A_s$ or $\Gamma \vdash^*_{NBx} \neg A_{\overline{s}}$.

SGT Γ is a SCAPEGOAT set for $\rightarrow$ iff for every formula of the form $\neg(A \rightarrow B)_s$, if $\Gamma \vdash^*_{NBx} \neg(A \rightarrow B)_s$ then there are y and z such that $\Gamma \vdash^*_{NBx} \overline{s}.y.z$, $\Gamma \vdash^*_{NBx} A_y$ and $\Gamma \vdash^*_{NBx} \neg B_{\overline{z}}$.

Γ is a SCAPEGOAT set for $>$ iff for every formula of the form $\neg(A > B)_s$, if $\Gamma \vdash^*_{NBx} \neg(A > B)_s$ then there is some y such that $\Gamma \vdash^*_{NBx} A_{\overline{s}/y}$ and $\Gamma \vdash^*_{NBx} \neg B_{\overline{y}}$.

Γ is a SCAPEGOAT set for C9/C10 iff for any access pair $s.t.u$ / $u.v.w$, if $\Gamma \vdash^*_{NBx} s.t.u$ and $\Gamma \vdash^*_{NBx} u.v.w$, then there is a y such that $\Gamma \vdash^*_{NBx} s.v.y$ and $\Gamma \vdash^*_{NBx} t.y.w$, and a z such that $\Gamma \vdash^*_{NBx} t.v.z$ and $\Gamma \vdash^*_{NBx} s.z.w$.

Γ is a SCAPEGOAT set for C12 iff for any access relation $s.t.u$, if $\Gamma \vdash^*_{NBx} s.t.u$, then there is a y such that $\Gamma \vdash^*_{NBx} s.t.y$ and $\Gamma \vdash^*_{NBx} y.t.u$.

C(Γ') For Γ with unsubscripted formulas and the corresponding Γ_0, we construct Γ' as follows. Set $\Omega_0 = \Gamma_0$. By L8.3, there is an enumeration, $\mathcal{P}_1, \mathcal{P}_2 \ldots$ of all the subscripted formulas, together with all the access relations $s.t.u$ if C12 is in *Bx*, and pairs $s.t.u / u.v.w$ if C9 and C10 are in *Bx*; let $\mathcal{E}_0$ be this enumeration. Then for the first expression $\mathcal{P}$ in $\mathcal{E}_{i-1}$ such that all its subscripts are 0 or introduced in Ω_{i-1}, let $\mathcal{E}_i$ be like $\mathcal{E}_{i-1}$ but without $\mathcal{P}$, and set,

$\Omega_i = \Omega_{i-1}$	if $\Omega_{i-1} \cup \{\mathcal{P}\}$ is inconsistent
$\Omega_{i^*} = \Omega_{i-1} \cup \{\mathcal{P}\}$	if $\Omega_{i-1} \cup \{\mathcal{P}\}$ is consistent

and

$\Omega_i = \Omega_{i^*}$	if $\mathcal{P}$ is not of the form $\neg(P \rightarrow Q)_s$, $\neg(P > Q)_s$, $s.t.u / u.v.w$, or $s.t.u$
$\Omega_i = \Omega_{i^*} \cup \{\overline{s}.y.z, P_y \neg Q_{\overline{z}}\}$	if $\mathcal{P}$ is of the form $\neg(P \rightarrow Q)_s$
$\Omega_i = \Omega_{i^*} \cup \{P_{\overline{s}/y}, \neg Q_{\overline{y}}\}$	if $\mathcal{P}$ is of the form $\neg(P > Q)_s$
$\Omega_i = \Omega_{i^*} \cup \{s.v.y, t.y.w, t.v.z, s.z.w\}$	if $\mathcal{P}$ is of the form $s.t.u / u.v.w$
$\Omega_i = \Omega_{i^*} \cup \{s.t.y, y.t.u\}$	if $\mathcal{P}$ is of the form $s.t.u$

-where y and z are the first subscripts not introduced in Ω_{i^*}

then

$$\Gamma' = \bigcup_{i \geqslant 0} \Omega_i$$

Note that there are always sure to be subscripts y and z not in Ω_{i^*} insofar as there are infinitely many subscripts, and at any stage only finitely many expressions are added – the only subscripts in the initial Ω_0 being 0. Suppose s is introduced in Γ'; then there is some Ω_i in which it is first introduced; and any expression $\mathcal{P}_j$ in the original enumeration that introduces subscript s is sure to be "considered" for inclusion at a subsequent stage.

L8.4 For any s introduced in Γ', Γ' is s-maximal.

Suppose s is introduced in Γ' but Γ' is not s-maximal. Then there is some A_s such that $\Gamma' \nvdash^*_{NBx} A_s$ and $\Gamma' \nvdash^*_{NBx} \neg A_{\overline{s}}$. For any i, each member of Ω_{i-1} is in Γ'; so if $\Omega_{i-1} \vdash^*_{NBx} \neg A_{\overline{s}}$ then $\Gamma' \vdash^*_{NBx} \neg A_{\overline{s}}$; but $\Gamma' \nvdash^*_{NBx} \neg A_{\overline{s}}$; so $\Omega_{i-1} \nvdash^*_{NBx} \neg A_{\overline{s}}$; so since s is introduced in Γ', by L8.2, $\Gamma' \cup \{A_s\}$ is consistent; so there is a stage in the construction that sets $\Omega_{i^*} = \Omega_{i-1} \cup \{A_s\}$; so by construction, $A_s \in \Gamma'$; so $\Gamma' \vdash^*_{NBx} A_s$. This is impossible; reject the assumption: Γ' is s-maximal.

L8.5 If Γ_0 is consistent, then each Ω_i is consistent.

Suppose Γ_0 is consistent.

Basis: $\Omega_0 = \Gamma_0$ and Γ_0 is consistent; so Ω_0 is consistent.

Assp: For any $i, 0 \leqslant i < k$, Ω_i is consistent.

Show: Ω_k is consistent.

We know that Ω_k is either (i) Ω_{k-1}, (ii) $\Omega_{k^*} = \Omega_{k-1} \cup \{\mathcal{P}\}$, (iii) $\Omega_{k^*} \cup \{\overline{s}.y.z, P_y, \neg Q_{\overline{z}}\}$, (iv) $\Omega_{k^*} \cup \{P_{\overline{s}/y}, \neg Q_{\overline{y}}\}$, (v) $\Omega_{k^*} \cup \{s.v.y, t.y.w, t.v.z, s.z.w\}$, or (vi) $\Omega_{k^*} \cup \{s.t.y, y.t.u\}$.

(i) Suppose Ω_k is Ω_{k-1}. By assumption, Ω_{k-1} is consistent; so Ω_k is consistent.

(ii) Suppose Ω_k is $\Omega_{k^*} = \Omega_{k-1} \cup \{\mathcal{P}\}$. Then by construction, $\Omega_{k-1} \cup \{\mathcal{P}\}$ is consistent; so Ω_k is consistent.

(iii) Suppose Ω_k is $\Omega_{k^*} \cup \{\overline{s}.y.z, P_y, \neg Q_{\overline{z}}\}$. In this case, as above, Ω_{k*} is consistent and by construction, $\neg(P \to Q)_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are A_x and $\neg A_{\overline{x}}$ such that $\Omega_{k^*} \cup \{\overline{s}.y.z, P_y, \neg Q_{\overline{z}}\} \vdash^*_{NBx} A_x$ and $\Omega_{k^*} \cup \{\overline{s}.y.z, P_y, \neg Q_{\overline{z}}\} \vdash^*_{NBx} \neg A_{\overline{x}}$. So reason as follows,

1	Ω_{k^*}	
2	$\quad \overline{s}.y.z$	A (g, $\to$I)
3	$\quad P_y$	
4	$\quad\quad \neg Q_{\overline{z}}$	A (c, $\neg$E)
5	$\quad\quad A_x$	from $\Omega_{k^*} \cup \{\overline{s}.y.z, P_y, \neg Q_{\overline{z}}\}$
6	$\quad\quad \neg A_{\overline{x}}$	from $\Omega_{k^*} \cup \{\overline{s}.y.z, P_y, \neg Q_{\overline{z}}\}$
7	$\quad Q_z$	4-6 $\neg$E
8	$(P \to Q)_{\overline{s}}$	2-7 $\to$I

where, by construction, y and z are not introduced Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NBx} (P \to Q)_{\overline{s}}$; but $\neg(P \to Q)_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{NBx} \neg(P \to Q)_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

(iv) Suppose Ω_k is $\Omega_{k^*} \cup \{P_{\overline{s}/y}, \neg Q_{\overline{y}}\}$. In this case, as above, Ω_{k*} is consistent and by construction, $\neg(P > Q)_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are A_x and $\neg A_{\overline{x}}$ such that $\Omega_{k^*} \cup$

$\{P_{\overline{s}/y}, \neg Q_{\overline{y}}\} \vdash^*_{NBx} A_x$ and $\Omega_{k^*} \cup \{P_{\overline{s}/y}, \neg Q_{\overline{y}}\} \vdash^*_{NBx} \neg A_{\overline{x}}$. So reason as follows,

1	Ω_{k^*}	
2	$\quad P_{\overline{s}/y}$	A (g, >I)
3	$\quad\quad \neg Q_{\overline{y}}$	A (c, ¬E)
4	$\quad\quad A_x$	from $\Omega_{k^*} \cup \{P_{\overline{s}/y}, \neg Q_{\overline{y}}\}$
5	$\quad\quad \neg A_{\overline{x}}$	from $\Omega_{k^*} \cup \{P_{\overline{s}/y}, \neg Q_{\overline{y}}\}$
6	$\quad Q_y$	3-5 ¬E
8	$(P > Q)_{\overline{s}}$	2-6 >I

where, by construction, y is not introduced Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NBx} (P > Q)_{\overline{s}}$; but $\neg(P > Q)_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{NBx} \neg(P > Q)_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

(v) Suppose Ω_k is $\Omega_{k^*} \cup \{s.v.y, t.y.w, t.v.z, s.z.w\}$. In this case, as above, Ω_{k*} is consistent and by construction, $s.t.u, u.v.w \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are A_x and $\neg A_{\overline{x}}$ such that $\Omega_{k^*} \cup \{s.v.y, t.y.w, t.v.z, s.z.w\} \vdash^*_{NBx} A_x$ and in addition, $\Omega_{k^*} \cup \{s.v.y, t.y.w, t.v.z, s.z.w\} \vdash^*_{NBx} \neg A_{\overline{x}}$. So reason as follows,

1	Ω_{k^*}	
2	$s.t.u$	member of Ω_{k^*}
3	$u.v.w$	member of Ω_{k^*}
4	$\quad s.v.y$	A (g, AM9)
5	$\quad t.y.w$	
6	$\quad\quad t.v.z$	A (g, AM10)
7	$\quad\quad s.z.w$	
8	$\quad\quad\quad (A \to A)_0$	A (c, ¬I)
9	$\quad\quad\quad A_x$	from $\Omega_{k^*} \cup \{s.v.y, t.y.w, t.v.z, s.z.w\}$
10	$\quad\quad\quad \neg A_{\overline{x}}$	from $\Omega_{k^*} \cup \{s.v.y, t.y.w, t.v.z, s.z.w\}$
11	$\quad\quad \neg(A \to A)_{0\#}$	8-10 ¬I
12	$\quad \neg(A \to A)_{0\#}$	2,3,6-11 AM10
13	$\neg(A \to A)_{0\#}$	2,3,4-12 AM9

where, by construction, y and z are not introduced Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NBx} \neg(A \to A)_{0\#}$; but $\vdash^*_{NBx} (A \to A)_0$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

(vi) Similarly.

For any i, Ω_i is consistent.

L8.6 If Γ_0 is consistent, then Γ' is consistent.

Suppose Γ_0 is consistent, but Γ' is not; from the latter, there is some P_s such that $\Gamma' \vdash^*_{NBx} P_s$ and $\Gamma' \vdash^*_{NBx} \neg P_{\overline{s}}$. Consider derivations D1 and D2 of these results, and the premises $\mathcal{P}_i \ldots \mathcal{P}_j$ of these derivations. By construction, there is an Ω_k with each of these premises as a member; so D1 and D2 are derivations from Ω_k; so Ω_k is not consistent. But since Γ_0 is consistent, by L8.5, Ω_k is consistent. This is impossible; reject the assumption: if Γ_0 is consistent then Γ' is consistent.

L8.7 If Γ_0 is consistent, then Γ' is a scapegoat set for $\rightarrow$, $>$ and, in the appropriate systems, for C9/C10 and C12.

For $\rightarrow$. Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{NBx} \neg(P \rightarrow Q)_s$. By L8.6, Γ' is consistent; and by the constraints on subscripts, s is introduced in Γ'. Since $\Gamma' \vdash^*_{NBx} \neg(P \rightarrow Q)_s$, Γ' has just the same consequences as $\Gamma' \cup \{\neg(P \rightarrow Q)_s\}$; so $\Gamma' \cup \{\neg(P \rightarrow Q)_s\}$ is consistent, and for any Ω_j, $\Omega_j \cup \{\neg(P \rightarrow Q)_s\}$ is consistent. So there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{\neg(P \rightarrow Q)_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{\overline{s}.y.z, P_y, \neg Q_{\overline{z}}\}$; so by construction, $\overline{s}.y.z, P_y, \neg Q_{\overline{z}} \in \Gamma'$; so $\Gamma' \vdash^*_{NBx} \overline{s}.y.z$, $\Gamma' \vdash^*_{NBx} P_y$ and $\Gamma' \vdash^*_{NBx} \neg Q_{\overline{z}}$. So Γ' is a scapegoat set for $\rightarrow$.

For $>$. Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{NBx} \neg(P > Q)_s$. By L8.6, Γ' is consistent; and by the constraints on subscripts, s is introduced in Γ'. Since $\Gamma' \vdash^*_{NBx} \neg(P > Q)_s$, Γ' has just the same consequences as $\Gamma' \cup \{\neg(P > Q)_s\}$; so $\Gamma' \cup \{\neg(P > Q)_s\}$ is consistent, and for any Ω_j, $\Omega_j \cup \{\neg(P > Q)_s\}$ is consistent. So there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{\neg(P > Q)_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{P_{\overline{s}/y}, \neg Q_{\overline{y}}\}$; so by construction, $P_{\overline{s}/y}, \neg Q_{\overline{y}} \in \Gamma'$; so $\Gamma' \vdash^*_{NBx} P_{\overline{s}/y}$ and $\Gamma' \vdash^*_{NBx} \neg Q_{\overline{y}}$. So Γ' is a scapegoat set for $>$.

For C9/C10. Suppose Γ_0 is consistent, $\Gamma' \vdash^*_{NBx} s.t.u$ and $\Gamma' \vdash^*_{NBx} u.v.w$. By L8.6, Γ' is consistent; and by the constraints on subscripts, s, t, u, v and w are introduced in Γ'. Since $\Gamma' \vdash^*_{NBx} s.t.u$, and $\Gamma' \vdash^*_{NBx} u.v.w$, Γ' has just the same consequences as $\Gamma' \cup \{s.t.u, u.v.w\}$; so $\Gamma' \cup \{s.t.u, u.v.w\}$ is consistent, and for any Ω_j, $\Omega_j \cup \{s.t.u, u.v.w\}$ is consistent. So there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{s.t.u, u.v.w\}$ and $\Omega_i = \Omega_{i^*} \cup \{s.v.y, t.y.w, t.v.z, s.z.w\}$; so by construction, $s.v.y$, $t.y.w$, $t.v.z$, $s.z.w \in \Gamma'$; so there is a y such that $\Gamma' \vdash^*_{NBx} s.v.y$ and $\Gamma' \vdash^*_{NBx} t.y.w$, and there is a z such that $\Gamma' \vdash^*_{NBx} t.v.z$ and $\Gamma' \vdash^*_{NBx} s.z.w$. So Γ' is a scapegoat set for C9/C10. And similarly for C12.

C(I) We construct an interpretation $I_{Bx} = \langle W, N, R, *, v\rangle$ or $\langle W, N, R, \{R_A \mid A \in \mathfrak{I}\}, *, v\rangle$ based on Γ' as follows. Let W have a member w_s corresponding to each subscript s introduced in Γ', except that if $\Gamma' \vdash^*_{NBx} s \simeq t$ then $w_s = w_t$ and $w_{\overline{s}} = w_{\overline{t}}$ (we might do this, in the usual way, by beginning with equivalence classes on subscripts). Then set $N = \{w_0\}$; $\langle w_s, w_t, w_u\rangle \in R$ iff $\Gamma' \vdash^*_{NBx} s.t.u$; $\langle w_s, w_t\rangle \in R_A$ iff $\Gamma' \vdash^*_{NBx} A_{s/t}$; $* = \{\langle w_s, w_{\overline{s}}\rangle \mid s \text{ is introduced in } \Gamma'\}$; and $v_{w_s}(p) = 1$ iff $\Gamma' \vdash^*_{NBx} p_s$.

Note that the specification is consistent: Suppose $w_s = w_t$; then by construction, $\Gamma' \vdash^*_{NBx} s \simeq t$; so by $\simeq$E, $\Gamma' \vdash^*_{NBx} p_s$ iff $\Gamma' \vdash^*_{NBx} p_t$; so $v_{w_s}(p) =$

$v_{w_t}(p)$; and similarly in other cases. Also, the $*$-function has the right form, as $s, \overline{s}$ are introduced in pairs, and $\langle w_s, w_{s\#}\rangle \in *$ iff $\langle w_{s\#}, w_s\rangle \in *$.

L8.8 If Γ_0 is consistent then for I_{Bx} constructed as above, and for any s introduced in Γ', $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NBx} A_s$.

Suppose Γ_0 is consistent and s is introduced in Γ'. By L8.4, Γ' is s-maximal. By L8.6 and L8.7, Γ' is consistent and a scapegoat set for $\rightarrow$ and $>$. Now by induction on the number of operators in A_s,

Basis: If A_s has no operators, then it is a parameter p_s and by construction, $v_{w_s}(p) = 1$ iff $\Gamma' \vdash^*_{NBx} p_s$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NBx} A_s$.

Assp: For any i, $0 \leqslant i < k$, if A_s has i operators, then $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NBx} A_s$.

Show: If A_s has k operators, then $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NBx} A_s$.
If A_s has k operators, then it is of the form $\neg P_s$, $(P \wedge Q)_s$, $(P \vee Q)_s$, $(P \rightarrow Q)_s$, or $(P > Q)_s$ where P and Q have $< k$ operators.

($\neg$) A_s is $\neg P_s$. (i) Suppose $v_{w_s}(A) = 1$; then $v_{w_s}(\neg P) = 1$; so by TB($\neg$), $v_{w_s^*}(P) = 0$; so by construction, $v_{w_{\overline{s}}}(P) = 0$; so by assumption, $\Gamma' \nvdash^*_{NBx} P_{\overline{s}}$; so by s-maximality, $\Gamma' \vdash^*_{NBx} \neg P_s$, where this is to say, $\Gamma' \vdash^*_{NBx} A_s$. (ii) Suppose $\Gamma' \vdash^*_{NBx} A_s$; then $\Gamma' \vdash^*_{NBx} \neg P_s$; so by consistency, $\Gamma' \nvdash^*_{NBx} P_{\overline{s}}$; so by assumption, $v_{w_{\overline{s}}}(P) = 0$; so by construction, $v_{w_s^*}(P) = 0$; so by TB($\neg$), $v_{w_s}(\neg P) = 1$, where this is to say, $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NBx} A_s$.

($\wedge$)

($\vee$)

($\rightarrow$) A_s is $(P \rightarrow Q)_s$. (i) Suppose $v_{w_s}(A) = 1$ but $\Gamma' \nvdash^*_{NBx} A_s$; then $v_{w_s}(P \rightarrow Q) = 1$ but $\Gamma' \nvdash^*_{NBx} (P \rightarrow Q)_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NBx} \neg(P \rightarrow Q)_{\overline{s}}$; so, since Γ' is a scapegoat set for $\rightarrow$, there are some y and z such that $\Gamma' \vdash^*_{NBx} s.y.z$, $\Gamma' \vdash^*_{NBx} P_y$ and $\Gamma' \vdash^*_{NBx} \neg Q_{\overline{z}}$; from the latter, by consistency, $\Gamma' \nvdash^*_{NBx} Q_z$; so by assumption, $v_{w_y}(P) = 1$ and $v_{w_z}(Q) = 0$; but since $\Gamma' \vdash^*_{NBx} s.y.z$, by construction, $\langle w_s, w_y, w_z\rangle \in R$; so by TB($\rightarrow$), $v_{w_s}(P \rightarrow Q) = 0$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$ then $\Gamma' \vdash^*_{NBx} A_s$.
(ii) Suppose $\Gamma' \vdash^*_{NBx} A_s$ but $v_{w_s}(A) = 0$; then $\Gamma' \vdash^*_{NBx} (P \rightarrow Q)_s$ but $v_{w_s}(P \rightarrow Q) = 0$. From the latter, by TB($\rightarrow$), there are some $w_t, w_u \in W$ such that $\langle w_s, w_t, w_u\rangle \in R$ and $v_{w_t}(P) = 1$ but $v_{w_u}(Q) = 0$; so by assumption, $\Gamma' \vdash^*_{NBx} P_t$ and $\Gamma' \nvdash^*_{NBx} Q_u$; so by s-maximality, $\Gamma' \vdash^*_{NBx} \neg Q_{\overline{u}}$. Since $\langle w_s, w_t, w_u\rangle \in R$, by construction, $\Gamma' \vdash^*_{NBx} s.t.u$; so by reasoning as follows,

1	Γ'	
2	$(P \to Q)_s$	A (c, $\neg$I)
3	$s.t.u$	from Γ'
4	P_t	from Γ'
5	Q_u	3,2,4 $\to$E
6	$\neg Q_{\overline{u}}$	from Γ'
7	$\neg(P \to Q)_{\overline{s}}$	2-6 $\neg$I

$\Gamma' \vdash^*_{NBx} \neg(P \to Q)_{\overline{s}}$; so by consistency, $\Gamma' \nvdash^*_{NBx} (P \to Q)_s$. This is impossible; reject the assumption: if $\Gamma' \vdash_{NBx} A_s$ then $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NBx} A_s$.

(>) A_s is $(P > Q)_s$. (i) Suppose $v_{w_s}(A) = 1$ but $\Gamma' \nvdash^*_{NBx} A_s$; then $v_{w_s}(P > Q) = 1$ but $\Gamma' \nvdash^*_{NBx} (P > Q)_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NBx} \neg(P > Q)_{\overline{s}}$; so, since Γ' is a scapegoat set for $>$, there is some y such that $\Gamma' \vdash^*_{NBx} P_{s/y}$, and $\Gamma' \vdash^*_{NBx} \neg Q_{\overline{y}}$; from the first of these, by construction, $\langle w_s, w_y \rangle \in R_P$; and from the second, by consistency, $\Gamma' \nvdash^*_{NBx} Q_y$; so by assumption, $v_{w_y}(Q) = 0$; so by TB(>), $v_{w_s}(P > Q) = 0$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$ then $\Gamma' \vdash^*_{NBx} A_s$.

(ii) Suppose $\Gamma' \vdash^*_{NBx} A_s$ but $v_{w_s}(A) = 0$; then $\Gamma' \vdash^*_{NBx} (P > Q)_s$ but $v_{w_s}(P > Q) = 0$. From the latter, by TB(>), there is a w_t such that $\langle w_s, w_t \rangle \in R_P$, and $v_{w_t}(Q) = 0$; so by assumption, $\Gamma' \nvdash^*_{NBx} Q_t$; so by s-maximality, $\Gamma' \vdash^*_{NBx} \neg Q_{\overline{t}}$. Since $\langle w_s, w_t \rangle \in R_P$, by construction, $\Gamma' \vdash^*_{NBx} P_{s/t}$; so by reasoning as follows,

1	Γ'	
2	$(P > Q)_s$	A (c, $\neg$I)
3	$P_{s/t}$	from Γ'
4	Q_t	2,3 >E
5	$\neg Q_{\overline{t}}$	from Γ'
6	$\neg(P > Q)_{\overline{s}}$	2-5 $\neg$I

$\Gamma' \vdash^*_{NBx} \neg(P > Q)_{\overline{s}}$; so by consistency, $\Gamma' \nvdash^*_{NBx} (P > Q)_s$. This is impossible; reject the assumption: if $\Gamma' \vdash_{NBx} A_s$ then $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NBx} A_s$.

For any A_s, $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NBx} A_s$.

L8.9 If Γ_0 is consistent, then I_{B_x} constructed as above is a *Bx* interpretation.

In each case, we need to show that relevant constraints are met. Suppose Γ_0 is consistent. By L8.7 Γ' is a scapegoat set for C9/C10 and C12 in those systems.

(NC) Suppose $\langle w_0, w_s, w_t \rangle \in R$; then by construction, $\Gamma' \vdash^*_{NBx} 0.s.t$; so by 0E, $\Gamma' \vdash^*_{NBx} s \simeq t$; so by construction, $w_s = w_t$. Suppose $w_s = w_t$; then by construction, $\Gamma' \vdash^*_{NBx} s \simeq t$; so by 0I, $\Gamma' \vdash^*_{NBx} 0.s.t$;

so by construction, $\langle w_0, w_s, w_t\rangle \in R$. So $\langle w_0, w_s, w_t\rangle \in R$ iff $w_s = w_t$; and, since $N = \{w_0\}$, NC is satisfied.

(C8) If C8 is in *Bx*, then AM8 is in *NBx*. Suppose $\langle w_s, w_t, w_u\rangle \in R$; then by construction, $\Gamma' \vdash^*_{NBx} s.t.u$; so by AM8, $\Gamma' \vdash^*_{NBx} s.\overline{u}.\overline{t}$; so by construction, $\langle w_s, w_{\overline{u}}, w_{\overline{t}}\rangle \in R$; so by construction, $\langle w_s, w^*_u, w^*_t\rangle \in R$. So C8 is satisfied.

(C9/10) Suppose there is a w_u such that $\langle w_s, w_t, w_u\rangle \in R$ and $\langle w_u, w_v, w_w\rangle \in R$; then by construction, $\Gamma' \vdash^*_{NBx} s.t.u$ and $\Gamma' \vdash^*_{NBx} u.v.w$; so, since Γ' is a C9/C10 scapegoat set, there is a y such that $\Gamma' \vdash^*_{NBx} s.v.y$ and $\Gamma' \vdash^*_{NBx} t.y.w$, and there is a z such that $\Gamma' \vdash^*_{NBx} t.v.z$ and $\Gamma' \vdash^*_{NBx} s.z.w$; so by construction, $\langle w_s, w_v, w_y\rangle \in R$, $\langle w_t, w_y, w_w\rangle \in R$, $\langle w_t, w_v, w_z\rangle \in R$ and $\langle w_s, w_z, w_w\rangle \in R$. So C9 and C10 are satisfied.

(C12) Similarly.

(C13) If C13 is in *Bx*, then AM13 is in *NBx*. Suppose $\langle w_s, w_t, w_u\rangle \in R$ and $\langle w_u, w_v, w_w\rangle \in R$; then by construction, $\Gamma' \vdash^*_{NBx} s.t.u$ and $\Gamma' \vdash^*_{NBx} u.v.w$; so by AM13, $\Gamma' \vdash^*_{NBx} s.v.w$; so by construction, $\langle w_s, w_v, w_w\rangle \in R$. So C13 is satisfied.

($\preceq$) If ($\preceq$) is in *Bx*, then AM$\preceq$ is in *NBx*. (i) Suppose $\langle w_s, w_t, w_u\rangle \in R$ and $v_{w_s}(p) = 1$; then by construction, $\Gamma' \vdash^*_{NBx} s.t.u$ and $\Gamma' \vdash^*_{NBx} p_s$; so by AM$\preceq$, $\Gamma' \vdash^*_{NBx} p_u$; so by construction, $v_{w_u}(p) = 1$. (ii) Suppose $\langle w_s, w_t, w_u\rangle \in R$ and $v_{w^*_u}(p) = 1$; then by construction, $v_{w_{\overline{u}}}(p) = 1$ so by construction again, $\Gamma' \vdash^*_{NBx} s.t.u$ and $\Gamma' \vdash^*_{NBx} p_{\overline{u}}$; so by AM$\preceq$, $\Gamma' \vdash^*_{NBx} p_{\overline{s}}$; so by construction, $v_{w_{\overline{s}}}(p) = 1$; and by construction again, $v_{w^*_s}(p) = 1$. So C13 is satisfied.

(1) If condition (1) is in *Bx*, then AMP1 is in *NBx*. Suppose $w_t \in f_A(w_0)$; then $\langle w_0, w_t\rangle \in R_A$; so by construction, $\Gamma' \vdash^*_{NBx} A_{0/t}$; so by AMP1, $\Gamma' \vdash^*_{NBx} A_t$; so by L8.8, $v_{w_t}(A) = 1$; so $w_t \in [A]$. So $f_A(w_0) \subseteq [A]$ and (1) is satisfied.

(2) If condition (2) is in *Bx*, then AMP2 is in *NBx*. Suppose $w_0 \in [A]$; then $v_{w_0}(A) = 1$; so by L8.8, $\Gamma' \vdash^*_{NBx} A_0$; so by AMP2, $\Gamma' \vdash^*_{NBx} A_{0/0}$; so by construction, $\langle w_0, w_0\rangle \in R_A$; so $w_0 \in f_A(w_0)$ and (2) is satisfied.

MAP For any $w_s \in W$, set $m(s) = w_s$; otherwise $m(s)$ is arbitrary.

L8.10 If Γ_0 is consistent, then $v_m(\Gamma_0) = 1$.

Reasoning parallel to that for L2.10 of *NK*α.

Main result: Suppose $\Gamma \models_{Bx} A$ but $\Gamma \nvdash_{NBx} A$. Then $\Gamma_0 \models^*_{Bx} A_0$ but $\Gamma_0 \nvdash^*_{NBx} A_0$. By (DN), if $\Gamma_0 \vdash^*_{NBx} \neg\neg A_0$, then $\Gamma_0 \vdash^*_{NBx} A_0$; so $\Gamma_0 \nvdash^*_{NBx} \neg\neg A_0$; so by L8.2, $\Gamma_0 \cup \{\neg A_{\overline{0}}\}$ is consistent; so by L8.9 and L8.10, there is a *Bx* interpretation with v and

m constructed as above such that $v_m(\Gamma_0 \cup \{\neg A_{\overline{0}}\}) = 1$; so $v_{m(\overline{0})}(\neg A) = 1$; so by construction, $v_{m_0^*}(\neg A) = 1$; so by TB($\neg$), $v_{m(0)}(A) = 0$; so $v_m(\Gamma_0) = 1$ and $v_{m(0)}(A) = 0$; so by VBx*, $\Gamma_0 \not\models^*_{Bx} A_0$. This is impossible; reject the assumption: if $\Gamma \models_{Bx} A$, then $\Gamma \vdash_{NBx} A$.

9 QUANTIFIED MODAL LOGICS: *Fnα*

Quantified modal logic raises many issues in the metaphysics of possible worlds and modality. As graphically exhibited by the nineteen (!) branches of a tree diagram on the second page of Garson's excellent survey [3], there are many issues and options for formal logic as well. This last section is a bare introduction to the topic. I exhibit a couple of concerns associated with "variable domains," and consider some ways free logic might be adapted in response. Access is constrained as for normal modal logics from before.

When one moves from ordinary sentential logic to quantified logic, one moves from a simple interpretation which assigns a truth value to each parameter, to interpretations which include a ***universe*** of objects, with ***assignments*** to constants and relation symbols. It is natural to think we could do something similar in the transition from sentential to quantified modal logic. Thus, for example, we might say an interpretation is $\langle W, U, D, R, v\rangle$ where W is a set of worlds, U a set of objects, D a function from W to subsets of U, R a subset of W^2, and v a function which assigns a member of U to each constant symbol, and a subset of U^n to each n-place relation symbol at each world. Then, intuitively, for $w \in W$, $D(w)$ says which things exist in world w. And v says which things are assigned to constants and to relation symbols at worlds. Thus, we might have $v(b) = \text{Bill}$, $v_w(H^1) = \{\text{Bill, Hill}\}$ and $v_x(H^1) = \{\text{Hill, Jill}\}$; so that Hb turns out true at w but false at x – and, depending on access, we could proceed in the usual way to say that $\Diamond Hb$ at some world, or whatever.

Variable Domains. Here is a first concern: It is natural to think that Bill does not exist at every world – that D varies from one world to the next. And it is natural to think that 'everybody is happy' should come out true at w just when all the people ***at*** w are happy, and 'somebody is happy' should come out true just when someone at w is happy. For this, for evaluation at w, quantifiers need to be restricted to the members of $D(w)$. So far, so good. But consider the following argument, proceeding by standard quantifier rules (with subscripts applied in the usual way).

1	0.1	A (g, □I)
2	$(b = b)_1$	=I
3	$\exists x(x = b)_1$	2 ∃I
4	$\Box\exists x(x = b)_0$	1-3 □I

It is thus (apparently) a theorem that Bill exists at every accessible world – so that Bill turns out to be a necessary being. Theological concerns to the side, something seems to have gone awry: for we began with precisely the assumption that Bill does *not* exist at every world.

Though its original motivation is not from possible worlds, quantified *free* logic is designed to accommodate interpretations with a universe U of objects greater than the domain D over which quantifiers range. The idea seems to have been that there are objects which do not exist (Pegasus, or the like). Whatever sense is to be made of this, from our assumptions, there would seem to be a straightforward application to the modal context, where Bill is a member of some, but not every D. To accommodate this sort of thing, relative to the classical case, free logic imposes constraints on the quantifier rules. We may thus introduce a predicate E for existence, with quantifier rules as follows,

∀I
| | Ea
| |—
| |
| | $P^{x}/_{a}$
| $\forall xP$

where a does not appear in any undischarged premise or assumption or in P

∀E
| $\forall xP$
| Ea
|
| $P^{x}/_{a}$

∃I
| $P^{x}/_{a}$
| Ea
|
| $\exists xP$

∃E
| $\exists xP$
| | $P^{x}/_{a}$
| | Ea
| |—
| |
| | Q
| Q

where a does not appear in any undischarged premise or assumption, in P or in Q

Then ∃I in our problematic derivation is blocked, insofar as Eb is not available. Of course, we might reason along the following lines,

1	0.1	A (g, □I)
2	Eb_1	A (g, ⊃I)
3	$(b = b)_1$	=I
4	$\exists x(x = b)_1$	2,3 ∃I
5	$[Eb \supset \exists x(x = b)]_1$	2-4 ⊃I
6	$\Box[Eb \supset \exists x(x = b)]_0$	1-5 □I

for the result that, necessarily, if Bill exists, then something is identical to Bill. But this seems right. So, subject to details, we seem to have the makings of a reasonable way out. (Notice that we have already seen a version of free logic for □ and ◇ as quantifiers over subscripts. Thus, where we see s.t as a sort of existence claim, these rules for ∀ and ∃ appear as parallel to ones we have seen for □ and ◇. In the modal case, we require the existence constraint insofar as the domain of worlds to which a given w has access may turn out to be empty.)

De Re / De Dicto. Much philosophical debate surrounds *de re* as opposed to *de dicto* modality. Formally an expression is *de re* (*of the thing*) iff a subformula of it has a constant or free variable in the scope of a modal operator. And expression is *de dicto* (*of the saying*) iff it is not *de re*. Thus, for example, □Hb and ∃x□Hx are *de re*; □∃xHx is *de dicto*. (Discussion of *quantifying in* by Quine and others is of *de re* modality, insofar as the quantifier reaches *across* the modal operator.) Evaluation of □∃xHx seems straightforward enough: □∃xHx should be true just in case there is no accessible $w \in W$ such that $v_w(H)$ is empty. But consider □Hb; this will be true just in case *Bill* is in the extension of H at every accessible world. Similarly, $\exists x\Box Hx$ will be true at w just in case there is something in w such that ***it*** is in the extension of H at every accessible world. Perhaps it is intuitive that a given thing could have been different ways, and so appears in different worlds. Even so, there are serious philosophical questions about how a thing has its modal features, and so appears in the worlds it does.[8]

Suppose we allow that a thing may appear in different worlds. As soon as we set things up this way, another problem emerges for variable domains. We have said that $\Box Hb$ will be true just in case Bill is in the extension of H at every accessible world. But Bill does not exist at every accessible world – and how are we to evaluate Hb at worlds where Bill does not exist (and similarly for $\exists xHx$)? Here are three responses: (i) A standard response, perhaps because it is technically straightforward, is to say that non-existence at a world need not prevent a thing's being in the extension of a predicate there – to let $v_w(H)$ be any subset of U. Another reaction is to deny that an object can be in the extension of (ordinary) predicates at a world where it does not exist – this is to restrict $v_w(H)$ to subsets of $D(w)$. Then (iia) we might let Hb be ***false*** at worlds where Bill does not exist. Alternatively, (iib), along lines from previous sections, we might say Hb is neither true nor false at worlds where Bill does not exist. Options (iia) and (iib) seem compatible with "serious actualism" as defended by Alvin Plantinga, though (iia) is like the one he explicitly endorses.[9] In the following, I develop a version of free quantified modal logic compatible with any of these options – and fairly fine-grained combinations of them as well. If options are limited to just (i), or to just (i) and (iia), the logic remains classical, and obvious simplifications are possible. It is left as an exercise to work out the details of such simplifications.

9.1 LANGUAGE / SEMANTIC NOTIONS

LFNα The VOCABULARY consists of variables $x_1, x_2 \ldots$; constants $c_1, c_2 \ldots$; operators $\neg$, $\wedge$, $\forall$, $\Box$; and relation symbols, E, $=$, R^1_1, $R^1_2 \ldots$, R^2_1, $R^2_2 \ldots$, etc. The number of "places" in a relation symbol is indicated by superscript, where '=' is always two-place, and 'E' one-place. Any variable or constant is a TERM. If $Q^n t_1 \ldots t_n$ is a n-place relation symbol followed by n terms, it is a FORMULA. If x is a variable and A and B are formulas, then $\neg A$, $(A \wedge B)$, $\forall xA$ and $\Box A$ are formulas. Variables are bound and free in the usual way. A is a SENTENCE iff it is a formula with no free variables. We allow overlines as before, and the usual abbreviations, including $\vee$, $\supset$, $\equiv$, $\exists$, and $\Diamond$.

IFNα An INTERPRETATION is $\langle W, U, D, R, P, v\rangle$ where W is a set of worlds, U a set of objects, D a function from W to subsets of U, R a subset of W^2, and v a function such that for any constant c, $v(c) \in U$ and for

[8] The literature is immense. Quine's [7] is a classic. For a contribution of my own, see [10].
[9] See, e.g., [5]. But compare his [6, n.3]

any n-place relation symbol Q^n and $w \in W$, $v_w(Q^n) \subseteq U^n$. P is a function that maps each n-place relation symbol Q^n to some member of $\{0,1,2\}^n$; so, for example, P might map some Q^4 to $\langle 1,0,0,2\rangle$. Where $P(Q^n) = \langle a_1 \dots a_n\rangle$, say $P(Q^n)_i = a_i$. Then we require as an *existence* (presupposition) requirement on v that,

EP If $\langle u_1 \dots u_n\rangle \in v_w(Q^n)$, then for any i, $1 \leqslant i \leqslant n$, if $P(Q^n)_i \geqslant 1$, then $u_i \in D(w)$

Additionally, require that $P(E) = \langle 0\rangle$ and $v_w(E) = D(w)$; if $P(=) = \langle 0,0\rangle$, then $v_w(=) = \{\langle u,u\rangle | u \in U\}$, and otherwise $v_w(=) = \{\langle u,u\rangle | u \in D(w)\}$. In addition, where α is empty or some combination of the following,

η	For any x, there is a y such that xRy	extendability
ρ	for all x, xRx	reflexivity
σ	for all x, y, if xRy then yRx	symmetry
τ	for all x, y, z, if xRy and yRz then xRz	transitivity

and, as sample versions of the presupposition constraint, where n is one of the following,

(0) For any relation symbol Q^n and any i, $1 \leqslant i \leqslant n$, $P(Q^n)_i = 0$

(1) For any relation symbol Q^n other than E and any i, $1 \leqslant i \leqslant n$, $P(Q^n)_i = 1$

(2) For any relation symbol Q^n other than E and any i, $1 \leqslant i \leqslant n$, $P(Q^n)_i = 2$

$\langle W, U, D, R, P, v\rangle$ is an $Fn\alpha$ interpretation when R meets the constraints from α, and P meets the constraint from n. Obviously, many other options are available for the constraints α and n.

Given an interpretation with its P and v, say $\langle u_1 \dots u_n\rangle \in \bar{v}_w(Q^n)$ just in case either $\langle u_1 \dots u_n\rangle \in v_w(Q^n)$ or for some i, $1 \leqslant i \leqslant n$, $P(Q^n)_i = 2$ and $u_i \notin D(w)$. When $n = 0$ or $n = 1$, $v_w(Q^n)$ is the same as $\bar{v}_w(Q^n)$. But when $n = 2$, for relation symbols other than E, $\bar{v}_w(Q^n)$ includes also any n-tuple with a member not in $D(w)$.

A variable designation assignment δ assigns each variable a member of U; $\delta[x|u]$ is like δ except that x is assigned to u; corresponding to a variable assignment δ ($\delta[x|u]$) the term assignment Δ ($\Delta[x|u]$) is like δ ($\delta[x|u]$) for variables, and v for constants. As before, define a function h based on v, writing $h_w(/A/)/\!/\delta = 1$ to indicate that $/A/$ is SATISFIED at w on h with variable assignment δ.

HF For assignments to formulas,

(R) $h_w(/Q^n t_1 \dots t_t/)/\!/\delta = 1$ if $\langle \Delta(t_1) \dots \Delta(t_2)\rangle \in /v/_w(Q^n)$, and 0 otherwise.

(¬) $h_w(/\neg A/)/\!/\delta = 1$ if $h_w(\backslash A\backslash)/\!/\delta = 0$, and 0 otherwise.

(∧) $h_w(/A \wedge B/)/\!/\delta = 1$ if $h_w(/A/)/\!/\delta = 1$ and $h_w(/B/)/\!/\delta = 1$, and 0 otherwise.

(∀) $h_w(/\forall xA/)/\!/\delta = 1$ if for any $u \in D(w)$, $h_w(/A/)/\!/\delta[x|u] = 1$, and 0 otherwise.

(□) $h_w(/\Box A/)/\!/\delta = 1$ if all $x \in W$ such that wRx have $h_x(/A/)/\!/\delta = 1$, and 0 otherwise.

$h_w(/A/) = 1$ ($/A/$ ***holds*** on h at w) iff for any δ, $h_w(/A/)/\!/\delta = 1$. And $h_w(\Gamma) = 1$ iff for each $/A/ \in \Gamma$, $h_w(/A/) = 1$. Then, where the members of Γ and A are sentences,

VFNα $\Gamma \models_{Fn\alpha} /A/$ iff there is no *Fnα* interpretation $\langle W, U, D, R, P, v\rangle$ and $w \in W$ such that $h_w(\Gamma) = 1$ and $h_w(/A/) = 0$.

Set relation symbol E to the side: Then for *F0α* and *F1α*, we have $h_w(/Q^n t_1 \ldots t_n/)/\!/\delta = 1$ iff $\langle \Delta(t_1) \ldots \Delta(t_n)\rangle \in v_w(Q^n)$ – where, for *F1α* each of $\Delta(t_1) \ldots \Delta(t_n)$ is required to be in $D(w)$. For *F2α*, $h_w(Q^n t_1 \ldots t_n)/\!/\delta = 1$ if and only if $\langle \Delta(t_1) \ldots \Delta(t_n)\rangle \in v_w(Q^n)$ – where again, each of $\Delta(t_1) \ldots \Delta(t_n)$ is required to be in $D(w)$; but $h_w(\overline{Q^n t_1 \ldots t_n})/\!/\delta = 0$ iff $\langle \Delta(t_1) \ldots \Delta(t_n)\rangle \notin v_w(Q^n)$ and each of $\Delta(t_1) \ldots \Delta(t_n)$ is in $D(w)$. So P works as an existence presupposition function for (each place of) each relation symbol: on the *F0α* option, there are no existence presuppositions; on the *F1α* option, there is an existence requirement for truth but not falsity; with *F2α*, there is an existence requirement for both truth and falsity. Insofar as truth and non-falsity are matched for the *F0α* and *F1α* options, the logic is essentially classical. However, for *F2α*, since expressions may be neither true nor false (but never both), the logic is like *MK3* from section 6.

As an example of reasoning with these definitions, here is an argument to show, $\exists y\Diamond\neg Ey \models_{F1\alpha} \neg\forall x\Box Hx$. We suppose derived clauses to HF are spelled out in the usual way.

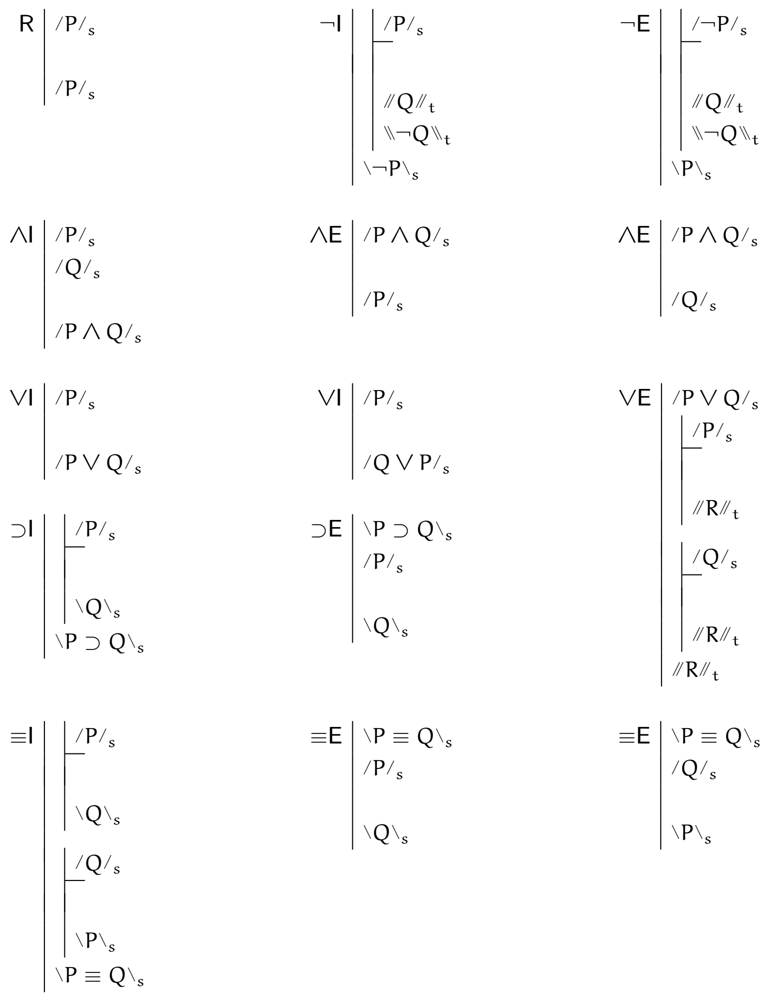
> Suppose $\exists y\Diamond\neg Ey \not\models_{F1\alpha} \neg\forall x\Box Hx$; then by VFNα there is an *F1α* interpretation $\langle W, U, D, R, P, v\rangle$ and $w \in W$ such that $h_w(\exists y\Diamond\neg Ey) = 1$ but $h_w(\neg\forall x\Box Hx) = 0$; from the latter, there is a δ where $h_w(\neg\forall x\Box Hx)/\!/\delta = 0$; then with the former, $h_w(\exists y\Diamond\neg Ey)/\!/\delta = 1$. From this, by HF(∃), there is some $u \in D(w)$ such that $h_w(\Diamond\neg Ey)/\!/\delta[y|u] = 1$; so by HF(◇), there is some $a \in W$ such that wRa and $h_a(\neg Ey)/\!/\delta[y|u] = 1$; so by HF(¬), $h_a(\overline{Ey})/\!/\delta[y|u] = 0$; so by HF(R), $\Delta[y|u](y) \notin \overline{v}_a(E)$; so by the construction of $\overline{v}$, $\Delta[y|u](y) \notin v_a(E)$; so $\Delta[y|u](y) \notin D(a)$; but $\Delta[y|u](y) = \delta[y|u](y) = u$; so $u \notin D(a)$. Since $h_w(\neg\forall x\Box Hx)/\!/\delta = 0$, by HF(¬), $h_w(\overline{\forall x\Box Hx})/\!/\delta = 1$; so by HF(∀), for any $v \in D(w)$, $h_w(\overline{\Box Hx})/\!/\delta[x|v] = 1$; so $h_w(\overline{\Box Hx})/\!/\delta[x|u] = 1$; so by HF(□), for any $b \in W$ such that wRb, $h_b(\overline{Hx})/\!/\delta[x|u] = 1$; so $h_a(\overline{Hx})/\!/\delta[x|u] = 1$; so by

> HF(R), $\vartriangle[x|u](x) \in \overline{\nu}_a(H)$; so, since this is *F1*$\alpha$, $\vartriangle[x|u](x) \in \nu_a(H)$; so by EP, $\vartriangle[x|u](x) \in D(a)$; but $\vartriangle[x|u](x) = \delta[x|u](x) = u$; so $u \in D(a)$. This is impossible; reject the assumption: $\exists y\Diamond\neg Ey \models_{F1\alpha} \neg\forall x\Box Hx$.

The argument does not go through in *F0*α insofar as we cannot move by EP from $\vartriangle[x|u](x) \in \nu_a(H)$ to $\vartriangle[x|u](x) \in D(a)$. It does not go through in *F2*α because we cannot move from $\vartriangle[x|u](x) \in \overline{\nu}_a(H)$ to $\vartriangle[x|u](x) \in \nu_a(H)$. However, as one can show by parallel reasoning, $\exists y\Diamond\neg Ey \models_{F2\alpha} \overline{\neg\forall x\Box Hx}$.

9.2 NATURAL DERIVATIONS: *NFn*α

Allow expressions with integer subscripts and overlines and, as before, expressions of the sort, s.t. Begin with a natural combination of rules from free logic with ones we have seen before, where rules for $\vee$, $\supset$, $\equiv$, $\exists$ and $\Diamond$ are derived.



$$\forall I \;\; \begin{array}{|l} \begin{array}{|l} Ea_s \\ \hline \\ //P^x/_a//_s \end{array} \\ //\forall xP//_s \end{array} \qquad \forall E \;\; \begin{array}{|l} //\forall xP//_s \\ Ea_s \\ \\ //P^x/_a//_s \end{array} \qquad \exists I \;\; \begin{array}{|l} //P^x/_a//_s \\ Ea_s \\ \\ //\exists xP//_s \end{array} \qquad \exists E \;\; \begin{array}{|l} //\exists xP//_s \\ \begin{array}{|l} //P^x/_a//_s \\ Ea_s \\ \hline \\ /Q/_u \end{array} \\ /Q/_u \end{array}$$

∀I: where a does not appear in any undischarged premise or assumption or in P

∃E: where a does not appear in any undischarged premise or assumption, in P or in Q

$$\Box I \;\; \begin{array}{|l} \begin{array}{|l} s.t \\ \hline \\ /P/_t \end{array} \\ /\Box P/_s \end{array} \qquad \Box E \;\; \begin{array}{|l} /\Box P/_s \\ s.t \\ \\ /P/_t \end{array} \qquad \Diamond I \;\; \begin{array}{|l} /P/_t \\ s.t \\ \\ /\Diamond P/_s \end{array} \qquad \Diamond E \;\; \begin{array}{|l} /\Diamond P/_s \\ \begin{array}{|l} s.t \\ /P/_t \\ \hline \\ //Q//_u \end{array} \\ //Q//_u \end{array}$$

□I: where t does not appear in any undischarged premise or assumption

◇E: where t does not appear in any undischarged premise or assumption and is not u

$$AM\eta \;\; \begin{array}{|l} \begin{array}{|l} s.t \\ \hline \\ /P/_u \end{array} \\ /P/_u \end{array} \qquad AM\rho \;\; \begin{array}{|l} \\ \\ s.s \end{array} \qquad AM\sigma \;\; \begin{array}{|l} s.t \\ \\ t.s \end{array} \qquad AM\tau \;\; \begin{array}{|l} s.t \\ t.u \\ s.u \end{array}$$

AMη: where t does not appear in any undischarged premise or assumption and is not u

Every subscript is 0, appears in a premise, or in the t place of an assumption for □I, ◇E or AMη. Now, for relation symbol Q^n, let $P1[Q^n t_1 \ldots t_n]_s$ be the conjunction $(\top \wedge Et_a \wedge \cdots \wedge Et_b)_s$ for each t_i such that $P(Q^n)_i \geqslant 1$. And let $P2[Q^n t_1 \ldots t_n]_s$ be the conjunction $(\top \wedge Et_a \wedge \cdots \wedge Et_b)_s$ for each t_i such that $P(Q^n)_i = 2$. Note that $\top_s$ can be asserted at any stage in a derivation. Then allow,

$$D \;\; \begin{array}{|l} P_s \\ \\ \overline{P}_s \end{array} \qquad =I \;\; \begin{array}{|l} P1[a=a]_s \\ \\ (a=a)_s \end{array} \qquad =E \;\; \begin{array}{|l} (a=b)_s \\ //Q//_t \\ \\ //Q^a/_b//_t \end{array} \;\; \begin{array}{|l} (b=a)_s \\ //Q//_t \\ \\ //Q^a/_b//_t \end{array}$$

=E: a single instance of a replaced by b

$$P1I \;\; \begin{array}{|l} Q_s \\ \\ P1[Q]_s \end{array} \qquad P2I \;\; \begin{array}{|l} \neg Q_s \\ \\ P2[Q]_s \end{array} \qquad P2E \;\; \begin{array}{|l} P2[Q]_s \\ \overline{Q}_s \\ \\ Q_s \end{array}$$

Where for =E, P1I, P2I and P2E, Q is an atomic, $R^n a_1 \ldots a_n$.

Notice that P2E is a constrained version of (U) from section 6. Informally, where $P1[Q^n t_1 \ldots t_n]_s$ or $P2[Q^n t_1 \ldots t_n]_s$ is other than just $\top$, let us drop $\top$ for the equivalent conjunction. Then, in any case, $P1(E) = P2(E) = \top$. Otherwise, in *NF0α*, $P1[Q^n t_1 \ldots t_n] = P2[Q^n t_1 \ldots t_n] = \top$. In *NF1α*, $P1[Q^n t_1 \ldots t_n] =$

$Et_1 \wedge \cdots \wedge Et_n$ and $P2[Q^n t_1 \ldots t_n] = \top$. In *NF2α*, we have $P1[Q^n t_1 \ldots t_n] = P2[Q^n t_1 \ldots t_n] = Et_1 \wedge \cdots \wedge Et_n$. When $P1[Q^n t_1 \ldots t_n] = \top$, the premise for =I goes trivial, as does the conclusion from P1I. Similarly, when $P2[Q^n t_1 \ldots t_n] = \top$, the conclusion of P2I is trivial, and P2E works like (U) for the relevant atomic. Where the members of Γ and $/A/$ are sentences without subscripts, let Γ_0 be the members of Γ, each with subscript 0. Then,

NFNα $\Gamma \vdash_{NFn\alpha} /A/$ iff there is an NFnα derivation of $/A/_0$ from the members of Γ_0.

Notice that our notions of validity are defined for sentences. We get the different derivation systems insofar as AM rules may differ, and insofar as $P1[Q]$ and $P2[Q]$ are different expressions. On occasion, arguments will go through no matter what presupposition constraints are in play. In this case, to show $\Gamma \vdash_{NFn\alpha} /A/$, apply the rules so that they would apply no matter what the constraints are. Thus apply =I and P2E as though $P(Q^n)_i$ is always 2, and P1I and P2I as though it is 0 (so the latter two rules effectively drop out).

As above, rules for $\vee$, $\supset$, $\equiv$, $\exists$ and $\Diamond$ are derived. As examples, here are derivations for $\exists$I and $\exists$E.

$\exists$I

1	$/\!/P^{x}/_{a}/\!/_{s}$	P
2	Ea_s	P
3	$\backslash\!\backslash\forall x\neg P\backslash\!\backslash_{s}$	A (*c*, ¬I)
4	$\backslash\!\backslash\neg P^{x}/_{a}\backslash\!\backslash_{s}$	2,3 ∀E
5	$/\!/P^{x}/_{a}/\!/_{s}$	1 R
6	$/\!/\neg\forall x\neg P/\!/_{s}$	3-5 ¬I
7	$/\!/\exists xP/\!/_{s}$	6 abv

$\exists$E

1	$/\!/\exists xP/\!/_{s}$	P
2	$/\!/\neg\forall x\neg P/\!/_{s}$	1 abv
3	$\backslash\neg Q\backslash_{u}$	A (*c*, ¬E)
4	Ea_s	A (*g*, ∀I)
5	$/\!/P^{x}/_{a}/\!/_{s}$	A (*c*, ¬I)
	⋮	with 1,4,5
6	$/Q/_{u}$	as for ∃E
7	$\backslash\neg Q\backslash_{u}$	3 R
8	$\backslash\!\backslash\neg P^{x}/_{a}\backslash\!\backslash_{s}$	5-7 ¬I
9	$\backslash\!\backslash\forall x\neg P\backslash\!\backslash_{s}$	4-8 ∀I
10	$/\!/\neg\forall x\neg P/\!/_{s}$	2 R
11	$/Q/_{u}$	3-10 ¬E

In addition, we allow standard two-way rules (including MN) with overlines and subscripts constant throughout. Include among two-way rules,

QN $/\forall xP/_s \triangleleft\triangleright /\neg\exists x\neg P/_s$ $\quad$ $/\neg\forall xP/_s \triangleleft\triangleright /\exists x\neg P/_s$
$/\exists xP/_s \triangleleft\triangleright /\neg\forall x\neg P/_s$ $\quad$ $/\neg\exists xP/_s \triangleleft\triangleright /\forall x\neg P/_s$

Allow MT, NB and DS in the forms,

MT: $/P \supset Q/_s$, $\backslash\neg Q\backslash_s$ ⟹ $/\neg P/_s$

NB: $/P \equiv Q/_s$, $\backslash\neg P\backslash_s$ ⟹ $/\neg Q/_s$ $\qquad$ $/P \equiv Q/_s$, $\backslash\neg Q\backslash_s$ ⟹ $/\neg P/_s$

DS: $/P \vee Q/_s$, $\backslash\neg P\backslash_s$ ⟹ $/Q/_s$ $\qquad$ $/P \vee Q/_s$, $\backslash\neg Q\backslash_s$ ⟹ $/P/_s$

As further examples, here are some simple results emphasizing behavior for existence and identity.

$a = b \vdash_{NF0\alpha} \Box(a = b)$

1	$(a = b)_0$	P
2	0.1	A (g, $\Box$I)
3	$\top_1$	taut
4	$(a = a)_1$	3 =I
5	$(a = b)_1$	1,3 =E
6	$\Box(a = b)_0$	2-5 $\Box$I

In *NF0*α, the premise for =I is trivial. Notice that this derivation does not go through for *NF1*α and *NF2*α, where =I requires a substantive premise. As above, we may, however show the following (and similarly for *NF2*α),

$a = b \vdash_{NF1\alpha} \Box[Ea \supset (a = b)]$

1	$(a = b)_0$	P
2	0.1	A (g, $\Box$I)
3	$\overline{Ea}_1$	A (g, $\supset$I)
4	$\top_1$	taut
5	Ea_1	3,4 P2E
6	$(Ea \wedge Ea)_1$	5,5 $\wedge$I
7	$(a = a)_1$	6 =I
8	$(a = b)_1$	1,7 =E
9	$[Ea \supset (a = b)]_1$	3-8 $\supset$I
10	$\Box[Ea \supset (a = b)]_0$	2-9 $\Box$I

The premise for P2E is trivial, as always for relation symbol E. But the premise for =I is not. In this case, the terms are the same so, following the rule (but dropping $\top$), the required premise is obtained at (6). In these systems, then, if a is equal to b, a is essentially equal to b.

Here is a case considered semantically above,

$\exists y\Diamond\neg Ey \vdash_{NF1\alpha} \neg\forall x\Box Hx$

1	$\exists y\Diamond\neg Ex_0$	P
2	$\Diamond\neg Ea_0$	A (g, 1 ∃E)
3	Ea_0	
4	0.1	A (g, 2 ◇E)
5	$\neg Ea_1$	
6	$\overline{\forall x\Box Hx_0}$	A (g, ¬I)
7	$\overline{\Box Ha_0}$	6,3 ∀E
8	$\overline{Ha_1}$	4,7 □E
9	$\top_1$	taut
10	Ha_1	8,9 P2E
11	Ea_1	10 P1I
12	$\overline{\neg Ea_1}$	5 D
13	$\neg\forall x\Box Hx_0$	6-12 ¬I
14	$\neg\forall x\Box Hx_0$	2,4-13 ◇E
15	$\neg\forall x\Box Hx_0$	1,2-14 ∃E

Notice that, in *NF2*α we would not have (10) since we do not have Ea_1. And in *NF0*α, (10) would get us just $\top_1$ instead of Ea_1 (again with $\top$ dropped) and we would not have the contradiction. As in the semantic case, though, we can show,

$\exists y\Diamond\neg Ey \vdash_{NF2\alpha} \overline{\neg\forall x\Box Hx}$

1	$\exists y\Diamond\neg Ex_0$	P
2	$\Diamond\neg Ea_0$	A (g, 1 ∃E)
3	Ea_0	
4	0.1	A (g, 2 ◇E)
5	$\neg Ea_1$	
6	$\forall x\Box Hx_0$	A (g, ¬I)
7	$\Box Ha_0$	6,3 ∀E
8	Ha_1	4,7 □E
9	Ea_1	8 P1I
10	$\overline{\neg Ea_1}$	5 D
11	$\overline{\neg\forall x\Box Hx_0}$	6-10 ¬I
12	$\overline{\neg\forall x\Box Hx_0}$	2,4-11 ◇E
13	$\overline{\neg\forall x\Box Hx_0}$	1,2-12 ∃E

When there is a world where some a does not exist, on *F1*α, we can be sure that atomics go *false* for the thing at that world, so that the □ goes false as well. On *F2*α, we can be sure that atomics are *not true* for the thing at that world, so that □ is not true either. On *F0*α, there are no immediate consequences, insofar as atomics might go either way for the thing at that world.

Insofar as there is no parallel discussion for quantified modal logic in Priest's text, the following are offered as exercises which the student may find useful.

1. Produce an interpretation to show that, for any n, $\nvDash_{\mathrm{F}n\rho\sigma\tau} \forall x\Box Qx \supset \Box\forall x Qx$.

2. Produce an interpretation to show that, for any n, $\not\models_{Fn\rho\sigma\tau} \Box\forall xQx \supset \forall x\Box Qx$.

3. The formulas in (1) and (2) are instances of the *Barcan formula* and *converse Barcan formula* respectively (after Ruth Barcan Marcus). They play an important role in discussions of quantified modal logic, especially related to issues with which we began. Show that the formulas are valid in a system Fnc which is like *Fn*$\rho\sigma\tau$ except that it includes the constraint (c) that for any $a, b \in W$, $D(a) = D(b)$.

4. Give derivations to show each of the following.

   a. $\Box\forall xAx \vdash_{NFn\alpha} \forall x\Box(Ex \supset Ax)$
   b. $\exists x\forall y\Box Axy \vdash_{NF1\alpha} \Box\exists x\exists yAxy$
   c. $\vdash_{NFn\rho} \forall x\Box(Ax \wedge Bx) \supset \forall yAy$
   d. $\vdash_{NFn\alpha} (\Diamond\forall xAx \wedge \Box\exists xBx) \supset \Diamond\exists x(Ax \wedge Bx)$
   e. $\vdash_{NFn\alpha} \Box\forall x(Ax \vee Bx) \supset (\Box\forall xAx \vee \Diamond\exists xBx)$
   f. $\vdash_{NF1\alpha} (\Diamond\exists x\Box Ax \wedge \Box\forall x\Diamond Bx) \supset \Diamond\exists x\Diamond(Ax \wedge Bx)$
   g. $\forall x\neg\Diamond Ax \supset \Box\forall x\neg Ax \vdash_{NFn\alpha} \Diamond\exists xAx \supset \exists x\Diamond Ax$
   h. $\Diamond\exists xAx \supset \exists x\Diamond Ax \vdash_{NFn\alpha} \forall x\neg\Diamond Ax \supset \neg\Diamond\exists xAx$
   i. $\exists x\Diamond\neg Ax \vdash_{NF2\alpha} \Diamond\exists x\neg Ax$
   j. $\vdash_{NF0\alpha} \forall x\Box(Ax \vee \neg Ax)$

5. (i) Suppose no $P(Q^n)_i = 2$. Provide a revised version of our derivation rules which takes advantage of this simplification. Hint: it is possible to do away with overlines altogether. Why? (ii) Suppose $P(Q^n)_i$ is always 0. Provide a version of our derivation rules which takes advantage of this additional simplification.

9.3 SOUNDNESS AND COMPLETENESS

Preliminaries: Begin with generalized notions of validity. Given any model $\langle W, U, D, R, P, v\rangle$, let m be a map from subscripts into W. Then say $\langle W, U, D, R, P, v\rangle_m$ is $\langle W, U, D, R, P, v\rangle$ ***with*** map m. Then, where Γ is a set of expressions of our language for derivations, $h_m(\Gamma) = 1$ iff for each $/A/_s \in \Gamma$, $h_{m(s)}(/A/) = 1$ and for each $s.t \in \Gamma$, $\langle m(s), m(t)\rangle \in R$. Now expand notions of validity for subscripts, and alternate expressions as indicated in double brackets. Where the formulas in Γ and A are sentences,

VFNα* $\Gamma \models^*_{Fn\alpha} /A/_s$ $[\![s.t]\!]$ iff there is no *Fn*α interpretation $\langle W, U, D, R, P, v\rangle_m$ such that $h_m(\Gamma) = 1$ but $h_{m(s)}(/A/) = 0$ $[\![\langle m(s), m(t)\rangle \notin R]\!]$.

NFNα* $\Gamma \vdash^*_{NFn\alpha} /A/_s$ $[\![s.t]\!]$ iff there is an *NFn*α derivation of $/A/_s$ $[\![s.t]\!]$ from the members of Γ.

These notions reduce to the standard ones when all the members of Γ and A have subscript 0. As usual, for the following, cases omitted are left to the reader.

THEOREM 9.1 *NFnα is sound: If* $\Gamma \vdash_{NFn\alpha} /A/$ *then* $\Gamma \models_{Fn\alpha} /A/$.

L9.1 If $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{Fn\alpha} /P/_s$ $[\![s.t]\!]$, then $\Gamma' \models^*_{Fn\alpha} /P/_s$ $[\![s.t]\!]$.

L9.2 If δ and δ' agree on their assignments to variables free in $/P/$ then $h_w(/P/)/\!\!/\delta = h_w(/P/)/\!\!/\delta'$.

L9.3 If ν and ν' differ at most in assignments to terms that do not occur in $/P/$, then for the corresponding h and h', $h_w(/P/)/\!\!/\delta = h'_w(/P/)/\!\!/\delta$. Corollary: $h_w(/P/) = h'_w(/P/)$.

L9.4 If $\Delta(a) = u$, then $h_w(\backslash P^x/_a\backslash)/\!\!/\delta = h_w(\backslash P\backslash)/\!\!/\delta[x|u]$.

Demonstrations for L9.1 - L9.4 are all on the model of parallel results from classical logic.

Main result: For each line in a derivation let $\mathcal{P}_i$ be the expression on line i and Γ_i be the set of all premises and assumptions whose scope includes line i. We set out to show "generalized" soundness: if $\Gamma \vdash^*_{NFn\alpha} \mathcal{P}$ then $\Gamma \models^*_{Fn\alpha} \mathcal{P}$. As above, this reduces to the standard result when $\mathcal{P}$ and all the members of Γ are without overlines and have subscript 0. Suppose $\Gamma \vdash^*_{NFn\alpha} \mathcal{P}$. Then there is a derivation of $\mathcal{P}$ from premises in Γ where $\mathcal{P}$ appears under the scope of the premises alone. By induction on line number of this derivation, we show that for each line i of this derivation, $\Gamma_i \models^*_{Fn\alpha} \mathcal{P}_i$. The case when $\mathcal{P}_i = \mathcal{P}$ is the desired result.

Basis: $\mathcal{P}_1$ is a premise or an assumption $/A/_s$ $[\![s.t]\!]$. Then $\Gamma_1 = \{/A/_s\}$ $[\![\{s.t\}]\!]$; so for any $\langle W, U, D, R, P, \nu\rangle_m$, we have $h_m(\Gamma_1) = 1$ iff $h_{m(s)}(/A/) = 1$ $[\![\langle m(s), m(t)\rangle \in R]\!]$; so there is no $\langle W, U, D, R, P, \nu\rangle_m$ where $h_m(\Gamma_1) = 1$ but $h_{m(s)}(/A/) = 0$ $[\![\langle m(s), m(t)\rangle \notin R]\!]$. So by VFN$\alpha$*, it follows that $\Gamma_1 \models^*_{Fn\alpha} /A/_s$ $[\![s.t]\!]$, where this is just to say, $\Gamma_1 \models^*_{Fn\alpha} \mathcal{P}_1$.

Assp: For any $i, 1 \leqslant i < k, \Gamma_i \models^*_{Fn\alpha} \mathcal{P}_i$.

Show: $\Gamma_k \models^*_{Fn\alpha} \mathcal{P}_k$.

$\mathcal{P}_k$ is either a premise, an assumption, or arises from previous lines by R, $\wedge$I, $\wedge$E, $\neg$I, $\neg$E, $\forall$I, $\forall$E, $\Box$I, $\Box$E, D, =I, =E, P1I, P2I, P2E or, depending on the system, AMη, AMρ, AMσ, or AMτ. If $\mathcal{P}_k$ is a premise or an assumption, then as in the basis, $\Gamma_k \models^*_{Fn\alpha} \mathcal{P}_k$. So suppose $\mathcal{P}_k$ arises by one of the rules.

(R)

($\wedge$I) If $\mathcal{P}_k$ arises by $\wedge$I, then the picture is like this,

$$\begin{array}{r|l} i & /A/_s \\ j & /B/_s \\ & \\ k & /A \wedge B/_s \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $/A \wedge B/_s$. By assumption, $\Gamma_i \models^*_{Fn\alpha} /A/_s$ and $\Gamma_j \models^*_{Fn\alpha} /B/_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L9.1, $\Gamma_k \models^*_{Fn\alpha} /A/_s$ and $\Gamma_k \models^*_{Fn\alpha} /B/_s$. Suppose $\Gamma_k \not\models^*_{Fn\alpha} /A \wedge B/_s$; then by VFN$\alpha$*, there is an *Fn*$\alpha$ interpretation $\langle W, U, D, R, P, v\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(/A \wedge B/) = 0$. From the latter, there is a δ such that $h_{m(s)}(/A \wedge B/)/\!/\delta = 0$. From the former, by VFNα*, $h_{m(s)}(/A/) = 1$ and $h_{m(s)}(/B/) = 1$; so $h_{m(s)}(/A/)/\!/\delta = 1$ and $h_{m(s)}(/B/)/\!/\delta = 1$; so by HF($\wedge$), $h_{m(s)}(/A \wedge B/)/\!/\delta = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Fn\alpha} /A \wedge B/_s$, which is to say, $\Gamma_k \models^*_{Fn\alpha} \mathcal{P}_k$.

($\wedge$E)

($\neg$I) If $\mathcal{P}_k$ arises by $\neg$I, then the picture is like this,

$$\begin{array}{r|l|l} & & /A/_s \\ \cline{3-3} & & \\ i & & /\!/B/\!/_t \\ j & & \backslash\!\backslash\neg B\backslash\!\backslash_t \\ k & \multicolumn{2}{l}{\backslash\neg A\backslash_s} \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $\backslash\neg A\backslash_s$. By assumption, $\Gamma_i \models^*_{Fn\alpha} /\!/B/\!/_t$ and $\Gamma_j \models^*_{Fn\alpha} \backslash\!\backslash\neg B\backslash\!\backslash_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{/A/_s\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{/A/_s\}$; so by L9.1, $\Gamma_k \cup \{/A/_s\} \models^*_{Fn\alpha} /\!/B/\!/_t$ and $\Gamma_k \cup \{/A/_s\} \models^*_{Fn\alpha} \backslash\!\backslash\neg B\backslash\!\backslash_t$. Suppose $\Gamma_k \not\models^*_{Fn\alpha} \backslash\neg A\backslash_s$; then by VFN$\alpha$*, there is an *Fn*$\alpha$ interpretation $\langle W, U, D, R, P, v\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(\backslash\neg A\backslash) = 0$; from the latter, there is some δ such that $h_{m(s)}(\backslash\neg A\backslash)/\!/\delta = 0$; so by HF($\neg$), $h_{m(s)}(/A/)/\!/\delta = 1$; but a derivation is a sequence of sentences; so $\neg A$ and so A have no free variables; so by L9.2, for any δ', $h_{m(s)}(/A/)/\!/\delta' = 1$; so $h_{m(s)}(/A/) = 1$; so $h_m(\Gamma_k) = 1$ and $h_{m(s)}(/A/) = 1$; so $h_m(\Gamma_k \cup \{/A/_s\}) = 1$; so by VFN$\alpha$*, $h_{m(t)}(/\!/B/\!/) = 1$ and $h_{m(t)}(\backslash\!\backslash\neg B\backslash\!\backslash) = 1$; so for any δ, $h_{m(t)}(/\!/B/\!/)/\!/\delta = 1$ and $h_{m(t)}(\backslash\!\backslash\neg B\backslash\!\backslash)/\!/\delta = 1$; from the latter, by HF($\neg$), $h_{m(t)}(/\!/B/\!/)/\!/\delta = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Fn\alpha} \backslash\neg A\backslash_s$, which is to say, $\Gamma_k \models^*_{Fn\alpha} \mathcal{P}_k$.

($\neg$E)

($\forall$I) If $\mathcal{P}_k$ arises by $\forall$I, then the picture is like this,

$$\begin{array}{r|l|l} & & Ea_s \\ \cline{3-3} & & \\ i & & /\!/A^x/_a/\!/_s \\ k & \multicolumn{2}{l}{/\!/\forall xA/\!/_s} \end{array}$$

where $i < k$, a does not appear in any member of Γ_k (in any undischarged premise or assumption) or in A, and $\mathcal{P}_k$ is $/\!/\forall xA/\!/_s$. By assumption, $\Gamma_i \models^*_{Fn\alpha} /\!/A^x/_a/\!/_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{Ea_s\}$; so by L9.1, $\Gamma_k \cup \{Ea_s\} \models^*_{Fn\alpha} /\!/A^x/_a/\!/_s$. Suppose $\Gamma_k \not\models^*_{Fn\alpha} /\!/\forall xA/\!/_s$; then by VFNα*, there is an *Fnα* interpretation $I = \langle W, U, D, R, P, v\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(/\!/\forall xA/\!/) = 0$; from the latter, there is some δ such that $h_{m(s)}(/\!/\forall xA/\!/)/\!/\delta = 0$; so by HF(∀), there is some $u \in D(m(s))$ such that $h_{m(s)}(/\!/A/\!/)/\!/\delta[x|u] = 0$. Let $I' = \langle W, U, D, R, P, v'\rangle_m$ be like I except that $v'(a) = u$. Then since a does not occur in Γ_k, by the corollary to L9.3, it remains that $h'_m(\Gamma_k) = 1$; and since $u \in D(m(s))$, and $v'(a) = u$, $v'(a) \in D(m(s))$; so for arbitrary δ', $\Delta'(a) \in D(m(s))$; so $\Delta'(a) \in v'_{m(s)}(E)$; so by HF(R), $h'_{m(s)}(Ea)/\!/\delta' = 1$; and since δ' is arbitrary, $h'_{m(s)}(Ea) = 1$; so $h'_m(\Gamma_k \cup \{Ea_s\}) = 1$; so by VFNα*, $h'_{m(s)}(/\!/A^x/_a/\!/) = 1$; so $h'_{m(s)}(/\!/A^x/_a/\!/)/\!/\delta = 1$; but $\Delta(a) = v'(a) = u$; so by L9.4, $h'_{m(s)}(/\!/A/\!/)/\!/\delta[x|u] = 1$; so, since a does not occur in A, by L9.3, $h_{m(s)}(/\!/A/\!/)/\!/\delta[x|u] = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Fn\alpha} /\!/\forall xA/\!/_s$, which is to say, $\Gamma_k \models^*_{Fn\alpha} \mathcal{P}_k$.

(∀E) If $\mathcal{P}_k$ arises by ∀E, then the picture is like this,

i	$/\!/\forall xA/\!/_s$
j	Ea_s
k	$/\!/A^x/_a/\!/_s$

where $i, j < k$ and $\mathcal{P}_k$ is $/\!/A^x/_a/\!/_s$. By assumption, $\Gamma_i \models^*_{Fn\alpha} /\!/\forall xA/\!/_s$ and $\Gamma_j \models^*_{Fn\alpha} Ea_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L9.1, $\Gamma_k \models^*_{Fn\alpha} /\!/\forall xA/\!/_s$ and $\Gamma_k \models^*_{Fn\alpha} Ea_s$. Suppose $\Gamma_k \not\models^*_{Fn\alpha} /\!/A^x/_a/\!/_s$; then by VFNα*, there is an *Fnα* interpretation $\langle W, U, D, R, P, v\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(/\!/A^x/_a/\!/) = 0$. From the latter, there is a δ such that $h_{m(s)}(/\!/A^x/_a/\!/)/\!/\delta = 0$. With the former, by VFNα*, $h_{m(s)}(/\!/\forall xA/\!/) = 1$ and $h_{m(s)}(Ea) = 1$; from the second of these, $h_{m(s)}(Ea)/\!/\delta = 1$; so by HF(R), $\Delta(a) \in v_{m(s)}(E)$; so $\Delta(a) \in D(m(s))$; say $\Delta(a) = u$; then $u \in D(m(s))$. Since $h_{m(s)}(/\!/\forall xA/\!/) = 1$, we have $h_{m(s)}(/\!/\forall xA/\!/)/\!/\delta = 1$; so by HF(∀), for any $v \in D(m(s))$, $h_{m(s)}(/\!/A/\!/)/\!/\delta[x|v] = 1$; it follows that $h_{m(s)}(/\!/A/\!/)/\!/\delta[x|u] = 1$; so since $\Delta(a) = u$, by L9.4, $h_{m(s)}(/\!/A^x/_a/\!/)/\!/\delta = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Fn\alpha} /\!/A^x/_a/\!/_s$, which is to say, $\Gamma_k \models^*_{Fn\alpha} \mathcal{P}_k$.

(□I) If $\mathcal{P}_k$ arises by □I, then the picture is like this,

		$s.t$
i		$/A/_t$
k	$/\Box A/_s$	

where $i < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $/\Box A/_s$. By assumption, $\Gamma_i \models^*_{Fn\alpha} /A/_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{s.t\}$; so by L9.1, $\Gamma_k \cup \{s.t\} \models^*_{Fn\alpha} /A/_t$. Suppose $\Gamma_k \not\models^*_{Fn\alpha} /\Box A/_s$; then by VFNα*, there is an *Fnα* interpretation $\langle W, U, D, R, P, v\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(/\Box A/) = 0$; from the latter, there is some δ such that $h_{m(s)}(/\Box A/){/\!/}\delta = 0$; so by HF(□), there is some $w \in W$ such that $m(s)Rw$ and $h_w(/A/){/\!/}\delta = 0$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, U, D, R, P, v\rangle_{m'}$; since t does not appear in Γ_k, it remains that $h_{m'}(\Gamma_k) = 1$; and since $m'(t) = w$ and $m'(s) = m(s)$, $\langle m'(s), m'(t)\rangle \in R$; so $h_{m'}(\Gamma_k \cup \{s.t\}) = 1$; so by VFNα*, $h_{m'(t)}(/A/) = 1$; so $h_{m'(t)}(/A/){/\!/}\delta = 1$. But $m'(t) = w$; so $h_w(/A/){/\!/}\delta = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Fn\alpha} /\Box A/_s$, which is to say, $\Gamma_k \models^*_{Fn\alpha} \mathcal{P}_k$.

(□E) If $\mathcal{P}_k$ arises by □E, then the picture is like this,

i	$/\Box A/_s$
j	$s.t$
k	$/A/_t$

where $i, j < k$ and $\mathcal{P}_k$ is $/A/_t$. By assumption, $\Gamma_i \models^*_{Fn\alpha} /\Box A/_s$ and $\Gamma_j \models^*_{Fn\alpha} s.t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L9.1, $\Gamma_k \models^*_{Fn\alpha} /\Box A/_s$ and $\Gamma_k \models^*_{Fn\alpha} s.t$. Suppose $\Gamma_k \not\models^*_{Fn\alpha} /A/_t$; then by VFNα*, there is an *Fnα* interpretation $\langle W, U, D, R, P, v\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(t)}(/A/) = 0$. From the latter, there is a δ such that $h_{m(t)}(/A/){/\!/}\delta = 0$. With the former, by VFNα*, $h_{m(s)}(/\Box A/) = 1$ and $\langle m(s), m(t)\rangle \in R$; from the first of these, $h_{m(s)}(/\Box A/){/\!/}\delta = 1$; so by HF(□), for any $w \in W$ such that $m(s)Rw$, $h_w(/A/){/\!/}\delta = 1$; so $h_{m(t)}(/A/){/\!/}\delta = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Fn\alpha} /A/_t$, which is to say, $\Gamma_k \models^*_{Fn\alpha} \mathcal{P}_k$.

(D) If $\mathcal{P}_k$ arises by D, then the picture is like this,

i	A_s
k	$\overline{A}_s$

where $i < k$ and $\mathcal{P}_k$ is $\overline{A}_s$. By assumption, $\Gamma_i \models^*_{Fn\alpha} A_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L9.1, $\Gamma_k \models^*_{Fn\alpha} A_s$. Suppose $\Gamma_k \not\models^*_{Fn\alpha} \overline{A}_s$; then by VFNα*, there is an *Fnα* interpretation $\langle W, U, D, R, P, v\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(\overline{A}) = 0$. From the latter, there is a δ such that $h_{m(s)}(\overline{A}){/\!/}\delta = 0$. With the former, by VFNα*, $h_{m(s)}(A) = 1$; so $h_{m(s)}(A){/\!/}\delta = 1$. But for these interpretations, there is no formula A and $w \in W$ such that $h_w(A){/\!/}\delta = 1$ and $h_w(\overline{A}){/\!/}\delta = 0$.

Basis: Suppose A is an atomic $Q^n t_1 \ldots t_n$, and for some $w \in W$ and δ, $h_w(A){/\!/}\delta = 1$; then $h_w(Q^n t_1 \ldots t_n){/\!/}\delta = 1$; so by HF(R),

$\langle \Delta(t_1) \ldots \Delta(t_n) \rangle \in v_w(Q^n)$; so by the construction of $\bar{v}$, $\langle \Delta(t_1) \ldots \Delta(t_n) \rangle \in \bar{v}_w(Q^n)$; so by HF(R), $h_w(\overline{Q^n t_1 \ldots t_n}) /\!\!/ \delta = 1$; so $h_w(\overline{A}) /\!\!/ \delta = 1$.

Assp: For any i, $0 \leqslant i < k$, if A has i operators, and for some $w \in W$ and δ, $h_w(A) /\!\!/ \delta = 1$, then $h_w(\overline{A}) /\!\!/ \delta = 1$.

Show: If A has k operators, and for some $w \in W$ and δ, $h_w(A) /\!\!/ \delta = 1$, then $h_w(\overline{A}) /\!\!/ \delta = 1$.

If A has k operators, then A is of the form, $\neg P$, $P \wedge Q$, $\Box P$, or $\forall x P$, where P and Q have $< k$ operators.

($\neg$) Suppose A is $\neg P$ and for some $w \in W$ and δ, $h_w(A) /\!\!/ \delta = 1$; then $h_w(\neg P) /\!\!/ \delta = 1$; so by HF($\neg$), $h_w(\overline{P}) /\!\!/ \delta = 0$; so by assumption, $h_w(P) /\!\!/ \delta = 0$; so by HF($\neg$), $h_w(\overline{\neg P}) /\!\!/ \delta = 1$; so $h_w(\overline{A}) /\!\!/ \delta = 1$.

($\wedge$) Suppose A is $P \wedge Q$ and for some $w \in W$ and δ, $h_w(A) /\!\!/ \delta = 1$; then $h_w(P \wedge Q) /\!\!/ \delta = 1$; so by HF($\wedge$), $h_w(P) /\!\!/ \delta = 1$ and $h_w(Q) /\!\!/ \delta = 1$; so by assumption, $h_w(\overline{P}) /\!\!/ \delta = 1$ and $h_w(\overline{Q}) /\!\!/ \delta = 1$; so by HF($\wedge$), $h_w(\overline{P \wedge Q}) /\!\!/ \delta = 1$; so $h_w(\overline{A}) /\!\!/ \delta = 1$.

($\Box$) Suppose A is $\Box P$ and for some $w \in W$ and δ, $h_w(A) /\!\!/ \delta = 1$ but $h_w(\overline{A}) /\!\!/ \delta = 0$; then $h_w(\Box P) /\!\!/ \delta = 1$ but $h_w(\overline{\Box P}) /\!\!/ \delta = 0$; from the latter, by HF($\Box$), there is an $a \in W$ such that wRa and $h_a(\overline{P}) /\!\!/ \delta = 0$; so by assumption, $h_a(P) /\!\!/ \delta = 0$; so by HF($\Box$), $h_w(\Box P) /\!\!/ \delta = 0$. This is impossible; reject the assumption: if $h_w(A) /\!\!/ \delta = 1$ then $h_w(\overline{A}) /\!\!/ \delta = 1$.

($\forall$) Suppose A is $\forall x P$ and for some $w \in W$ and δ, $h_w(A) /\!\!/ \delta = 1$ but $h_w(\overline{A}) /\!\!/ \delta = 0$; then $h_w(\forall x P) /\!\!/ \delta = 1$ but $h_w(\overline{\forall x P}) /\!\!/ \delta = 0$; from the latter, by HF($\forall$), there is some $u \in D(w)$ such that $h_w(\overline{P}) /\!\!/ \delta[x|u] = 0$; so by assumption, $h_w(P) /\!\!/ \delta[x|u] = 0$; so by HF($\forall$), $h_w(\forall x P) /\!\!/ \delta = 0$. This is impossible; reject the assumption: if $h_w(A) /\!\!/ \delta = 1$ then $h_w(\overline{A}) /\!\!/ \delta = 1$.

For any A, if for some $w \in W$ and δ, $h_w(A) /\!\!/ \delta = 1$, then $h_w(\overline{A}) /\!\!/ \delta = 1$.

So, returning to the main case, $h_{m(s)}(\overline{A}) /\!\!/ \delta = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Fn\alpha} \overline{A}_s$, which is to say, $\Gamma_k \models^*_{Fn\alpha} \mathcal{P}_k$.

(=I) If $\mathcal{P}_k$ arises by =I, then the picture is like this,

$$\begin{array}{l|l} i & P1[a = a]_s \\ & \\ k & (a = a)_s \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is $(a = a)_s$. By assumption, $\Gamma_i \models^*_{Fn\alpha} P1[a = a]_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L9.1, $\Gamma_k \models^*_{Fn\alpha} P1[a = a]_s$. Suppose $\Gamma_k \not\models^*_{Fn\alpha} (a = a)_s$; then by VFNα*, there is an *Fnα* interpretation $\langle W, U, D, R, P, v \rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(a = a) = 0$.

From the latter, there is a δ such that $h_{m(s)}(a = a)/\!/\delta = 0$; so by HF(R), $\langle \Delta(a), \Delta(a)\rangle \notin v_{m(s)}(=)$. Now $P(=)$ is $\langle 0, 0\rangle$ or not; if $P(=) = \langle 0, 0\rangle$, then $v_{m(s)}(=) = \{\langle u, u\rangle \mid u \in U\}$; so $\langle \Delta(a), \Delta(a)\rangle \in v_{m(s)}(=)$; this is impossible, so $P(=) \neq \langle 0, 0\rangle$, and $v_{m(s)}(=) = \{\langle u, u\rangle \mid u \in D(m(s))\}$. But since $h_m(\Gamma_k) = 1$, by VFNα*, $h_{m(s)}(P1[a = a]) = 1$; so $h_{m(s)}(P1[a = a])/\!/\delta = 1$; so with HF(∧), for an i such that $P(=)_i \geqslant 1$, $h_{m(s)}(Ea)/\!/\delta = 1$, and by HF(R), $\Delta(a) \in v_{w(s)}(E)$, so that $\Delta(a) \in D(m(s))$; and since there is some such i, $\Delta(a) \in D(m(s))$; so $\langle \Delta(a), \Delta(a)\rangle \in v_{m(s)}(=)$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Fn\alpha} (a = a)_s$, which is to say, $\Gamma_k \models^*_{Fn\alpha} \mathcal{P}_k$.

(=E) If A_k arises by =E, then the picture is like this,

$$\begin{array}{r|l} i & (a_i = b)_s \\ j & /Q^n a_1 \ldots a_i \ldots a_n/_t \\ & \\ k & /Q^n a_1 \ldots b \ldots a_n/_t \end{array} \quad \text{or} \quad \begin{array}{r|l} i & (b = a_i)_s \\ j & /Q^n a_1 \ldots a_i \ldots a_n/_t \\ & \\ k & /Q^n a_1 \ldots b \ldots a_n/_t \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $/Q^n a_1 \ldots b \ldots a_n/_t$. In the first case, by assumption, $\Gamma_i \models^*_{Fn\alpha} (a_i = b)_s$ and $\Gamma_j \models^*_{Fn\alpha} /Q^n a_1 \ldots a_i \ldots a_n/_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L9.1, $\Gamma_k \models^*_{Fn\alpha} (a_i = b)_s$ and $\Gamma_k \models^*_{Fn\alpha} /Q^n a_1 \ldots a_i \ldots a_n/_t$. Suppose $\Gamma_k \not\models^*_{Fn\alpha} /Q^n a_1 \ldots b \ldots a_n/_t$; then by VFNα*, there is an *Fn*α interpretation $\langle W, U, D, R, P, v\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(t)}(/Q^n a_1 \ldots b \ldots a_n/) = 0$. From the latter, there is a δ such that $h_{m(t)}(/Q^n a_1 \ldots b \ldots a_n/)/\!/\delta = 0$; so by HF(R), $\langle \Delta(a_1) \ldots \Delta(b) \ldots \Delta(a_n)\rangle \notin /v/_{m(t)}(Q^n)$. But since $h_m(\Gamma_k) = 1$, by VFNα*, $h_{m(s)}(a_i = b) = 1$ and $h_{m(t)}(/Q^n a_1 \ldots a_i \ldots a_n/) = 1$; so $h_{m(s)}(a_i = b)/\!/\delta = 1$ and $h_{m(t)}(/Q^n a_1 \ldots a_i \ldots a_n/)/\!/\delta = 1$. From the first of these, by HF(R), $\langle \Delta(a_i), \Delta(b)\rangle \in v_{m(s)}(=)$; so, on either specification of $v_{m(s)}(=)$, $\Delta(a_i) = \Delta(b)$. From the second, by HF(R), $\langle \Delta(a_1) \ldots \Delta(a_i) \ldots \Delta(a_n)\rangle \in /v/_{m(t)}(Q^n)$; so $\langle \Delta(a_1) \ldots \Delta(b) \ldots \Delta(a_n)\rangle \in /v/_{m(t)}(Q^n)$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Fn\alpha} /Q^n a_1 \ldots b \ldots a_n/_t$, which is to say, $\Gamma_k \models^*_{Fn\alpha} \mathcal{P}_k$. And similarly in the other case.

(P1I) If $\mathcal{P}_k$ arises by P1I, then the picture is like this,

$$\begin{array}{r|l} i & (Q^n a_1 \ldots a_n)_s \\ & \\ k & P1[Q^n a_1 \ldots a_n]_s \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is $P1[Q^n a_1 \ldots a_n]_s$. By our assumption, $\Gamma_i \models^*_{Fn\alpha} (Q^n a_1 \ldots a_n)_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L9.1, we have $\Gamma_k \models^*_{Fn\alpha} (Q^n a_1 \ldots a_n)_s$. Suppose $\Gamma_k \not\models^*_{Fn\alpha} P1[Q^n a_1 \ldots a_n]_s$; then by VFNα*, there is some *Fn*α interpretation $\langle W, U, D, R, P, v\rangle_m$ such that $h_m(\Gamma_k) = 1$ but in which $h_{m(s)}(P1[Q^n a_1 \ldots a_n]) = 0$. From the latter, there is a δ such that $h_{m(s)}(P1[Q^n a_1 \ldots a_n])/\!/\delta = 0$; so, with HF(∧), there is some i such that $P(Q^n)_i \geqslant 1$ and $h_{m(s)}(Ea_i)/\!\!/\delta = 0$; so by HF(R), $\Delta(a_i) \notin v_{m(s)}(E)$; so $\Delta(a_i) \notin D(m(s))$. But since $h_m(\Gamma_k) = 1$, by VFɴα*, $h_{m(s)}(Q^n a_1 \dots a_n) = 1$; so $h_{m(s)}(Q^n a_1 \dots a_n)/\!\!/\delta = 1$; so by HF(R), $\langle \Delta(a_1) \dots \Delta(a_n) \rangle \in v_{m(s)}(Q^n a_1 \dots a_n)$; so by EP, for i with $P(Q^n)_i \geqslant 1$ as above, $\Delta(a_i) \in D(m(s))$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Fn\alpha} P1[Q^n a_1 \dots a_n]_s$, which is to say, $\Gamma_k \models^*_{Fn\alpha} \mathcal{P}_k$.

(P2I) If $\mathcal{P}_k$ arises by P2I, then the picture is like this,

$$\begin{array}{r|l} i & (\neg Q^n a_1 \dots a_n)_s \\ & \\ k & P2[Q^n a_1 \dots a_n]_s \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is $P2[Q^n a_1 \dots a_n]_s$. By assumption, we have $\Gamma_i \models^*_{Fn\alpha} (\neg Q^n a_1 \dots a_n)_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L9.1, $\Gamma_k \models^*_{Fn\alpha} (\neg Q^n a_1 \dots a_n)_s$. Suppose that $\Gamma_k \not\models^*_{Fn\alpha} P2[Q^n a_1 \dots a_n]_s$; then by VFɴα*, there is some *Fn*α interpretation $\langle W, U, D, R, P, v \rangle_m$ such that $h_m(\Gamma_k) = 1$ but in which $h_{m(s)}(P2[Q^n a_1 \dots a_n]) = 0$. From the latter, there is a δ such that $h_{m(s)}(P2[Q^n a_1 \dots a_n])/\!\!/\delta = 0$; so, with HF(∧), there is some i such that $P(Q^n)_i = 2$ and $h_{m(s)}(Ea_i)/\!\!/\delta = 0$; so by HF(R), $\Delta(a_i) \notin v_{m(s)}(E)$; so $\Delta(a_i) \notin D(m(s))$. But since $h_m(\Gamma_k) = 1$, by VFɴα*, $h_{m(s)}(\neg Q^n a_1 \dots a_n) = 1$; so $h_{m(s)}(\neg Q^n a_1 \dots a_n))/\!\!/\delta = 1$; so by HF(¬), $h_{m(s)}(\overline{Q^n a_1 \dots a_n}) = 0$; so by HF(R), $\langle \Delta(a_1) \dots \Delta(a_n) \rangle \notin \bar{v}_{m(s)}(Q^n a_1 \dots a_n)$; so, by the construction of $\bar{v}$, for i with $P(Q^n)_i = 2$ as above, $\Delta(a_i) \in D(m(s))$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Fn\alpha} P2[Q^n a_1 \dots a_n]_s$, which is to say, $\Gamma_k \models^*_{Fn\alpha} \mathcal{P}_k$.

(P2E) If $\mathcal{P}_k$ arises by P2E, then the picture is like this,

$$\begin{array}{r|l} i & P2[Q^n a_1 \dots a_n]_s \\ j & (\overline{Q^n a_1 \dots a_n})_s \\ & \\ k & (Q^n a_1 \dots a_n)_s \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $(Q^n a_1 \dots a_n)_s$. By assumption, we have $\Gamma_i \models^*_{Fn\alpha} P2[Q^n a_1 \dots a_n]_s$ and $\Gamma_j \models^*_{Fn\alpha} (\overline{Q^n a_1 \dots a_n})_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L9.1, $\Gamma_k \models^*_{Fn\alpha} P2[Q^n a_1 \dots a_n]_s$ and $\Gamma_k \models^*_{Fn\alpha} (\overline{Q^n a_1 \dots a_n})_s$. Suppose $\Gamma_k \not\models^*_{Fn\alpha} (Q^n a_1 \dots a_n)_s$; then by VFɴα*, there is som *Fn*α interpretation $\langle W, U, D, R, P, v \rangle_m$ such that $h_m(\Gamma_k) = 1$ but in which $h_{m(s)}(Q^n a_1 \dots a_n) = 0$. From the latter, there is a δ such that $h_{m(s)}(Q^n a_1 \dots a_n)/\!\!/\delta = 0$; so by HF(R), we have $\langle \Delta(a_1) \dots \Delta(a_n) \rangle \notin v_{m(s)}(Q^n)$. But since $h_m(\Gamma_k) = 1$, by VFɴα*, $h_{m(s)}(P2[Q^n a_1 \dots a_n]) = 1$ and $h_{m(s)}(\overline{Q^n a_1 \dots a_n}) = 1$; so $h_{m(s)}(P2[Q^n a_1 \dots a_n])/\!\!/\delta = 1$ and $h_{m(s)}(\overline{Q^n a_1 \dots a_n})/\!\!/\delta = 1$; from the second of these, by HF(R), $\langle \Delta(a_1) \dots \Delta(a_n) \rangle \in \bar{v}_{m(s)}(Q^n)$; so by the construction of $\bar{v}$, either $\langle \Delta(a_1) \dots \Delta(a_n) \rangle \in v_{m(s)}(Q^n)$ or there is some i such that $P(Q^n)_i = 2$ and $\Delta(a_i) \notin D(m(s))$; so there is some i such that $P(Q^n)_i = 2$ and $\Delta(a_i) \notin$

$D(m(s))$. But since $h_{m(s)}(P2[Q^n a_1 \dots a_n])/\!\!/\delta = 1$, with HF($\wedge$), any such i with $P(Q^n)_i = 2$ has $h_{m(s)}(Ea_i)/\!\!/\delta = 1$; so by HF(R), $\Delta(a_i) \in \nu_{m(s)}(E)$; so $\Delta(a_i) \in D(m(s))$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Fn\alpha} (Q^n a_1 \dots a_n)_s$, which is to say, $\Gamma_k \models^*_{Fn\alpha} \mathcal{P}_k$.

(AMη) If $\mathcal{P}_k$ arises by AMη, then the picture is like this,

$$\begin{array}{r|l}
 & \begin{array}{|l} s.t \\ \hline \\ \end{array} \\
i & \begin{array}{|l} /A/_u \end{array} \\
k & /A/_u
\end{array}$$

where $i < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption) and is not u, and $\mathcal{P}_k$ is $/A/_u$. Where this rule is included in *NFn*α, *Fn*α includes condition η. By assumption, $\Gamma_i \models^*_{Fn\alpha} /A/_u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{s.t\}$; so by L9.1, $\Gamma_k \cup \{s.t\} \models^*_{Fn\alpha} /A/_u$. Suppose $\Gamma_k \not\models^*_{Fn\alpha} /A/_u$; then by VFN$\alpha$*, there is a *Fn*$\alpha$ interpretation $\langle W, U, D, R, P, \nu\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(u)}(/A/) = 0$. By condition η, there is a $w \in W$ such that $m(s)Rw$; consider a map m' like m except that $m'(t) = w$, and consider $\langle W, U, D, R, P, \nu\rangle_{m'}$; since t does not appear in Γ_k, it remains that $h_{m'}(\Gamma_k) = 1$; and since $m'(s) = m(s)$ and $m'(t) = w$, $\langle m'(s), m'(t)\rangle \in R$; so $h_{m'}(\Gamma_k \cup \{s.t\}) = 1$; so by VFN$\alpha$*, $h_{m'(u)}(/A/) = 1$. But since $t \neq u$, $m'(u) = m(u)$; so $h_{m(u)}(/A/) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Fn\alpha} /A/_u$, which is to say, $\Gamma_k \models^*_{Fn\alpha} \mathcal{P}_k$.

(AMρ)

(AMσ) If $\mathcal{P}_k$ arises by AMσ, then the picture is like this,

$$\begin{array}{r|l}
i & s.t \\
 & \\
k & t.s
\end{array}$$

where $i < k$ and $\mathcal{P}_k$ is $t.s$. Where this rule is in *NFn*α, *Fn*α includes condition σ. By assumption, $\Gamma_i \models^*_{Fn\alpha} s.t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L9.1, $\Gamma_k \models^*_{Fn\alpha} s.t$. Suppose $\Gamma_k \not\models^*_{Fn\alpha} t.s$; then by VFN$\alpha$*, there is some *Fn*$\alpha$ interpretation $\langle W, U, D, R, P, \nu\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $\langle m(t), m(s)\rangle \notin R$; since $h_m(\Gamma_k) = 1$, by VFNα*, $\langle m(s), m(t)\rangle \in R$; and by condition σ, for any $\langle x, y\rangle \in R$, $\langle y, x\rangle \in R$; so $\langle m(t), m(s)\rangle \in R$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Fn\alpha} t.s$, which is to say, $\Gamma_k \models^*_{Fn\alpha} \mathcal{P}_k$.

(AMτ)

For any i, $\Gamma_i \models^*_{Fn\alpha} \mathcal{P}_i$.

THEOREM 9.2 *NFnα is complete: if* $\Gamma \models_{Fn\alpha} /A/$ *then* $\Gamma \vdash_{NFn\alpha} /A/$.

Suppose $\Gamma \models_{Fn\alpha} /A/$; then $\Gamma_0 \models^*_{Fn\alpha} /A/_0$; we show that $\Gamma_0 \vdash^*_{NFn\alpha} /A/_0$. As usual, this reduces to the standard notion. For the following, fix on some particular *Fnα*. Then definitions of *consistency* etc. are relative to it.

CON Γ is CONSISTENT iff there is no $/A/_s$ such that $\Gamma \vdash^*_{NFn\alpha} /A/_s$ and $\Gamma \vdash^*_{NFn\alpha} \backslash\neg A\backslash_s$.

L9.5 If s is 0 or appears in Γ, and $\Gamma \nvdash^*_{NFn\alpha} \backslash\neg P\backslash_s$, then $\Gamma \cup \{/P/_s\}$ is consistent.

Suppose s is 0 or appears in Γ and $\Gamma \nvdash^*_{NFn\alpha} \backslash\neg P\backslash_s$ but $\Gamma \cup \{/P/_s\}$ is inconsistent. Then there is some A_t such that $\Gamma \cup \{/P/_s\} \vdash^*_{NFn\alpha} /\!\!/A/\!\!/_t$ and $\Gamma \cup \{/P/_s\} \vdash^*_{NFn\alpha} \backslash\!\!\backslash\neg A\backslash\!\!\backslash_t$. But then we can argue,

1	Γ	
2	$/P/_s$	A (c, ¬I)
3	$/\!\!/A/\!\!/_t$	from $\Gamma \cup \{/P/_s\}$
4	$\backslash\!\!\backslash\neg A\backslash\!\!\backslash_t$	from $\Gamma \cup \{/P/_s\}$
5	$\backslash\neg P\backslash_s$	2-4 ¬I

where the assumption is allowed insofar as s is either 0 or appears in Γ; so $\Gamma \vdash^*_{NFn\alpha} \backslash\neg P\backslash_s$. But this is impossible; reject the assumption: if s is 0 or appears in Γ and $\Gamma \nvdash^*_{NFn\alpha} \backslash\neg P\backslash_s$, then $\Gamma \cup \{/P/_s\}$ is consistent.

LNG Let $\mathcal{L}$ be like our standard language except for the addition of constants $b_1, b_2 \ldots$, and let Γ'_0 be like Γ_0 except that its members are members of $\mathcal{L}$.

L9.6 For a derivation D (or set Σ) say $D^b/_a$ ($\Sigma^b/_a$) is like D (Σ) except that each of its members has instances of b replaced by a. Then if D is a derivation from Σ and a is a constant that does not appear in D, $D^b/_a$ is a derivation from $\Sigma^b/_a$.

L9.7 If Γ_0 is consistent, then Γ'_0 is consistent.

Demonstrations for L9.6 and L9.7 on the model of parallel results for classical logic.

L9.8 There is an enumeration of all the subscripted sentences, $\mathcal{P}_1\ \mathcal{P}_2 \ldots$

Proof by construction as usual.

MAX Γ is S-MAXIMAL iff for any A_s either $\Gamma \vdash^*_{NFn\alpha} /A/_s$ or $\Gamma \vdash^*_{NFn\alpha} \backslash\neg A\backslash_s$.

SGT Γ is a SCAPEGOAT set for $\Box$ iff for every formula of the form $/\neg\Box A/_s$, if $\Gamma \vdash^*_{NFn\alpha} /\neg\Box A/_s$ then there is some t such that $\Gamma \vdash^*_{NFn\alpha} s.t$ and $\Gamma \vdash^*_{NFn\alpha} /\neg A/_t$.

Γ is a SCAPEGOAT set for $\forall$ iff for every formula of the form $/\neg\forall x A/_s$, if $\Gamma \vdash^*_{NFn\alpha} /\neg\forall x A/_s$ then there is some a such that $\Gamma \vdash^*_{NFn\alpha} \mathsf{E}a_s$ and $\Gamma \vdash^*_{NFn\alpha} /\neg A^x/_a/_s$.

C(Γ'') For Γ with unsubscripted sentences and the corresponding Γ_0 and Γ_0', we construct Γ'' as follows. Set $\Omega_0 = \Gamma_0'$. By L9.8, there is an enumeration, $\mathcal{P}_1, \mathcal{P}_2 \ldots$ of all the subscripted sentences; let $\mathcal{E}_0$ be this enumeration. Then for the first $/A/_s$ in $\mathcal{E}_{i-1}$ such that s is 0 or included in Ω_{i-1}, let $\mathcal{E}_i$ be like $\mathcal{E}_{i-1}$ but without $/A/_s$, and set,

$\Omega_i = \Omega_{i-1}$ if $\Omega_{i-1} \vdash^*_{NFn\alpha} \backslash\neg A\backslash_s$

$\Omega_{i^*} = \Omega_{i-1} \cup \{/A/_s\}$ if $\Omega_{i-1} \nvdash^*_{NFn\alpha} \backslash\neg A\backslash_s$

and

$\Omega_i = \Omega_{i^*}$ if $/A/_s$ is not of the form $/\neg\Box P/_s$ or $/\neg\forall x P/_s$

$\Omega_i = \Omega_{i^*} \cup \{s.t, /\neg P/_t\}$ if $/A/_s$ is of the form $/\neg\Box P/_s$

-where t is the first subscript not included in Ω_{i^*}

$\Omega_i = \Omega_{i^*} \cup \{Eb_s, /\neg P^x/_b/_s\}$ if $/A/_s$ is of the form $/\neg\forall x P/_s$

-where b is the first new constant not included in Ω_{i^*}

then

$\Gamma'' = \bigcup_{i \geqslant 0} \Omega_i$

Note that there is always sure to be a subscript t not in Ω_{i^*} insofar as there are infinitely many subscripts, and at any stage only finitely many formulas are added – the only subscripts in the initial Ω_0 being 0. And similarly, there is sure to be a constant b not in Ω_{i^*} since the only constants in Ω_0 are from the original language. Suppose s appears in Γ''; then there is some Ω_i in which it is first appears; and any formula $\mathcal{P}_j$ in the original enumeration that has subscript s is sure to be "considered" for inclusion at a subsequent stage.

L9.9 For any s included in Γ'', Γ'' is s-maximal.

Suppose s is included in Γ'' but Γ'' is not s-maximal. Then there is some A_s such that $\Gamma'' \nvdash^*_{NFn\alpha} /A/_s$ and $\Gamma'' \nvdash^*_{NFn\alpha} \backslash\neg A\backslash_s$. For any i, each member of Ω_{i-1} is in Γ''; so if $\Omega_{i-1} \vdash^*_{NFn\alpha} \backslash\neg A\backslash_s$ then $\Gamma'' \vdash^*_{NFn\alpha} \backslash\neg A\backslash_s$; but $\Gamma'' \nvdash^*_{NFn\alpha} \backslash\neg A\backslash_s$; so $\Omega_{i-1} \nvdash^*_{NFn\alpha} \backslash\neg A\backslash_s$; so since s is included in Γ'', there is a stage in the construction that sets $\Omega_{i^*} = \Omega_{i-1} \cup \{/A/_s\}$; so by construction, $/A/_s \in \Gamma''$; so $\Gamma'' \vdash^*_{NFn\alpha} /A/_s$. This is impossible; reject the assumption: Γ'' is s-maximal.

L9.10 If Γ_0' is consistent, then each Ω_i is consistent.

Suppose Γ_0' is consistent.

Basis: $\Omega_0 = \Gamma_0'$ and Γ_0' is consistent; so Ω_0 is consistent.

Assp: For any i, $0 \leqslant i < k$, Ω_i is consistent.

Show: Ω_k is consistent.

Ω_k is either (i) Ω_{k-1}, (ii) $\Omega_{k^*} = \Omega_{k-1} \cup \{/A/_s\}$, (iii) $\Omega_{k^*} \cup \{s.t, /\neg P/_t\}$, or (iv) $\Omega_{k^*} \cup \{Eb_s, /\neg P^x/_b/_s\}$.

(i) Suppose Ω_k is Ω_{k-1}. By assumption, Ω_{k-1} is consistent; so Ω_k is consistent.

(ii) Suppose Ω_k is $\Omega_{k^*} = \Omega_{k-1} \cup \{/A/_s\}$. Then by construction, s is 0 or in Ω_{k-1} and $\Omega_{k-1} \nvdash^*_{NFn\alpha} \backslash\neg A\backslash_s$; so by L9.5, $\Omega_{k-1} \cup \{/A/_s\}$ is consistent; so Ω_k is consistent.

(iii) Suppose Ω_k is $\Omega_{k^*} \cup \{s.t, /\neg P/_t\}$. In this case, as above, Ω_{k*} is consistent and by construction, $/\neg\Box P/_s \in \Omega_{k^*}$. Suppose that Ω_k is inconsistent. Then there is some A_u such that $\Omega_{k^*} \cup \{s.t, /\neg P/_t\} \vdash^*_{NFn\alpha} /\!/A/\!/_u$ and $\Omega_{k^*} \cup \{s.t, /\neg P/_t\} \vdash^*_{NFn\alpha} \backslash\!\backslash\neg A\backslash\!\backslash_u$. In this case, reason as follows,

1	Ω_{k^*}	
2		$s.t$	A (g, $\Box$I)
3			$/\neg P/_t$	A (c, $\neg$E)
4			$/\!/A/\!/_u$	from $\Omega_{k^*} \cup \{s.t, /\neg P/_t\}$
5			$\backslash\!\backslash\neg A\backslash\!\backslash_u$	from $\Omega_{k^*} \cup \{s.t, /\neg P/_t\}$
6		$\backslash P\backslash_t$	3-5 $\neg$E
7	$\backslash\Box P\backslash_s$	2-6 $\Box$I

where, by construction, t is not in Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NFn\alpha} \backslash\Box P\backslash_s$; but $/\neg\Box P/_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{NFn\alpha} /\neg\Box P/_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

(iv) Suppose Ω_k is $\Omega_{k^*} \cup \{Eb_s, /$ŋ$P^x/_b/_s\}$. In this case, as above, Ω_{k^*} is consistent and by construction, $/\neg\forall xP/_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there is some A_u such that $\Omega_{k^*} \cup \{Eb_s, /P^x/_b/_s\} \vdash^*_{NFn\alpha} /\!/A/\!/_u$ and $\Omega_{k^*} \cup \{Eb_s, /P^x/_b/_s\} \vdash^*_{NFn\alpha} \backslash\!\backslash\neg A\backslash\!\backslash_u$. So reason as follows,

1	Ω_{k^*}	
2		Eb_s	A (g, $\forall$I)
3			$/\neg P^x/_b/_s$	A (c, $\neg$E)
4			$/\!/A/\!/_u$	from $\Omega_{k^*} \cup \{Eb_s, /\neg P^x/_b/_s\}$
5			$\backslash\!\backslash\neg A\backslash\!\backslash_u$	from $\Omega_{k^*} \cup \{Eb_s, /\neg P^x/_b/_s\}$
6		$\backslash P^x/_b\backslash_s$	3-5 $\neg$E
7	$\backslash\forall xP\backslash_s$	2-6 $\forall$I

where, by construction, b does not appear in Ω_{k^*} or in P. So $\Omega_{k^*} \vdash^*_{NFn\alpha} \backslash\forall xP\backslash_s$; but $/\neg\forall xP/_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{NFn\alpha} /\neg\forall xP/_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

For any i, Ω_i is consistent.

L9.11 If Γ'_0 is consistent, then Γ'' is consistent.

Reasoning parallel to L2.6 and L6.6.

L9.12 If Γ'_0 is consistent, then Γ'' is a scapegoat set for $\Box$ and $\forall$.

For ($\Box$). Suppose Γ'_0 is consistent and $\Gamma'' \vdash^*_{NFn\alpha} /\neg\Box P/_s$. By L9.11, Γ'' is consistent; and by the constraints on subscripts, s is included in Γ''. Since Γ'' is consistent, $\Gamma'' \nvdash^*_{NFn\alpha} \backslash\neg\neg\Box P\backslash_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{/\neg\Box P/_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{s.t, /\neg P/_t\}$; so by construction, $s.t \in \Gamma''$ and $/\neg P/_t \in \Gamma''$; so $\Gamma'' \vdash^*_{NFn\alpha} s.t$ and $\Gamma'' \vdash^*_{NFn\alpha} /\neg P/_t$. So Γ'' is a scapegoat set for $\Box$.

For $\forall$. Suppose Γ_0 is consistent and $\Gamma'' \vdash^*_{NFn\alpha} /\neg\forall xP/_s$. By L9.11, Γ'' is consistent; and by the constraints on subscripts, s is included in Γ''. Since Γ'' is consistent, $\Gamma'' \nvdash^*_{NFn\alpha} \backslash\neg\neg\forall xP\backslash_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{/\neg\forall xP/_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{Eb_s, /\neg P^x/_b/_s\}$; so by construction, $Eb_s \in \Gamma''$ and $/\neg P^x/_b/_s \in \Gamma''$; so $\Gamma'' \vdash^*_{NFn\alpha} Eb_s$ and $\Gamma'' \vdash^*_{NFn\alpha} /\neg P^x/_b/_s$. So Γ'' is a scapegoat set for $\forall$.

C(I) We construct an interpretation $I = \langle W, U, D, R, P, v\rangle$ based on Γ'' as follows. Consider an enumeration $a_1, a_2 \ldots$ of constants in $\mathcal{L}$ and say $i \simeq j$ iff $i = j$ or for some s, $\Gamma'' \vdash^*_{NFn\alpha} (a_i = a_j)_s$, and set $\bar{i} = \{j \mid i \simeq j\}$. Then for $I = \langle W, U, D, R, P, v\rangle$, let W have a member w_s corresponding to each subscript s included in Γ''. Set $U = \{\bar{i} \mid i \geqslant 1\}$, and $\bar{i} \in D(w_s)$ iff $\Gamma'' \vdash^*_{NFn\alpha} (Ea_i)_s$. And set $\langle w_s, w_t\rangle \in R$ iff $\Gamma'' \vdash^*_{NFn\alpha} s.t$. For any a_i, let $v(a_i) = \bar{i}$; and $\langle \bar{i} \ldots \bar{j}\rangle \in v_{w_s}(Q^n)$ iff $\Gamma'' \vdash^*_{NFn\alpha} (Q^n a_1 \ldots a_n)_s$. Set P directly from specification of the derivation system: if Et_i is a conjunct of an arbitrary $P2[Q^n t_1 \ldots t_n]_s$, then $P(Q^n)_i = 2$; if Et_i is a conjunct of an arbitrary $P1[Q^n t_1 \ldots t_n]_s$ but not $P2[Q^n t_1 \ldots t_n]_s$, then $P(Q^n)_i = 1$; otherwise $P(Q^n)_i = 0$.

Note that, for arbitrary δ, $\Delta(a_k) = v(a_k) = \bar{k}$.

L.13 $I = \langle W, U, D, R, P, v\rangle$ constructed as above is consistently specified.

reflexivity: For any i, $i \simeq i$. By construction.

symmetry: For any i and j, if $i \simeq j$, then $j \simeq i$. Suppose $i \simeq j$. If $i = j$, the result is immediate. So suppose $i \neq j$; then for some s, $\Gamma'' \vdash^*_{NFn\alpha} (a_i = a_j)_s$; so by =E, $\Gamma'' \vdash^*_{NFn\alpha} (a_i = a_i)_s$; and by =E again, $\Gamma'' \vdash^*_{NFn\alpha} (a_j = a_i)_s$; so $j \simeq i$.

transitivity: For any i, j, and k, if $i \simeq j$, and $j \simeq k$, then $i \simeq k$. Suppose $i \simeq j$ and $j \simeq k$. If $i = j$ or $j = k$, the result is immediate. So suppose $i \neq j$ and $j \neq k$; then for some s and t, $\Gamma'' \vdash^*_{NFn\alpha} (a_i = a_j)_s$ and $\Gamma'' \vdash^*_{NFn\alpha} (a_j = a_k)_t$; so by =E, $\Gamma'' \vdash^*_{NFn\alpha} (a_i = a_k)_s$; so $i \simeq k$.

self-membership: For any i, $i \in \bar{i}$. For any i, by reflexivity, $i \simeq i$; so $i \in \bar{i}$.

uniqueness: For any i, if $i \in \bar{h}$ and $i \in \bar{k}$, then $\bar{h} = \bar{k}$. Suppose there are h and k such that $i \in \bar{h}$ and $i \in \bar{k}$, but $h \neq k$. From the latter, there is some j such that $j \in \bar{h}$ and $j \notin \bar{k}$, or $j \in \bar{k}$ and $j \notin \bar{h}$; without loss of generality, suppose the former; then $h \simeq i$, $k \simeq i$ and $h \simeq j$; from the first, by symmetry, $i \simeq h$; so with the second, by transitivity, $k \simeq h$, and with the third, by transitivity again, $k \simeq j$; so $j \in \overline{k}$. This is impossible; reject the assumption: if $i \in \overline{h}$ and $i \in \overline{k}$, then $\overline{h} = \overline{k}$.

equality: For any i and j, $j \simeq i$ iff $\overline{i} = \overline{j}$. Suppose $j \simeq i$; then $i \in \overline{j}$; but by self-membership, $i \in \overline{i}$; so by uniqueness, $\overline{i} = \overline{j}$. Suppose $\overline{i} = \overline{j}$; by self-membership, $i \in \overline{i}$; so $i \in \overline{j}$; so $j \simeq i$.

Now for the main lemma:

D is consistently specified. Suppose otherwise; then for some s and $\overline{i} = \overline{j}$, $\overline{i} \in D(w_s)$ but $\overline{j} \notin D(w_s)$. So suppose $\overline{i} = \overline{j}$ and $\overline{i} \in D(w_s)$. If $i = j$, it is immediate that $\overline{j} \in D(w_s)$. So suppose $i \neq j$. Since $\overline{i} \in D(w_s)$, by construction, $\Gamma'' \vdash^*_{NFn\alpha} (Ea_i)_s$. And since $\overline{i} = \overline{j}$, by equality, $i \simeq j$; so, since i and j are distinct, for some t, $\Gamma'' \vdash^*_{NFn\alpha} (a_i = a_j)_t$; so by =E, $\Gamma'' \vdash^*_{NFn\alpha} (Ea_j)_s$; so by construction, $\overline{j} \in D(w_s)$.

$v_w(Q^n)$ is consistently specified. Suppose otherwise; then for some s and $\langle\overline{h}\ldots\overline{k}\rangle = \langle\overline{d}\ldots\overline{l}\rangle$, $\langle\overline{h}\ldots\overline{k}\rangle \in v_{w_s}(Q^n)$ but $\langle\overline{d}\ldots\overline{l}\rangle \notin v_{w_s}(Q^n)$. So suppose $\langle\overline{h}\ldots\overline{k}\rangle = \langle\overline{d}\ldots\overline{l}\rangle$ and $\langle\overline{h}\ldots\overline{k}\rangle \in v_{w_s}(Q^n)$. If $h = d$ and …and $k = l$, it is immediate that $\langle\overline{d}\ldots\overline{l}\rangle \in v_{w_s}(Q^n)$. So suppose some i in $h\ldots k$ is distinct from the corresponding j in $d\ldots l$. Since $\langle\overline{h}\ldots\overline{i}\ldots\overline{k}\rangle \in v_{w_s}(Q^n)$, by construction, $\Gamma'' \vdash^*_{NFn\alpha} (Q^n a_h \ldots a_i \ldots a_k)_s$. Since $\langle\overline{h}\ldots\overline{i}\ldots\overline{k}\rangle = \langle\overline{d}\ldots\overline{j}\ldots\overline{l}\rangle$, $\overline{i} = \overline{j}$; so by equality, $i \simeq j$; so since i and j are distinct, for some t, $\Gamma'' \vdash^*_{NFn\alpha} (a_i = a_j)_t$; so by =E, $\Gamma'' \vdash^*_{NFn\alpha} (Q^n a_h \ldots a_j \ldots a_k)_s$; and similarly for other members that are distinct; so $\Gamma'' \vdash^*_{NFn\alpha} (Q^n a_d \ldots a_j \ldots a_l)_s$; so by construction, $\langle\overline{d}\ldots\overline{l}\rangle \in v_{w_s}(Q^n)$.

$v(a_i)$ is consistently specified – any constant a_i is assigned to exactly one member of U. This follows immediately from self-membership and uniqueness.

L9.14 $I = \langle W, U, D, R, P, v\rangle$ constructed as above is such that $\langle\overline{i}\ldots\overline{j}\rangle \in \overline{v}_{w_s}(Q^n)$ iff $\Gamma'' \vdash^*_{NFn\alpha} (\overline{Q^n a_1 \ldots a_n})_s$.

(i) Suppose $\langle\overline{i}\ldots\overline{j}\rangle \in \overline{v}_{w_s}(Q^n)$; then by the construction of $\overline{v}$, either $\langle\overline{i}\ldots\overline{j}\rangle \in v_{w_s}(Q^n)$, or for some k in $i\ldots j$, $P(Q^n)_k = 2$ and $\overline{k} \notin D(w_s)$. In the first case, by construction, $\Gamma'' \vdash^*_{NFn\alpha} (Q^n a_i \ldots a_j)_s$; so by (D), it follows that $\Gamma'' \vdash^*_{NFn\alpha} (\overline{Q^n a_i \ldots a_j})_s$. In the second case, by construction, Ea_k is a conjunct of $P2[Q^n a_i \ldots a_j]$ and $\Gamma'' \nvdash^*_{NFn\alpha} (Ea_k)_s$; however, if $\Gamma'' \vdash^*_{NFn\alpha} \neg(Q^n a_i \ldots a_j)_s$, then by P2I, $\Gamma'' \vdash^*_{NFn\alpha} P2[Q^n a_i \ldots a_j]_s$, and by $\wedge$E, $\Gamma' \vdash^*_{NFn\alpha} (Ea_k)_s$; so we have $\Gamma'' \nvdash^*_{NFn\alpha} (\neg Q^n a_i \ldots a_j)_s$; but by L9.9, Γ'' is s-maximal; so $\Gamma'' \vdash^*_{NFn\alpha} (\overline{Q^n a_i \ldots a_j})_s$. So in either case, $\Gamma'' \vdash^*_{NFn\alpha} (\overline{Q^n a_i \ldots a_j})_s$.

(iib) Suppose $\Gamma'' \vdash^*_{NFn\alpha} (\overline{Q^n a_i \ldots a_j})_s$. Either $\Gamma'' \vdash^*_{NFn\alpha} P2[Q^n a_i \ldots a_j]_s$ or not. If $\Gamma'' \vdash^*_{NFn\alpha} P2[Q^n a_i \ldots a_j]_s$, by P2E, $\Gamma'' \vdash^*_{NFn\alpha} (Q^n a_i \ldots a_j)_s$; so by construction, $\langle\overline{i}\ldots\overline{j}\rangle \in v_{w_s}(Q^n)$; so by the construction of $\overline{v}$, $\langle\overline{i}\ldots\overline{j}\rangle \in$

$\bar{v}_{w_s}(Q^n)$. If $\Gamma'' \nvdash^*_{NFn\alpha}$ $P2[Q^n a_i \ldots a_j]_s$, there is some k in $i \ldots j$ such that $P(Q^n)_k = 2$, and $\Gamma'' \nvdash^*_{NFn\alpha} (Ea_k)_s$; from the latter, by construction, $\bar{k} \notin D(w_s)$; so by the construction of $\bar{v}$, $\langle \bar{i} \ldots \bar{j} \rangle \in \bar{v}_{w_s}(Q^n)$. So in either case, $\langle \bar{i} \ldots \bar{j} \rangle \in \bar{v}_{w_s}(Q^n)$.

L9.15 If Γ'_0 is consistent then for $\langle W, U, D, R, P, v \rangle$ constructed as above, any sentence $/A/$ in $\mathcal{L}$, and for any s included in Γ'', $h_{w_s}(/A/) = 1$ iff $\Gamma'' \vdash^*_{NFn\alpha}$ $/A/_s$.

Suppose Γ'_0 is consistent and s is included in Γ''. By L9.9, Γ'' is s-maximal. By L9.11 and L9.12, Γ'' is consistent and a scapegoat set for $\Box$ and $\forall$. Now by induction on the number of operators in $/A/_s$,

Basis: If $/A/_s$ has no operators, then it follows that it is is an atomic of the sort $/Q^n a_i \ldots a_j/$. $h_{w_s}(/A/) = 1$ iff $h_{w_s}(/Q^n a_i \ldots a_j/) = 1$; iff, for arbitrary δ, $h_{w_s}(/Q^n a_i \ldots a_j/) /\!\!/ \delta = 1$; by HF(R), iff $\langle \Delta(a_i) \ldots \Delta(a_j) \rangle \in /v/_{w_s}(Q^n)$; iff $\langle \bar{i} \ldots \bar{j} \rangle \in /v/_{w_s}(Q^n)$; by construction and L9.14, iff $\Gamma'' \vdash^*_{NFn\alpha} (/Q^n a_i \ldots a_j/)_s$; iff $\Gamma'' \vdash^*_{NFn\alpha}$ $/A/_s$.

Assp: For any i, $0 \leqslant i < k$, if $/A/_s$ has i operators, then $h_{w_s}(/A/) = 1$ iff $\Gamma'' \vdash^*_{NFn\alpha} /A/_s$.

Show: If $/A/_s$ has k operators, then $h_{w_s}(/A/) = 1$ iff $\Gamma'' \vdash^*_{NFn\alpha} /A/_s$.

If $/A/_s$ has k operators, then it is of the form $/\neg P/_s$, $/P \wedge Q/_s$, $/\Box P/_s$ or $/\forall x P/_s$, where P and Q have $< k$ operators.

($\neg$) $/A/_s$ is $/\neg P/_s$. (i) Suppose $h_{w_s}(/A/) = 1$; then $h_{w_s}(/\neg P/) = 1$; so for arbitrary δ, $h_{w_s}(/\neg P/) /\!\!/ \delta = 1$; so by HF($\neg$), $h_{w_s}(\backslash P \backslash) /\!\!/ \delta = 0$; so, $h_{w_s}(\backslash P \backslash) = 0$; so by assumption, $\Gamma'' \nvdash^*_{NFn\alpha} \backslash P \backslash_s$; so by s-maximality, $\Gamma'' \vdash^*_{NFn\alpha} /\neg P/_s$, where this is to say, $\Gamma'' \vdash^*_{NFn\alpha} /A/_s$. (ii) Suppose $\Gamma'' \vdash^*_{NFn\alpha} /A/_s$; then $\Gamma'' \vdash^*_{NFn\alpha} /\neg P/_s$; so by consistency, $\Gamma'' \nvdash^*_{NFn\alpha} \backslash P \backslash_s$; so by assumption, $h_{w_s}(\backslash P \backslash) = 0$; so there is a δ such that $h_{w_s}(\backslash P \backslash) /\!\!/ \delta = 0$; so by HF($\neg$), $h_{w_s}(/\neg P/) /\!\!/ \delta = 1$; and since $/A/_s$ has no free variables, by L9.2, $h_{w_s}(/\neg P/) = 1$, where this is to say, $h_{w_s}(/A/) = 1$. So $h_{w_s}(/A/) = 1$ iff $\Gamma'' \vdash^*_{NFn\alpha} /A/_s$.

($\wedge$)

($\Box$) $/A/_s$ is $/\Box P/_s$. (i) Suppose $h_{w_s}(/A/) = 1$, but $\Gamma'' \nvdash^*_{NFn\alpha} /A/_s$; then $h_{w_s}(/\Box P/) = 1$, but $\Gamma'' \nvdash^*_{NFn\alpha} /\Box P/_s$. From the latter, by maximality, $\Gamma'' \vdash^*_{NFn\alpha} \backslash \neg \Box P \backslash_s$; so, since Γ'' is a scapegoat set for $\Box$, there is some t such that $\Gamma'' \vdash^*_{NFn\alpha} s.t$ and $\Gamma'' \vdash^*_{NFn\alpha} \backslash \neg P \backslash_t$; from the first of these, by construction, $\langle w_s, w_t \rangle \in R$; and from the second, by consistency, $\Gamma'' \nvdash^*_{NFn\alpha} /P/_t$; so by assumption, $h_{w_t}(/P/) = 0$; so for some δ, $h_{w_t}(/P/) /\!\!/ \delta = 0$; so by HF($\Box$), $h_{w_s}(/\Box P/) /\!\!/ \delta = 0$; so $h_{w_s}(/\Box P/) = 0$. This is impossible; reject the assumption: if $h_{w_s}(/A/) = 1$, then $\Gamma'' \vdash^*_{NFn\alpha} /A/_s$.

(ii) Suppose $\Gamma'' \vdash^*_{NFn\alpha} /A/_s$, but $h_{w_s}(/A/) = 0$; then $\Gamma'' \vdash^*_{NFn\alpha} /\Box P/_s$, but $h_{w_s}(/\Box P/) = 0$. From the latter, there is a δ such that $h_{w_s}(/\Box P/)/\!\!/\delta = 0$; so by HF($\Box$), there is a $w_t \in W$ such that $w_s R w_t$ and $h_{w_t}(/P/)/\!\!/\delta = 0$; so $h_{w_t}(/P/) = 0$; so by assumption, $\Gamma'' \nvdash^*_{NFn\alpha} /P/_t$; but since $w_s R w_t$, by construction, $\Gamma'' \vdash^*_{NFn\alpha} s.t$; so by $\Box$E, $\Gamma'' \vdash^*_{NFn\alpha} /P/_t$. This is impossible; reject the assumption: if $\Gamma'' \vdash^*_{NFn\alpha} /A/_s$, then $h_{w_s}(/A/) = 1$. So $h_{w_s}(/A/) = 1$ iff $\Gamma'' \vdash^*_{NFn\alpha} /A/_s$.

($\forall$) $/A/_s$ is $/\forall x P/_s$. (i) Suppose $h_{w_s}(/A/) = 1$, but $\Gamma'' \nvdash^*_{NFn\alpha} /A/_s$; then $h_{w_s}(/\forall x P/) = 1$, but $\Gamma'' \nvdash^*_{NFn\alpha} /\forall x P/_s$. From the latter, by maximality, $\Gamma'' \vdash^*_{NFn\alpha} \backslash\neg\forall x P\backslash_s$; so, since Γ'' is a scapegoat set for $\forall$, there is some a_i such that $\Gamma'' \vdash^*_{NFn\alpha} (\mathrm{E}a_i)_s$ and $\Gamma'' \vdash^*_{NFn\alpha} \backslash\neg P^x/_{a_i}\backslash_s$; from the first of these, by construction, $\bar{i} \in D(w_s)$; and from the second, by consistency, $\Gamma'' \nvdash^*_{NFn\alpha} /P^x/_{a_i}/_s$; so by assumption, $h_{w_s}(/P^x/_{a_i}/) = 0$; so for some δ, $h_{w_s}(/P^x/_{a_i}/)/\!\!/\delta = 0$; but $v(a_i) = \bar{i}$; so $\Delta(a_i) = \bar{i}$; so by L9.4, $h_{w(s)}(/P/)/\!\!/\delta[x|\bar{i}] = 0$; so by HF($\forall$), $h_{w_s}(/\forall x P/)/\!\!/\delta = 0$; so $h_{w_s}(/\forall x P/) = 0$. This is impossible; reject the assumption: if $h_{w_s}(/A/) = 1$, then $\Gamma'' \vdash^*_{NFn\alpha} /A/_s$.

(ii) Suppose $\Gamma'' \vdash^*_{NFn\alpha} /A/_s$, but $h_{w_s}(/A/) = 0$; then $\Gamma'' \vdash^*_{NFn\alpha} /\forall x P/_s$, but $h_{w_s}(/\forall x P/) = 0$. From the latter, there is a δ such that $h_{w_s}(/\forall x P/)/\!\!/\delta = 0$; so by HF($\forall$), there is some $\bar{i} \in D(w_s)$ such that $h_{w_s}(/P/)/\!\!/\delta[x|\bar{i}] = 0$. Since $\bar{i} \in D(w_s)$, by construction, $\Gamma'' \vdash^*_{NFn\alpha} (\mathrm{E}a_i)_s$. And since $v(a_i) = \bar{i}$, $\Delta(a_i) = \bar{i}$; so by L9.4, $h_{w_s}(/P^x/_{a_i}/)/\!\!/\delta = 0$; so $h_{w_s}(/P^x/_{a_i}/) = 0$; so by assumption, $\Gamma'' \nvdash^*_{NFn\alpha} /P^x/_{a_i}/_s$. But since $\Gamma'' \vdash^*_{NFn\alpha} /\forall x P/_s$ and $\Gamma'' \vdash^*_{NFn\alpha} (\mathrm{E}a_i)_s$, by $\forall$E, $\Gamma'' \vdash^*_{NFn\alpha} /P^x/_{a_i}/_s$. This is impossible; reject the assumption: if $\Gamma'' \vdash^*_{NFn\alpha} /A/_s$, then $h_{w_s}(/A/) = 1$. So $h_{w_s}(/A/) = 1$ iff $\Gamma'' \vdash^*_{NFn\alpha} /A/_s$.

For any A_s, $h_{w_s}(/A/) = 1$ iff $\Gamma'' \vdash^*_{NFn\alpha} /A/_s$.

L9.16 If Γ'_0 is consistent, then $\langle W, U, D, R, P, v\rangle$ constructed as above is an *Fn*α interpretation.

Suppose Γ'_0 is consistent.

(EP) Suppose $\langle \bar{i} \dots \bar{j}\rangle \in v_{w_s}(Q^n)$ and for some k in $i \dots j$, $P(Q^n)_k \geqslant 1$. From the former, by construction, $\Gamma'' \vdash^*_{NFn\alpha} (Q^n a_i \dots a_j)_s$; so by P1I, $\Gamma'' \vdash^*_{NFn\alpha} P1[Q^n a_i \dots a_j]_s$. But since since $P(Q^n)_k \geqslant 1$, by construction, $\mathrm{E}a_k$ is a conjunct of $P1[Q^n a_i \dots a_j]$; so with $\wedge$E, $\Gamma'' \vdash^*_{NFn\alpha} (\mathrm{E}a_k)_s$; so by construction, $\bar{k} \in D(w_s)$, and EP is satisfied.

(E) By construction, $\bar{i} \in v_{w_s}(\mathrm{E})$ iff $\Gamma'' \vdash^*_{NFn\alpha} (\mathrm{E}a_i)_s$; by construction again, iff $\bar{i} \in D(w_s)$.

(=) Suppose $P(=) = \langle 0, 0\rangle$. (i) Suppose $\langle \bar{i}, \bar{j}\rangle \in v_{w_s}(=)$; then by construction, $\Gamma'' \vdash^*_{NFn\alpha} (a_i = a_j)_s$; so $i \simeq j$; so by equality $\bar{i} = \bar{j}$. (ii) Suppose $\bar{i} = \bar{j}$. Since $P(=) = \langle 0, 0\rangle$ it is trivial that $\Gamma'' \vdash^*_{NFn\alpha} P1[a_i = a_i]_s$; so by (=I), $\Gamma'' \vdash^*_{NFn\alpha} (a_i = a_i)_s$. And since $\bar{i} = \bar{j}$, by equality, $i \simeq j$; so either $i = j$ or for some t, $\Gamma'' \vdash^*_{NFn\alpha} (a_i = a_j)_t$. In the first case, since $\Gamma'' \vdash_{NFn\alpha} (a_i = a_i)_s$, $\Gamma'' \vdash_{NFn\alpha} (a_i = a_j)_s$ so that, by construction, $\langle \bar{i}, \bar{j}\rangle \in v_{w_s}(=)$. In the second case, by (=E), $\Gamma'' \vdash^*_{NFn\alpha} (a_i = a_j)_s$; and by construction, $\langle \bar{i}, \bar{j}\rangle \in v_{w_s}(=)$. In either case, then, $\langle \bar{i}, \bar{j}\rangle \in v_{w_s}(=)$. So the interpretation of (=) is as it should be.

Suppose $P(=) \neq \langle 0, 0\rangle$. (i) Suppose $\langle \bar{i}, \bar{j}\rangle \in v_{w_s}(=)$; then by construction, $\Gamma'' \vdash^*_{NFn\alpha} (a_i = a_j)_s$; so $i \simeq j$; so by equality, $\bar{i} = \bar{j}$. And since $\Gamma'' \vdash^*_{NFn\alpha} (a_i = a_j)_s$, by P1I, $\Gamma'' \vdash^*_{NFn\alpha} P1[a_i = a_j]_s$; so since $P(=)$ has some member $\neq 0$, with ∧E, either $\Gamma'' \vdash^*_{NFn\alpha} (Ea_i)_s$ or $\Gamma'' \vdash^*_{NFn\alpha} (Ea_j)_s$; so by construction, $\bar{i} \in D(w_s)$ or $\bar{j} \in D(w_s)$; so, since $\bar{i} = \bar{j}$, $\bar{i} \in D(w_s)$. (ii) Suppose $\bar{i} = \bar{j}$ and $\bar{i} \in D(w_s)$. From the latter, $\Gamma'' \vdash^*_{NFn\alpha} (Ea_i)_s$; so for any version of $P1[a_i = a_i]$, $\Gamma'' \vdash^*_{NFn\alpha} P1[a_i = a_i]_s$; so by (=)I, $\Gamma'' \vdash^*_{NFn\alpha} (a_i = a_i)_s$. From the former, by equality, $i \simeq j$; so either $i = j$ or for some t, $\Gamma'' \vdash^*_{NFn\alpha} (a_i = a_j)_t$. In the first case, since $\Gamma'' \vdash_{NFn\alpha} (a_i = a_i)_s$, $\Gamma'' \vdash_{NFn\alpha} (a_i = a_j)_s$ so that, by construction, $\langle \bar{i}, \bar{j}\rangle \in v_{w_s}(=)$. In the second case, by (=)E, $\Gamma'' \vdash^*_{NFn\alpha} (a_i = a_j)_s$; and by construction, $\langle \bar{i}, \bar{j}\rangle \in v_{w_s}(=)$. In either case, then, $\langle \bar{i}, \bar{j}\rangle \in v_{w_s}(=)$. So the interpretation of (=) is as it should be.

(η) Suppose α includes condition η and $w_s \in W$. Then, by construction, s is a subscript in Γ'; so by reasoning as follows,

1	Γ'	
2	$\quad$ s.t	A (g, AMη)
3	$\quad \top_t$	⊤ is a tautology
4	$\quad \Diamond\top_s$	2,3 ◇I
5	$\Diamond\top_s$	2-4 AMη
6	$\neg\Box\neg\top_s$	5 MN

$\Gamma' \vdash^*_{NFn\alpha} \neg\Box\neg\top_s$; but by L9.12, Γ' is a scapegoat set for □; so there is a t such that $\Gamma' \vdash^*_{NFn\alpha}$ s.t; so by construction, $\langle w_s, w_t\rangle \in R$ and η is satisfied.

(ρ) Suppose α includes condition ρ and $w_s \in W$. Then by construction, s is a subscript in Γ'; so by (AMρ), $\Gamma' \vdash^*_{NFn\alpha}$ s.s; so by construction, $\langle w_s, w_s\rangle \in R$ and ρ is satisfied.

(σ) Suppose α includes condition σ and $\langle w_s, w_t\rangle \in R$. Then by construction, $\Gamma' \vdash^*_{NFn\alpha}$ s.t so by (AMσ), $\Gamma' \vdash^*_{NFn\alpha}$ t.s; so by construction, $\langle w_t, w_s\rangle \in R$ and σ is satisfied.

(τ) Suppose α includes condition τ and $\langle w_s, w_t\rangle, \langle w_t, w_u\rangle \in R$. Then by construction, $\Gamma' \vdash^*_{NFn\alpha}$ s.t and $\Gamma' \vdash^*_{NFn\alpha}$ t.u; so by (AMτ), $\Gamma' \vdash^*_{NFn\alpha}$ s.u; so by construction, $\langle w_s, w_u\rangle \in R$ and τ is satisfied.

MAP For any $w_s \in W$, set $m(s) = w_s$; otherwise $m(s)$ is arbitrary. And let $I' = \langle W, U, D, R, P, v'\rangle$ be like I except without assignments to constants not in the original language. Clearly I' is an interpretatio for our our original language, and remains an *Fn*α interpretation.

L9.17 If Γ_0 is consistent, then $h'_m(\Gamma_0) = 1$.

Suppose Γ_0 is consistent and $/A/_0 \in \Gamma_0$; then $/A/_0 \in \Gamma'_0$ and by L9.7, Γ'_0 is consistent; then by construction, $/A/_0 \in \Gamma''$; so $\Gamma'' \vdash^*_{NFn\alpha} /A/_0$; so by L9.15, $h_{w_0}(/A/) = 1$; but $/A/$ is a sentence of the original language without extra constants; so by the corollary to L9.3, $h'_{w_0}(/A/) = 1$. And similarly for any $/A/_0 \in \Gamma_0$. But $m(0) = w_0$; so $h'_m(\Gamma_0) = 1$.

Main result: Suppose $\Gamma \models_{Fn\alpha} /A/$ but $\Gamma \nvdash_{NFn\alpha} /A/$. Then $\Gamma_0 \models^*_{Fn\alpha} /A/_0$ but $\Gamma_0 \nvdash^*_{NFn\alpha} /A/_0$. By (DN), if $\Gamma_0 \vdash^*_{NFn\alpha} /\neg\neg A/_0$, then $\Gamma_0 \vdash^*_{NFn\alpha} /A/_0$; so $\Gamma_0 \nvdash^*_{NFn\alpha} /\neg\neg A/_0$; so by L9.5, $\Gamma_0 \cup \{\backslash\neg A\backslash_0\}$ is consistent; so by L9.16 and L9.17, there is an *Fn*α interpretation $I' = \langle W, U, D, R, P, v'\rangle_m$ constructed as above such that $h'_m(\Gamma_0 \cup \{\backslash\neg A\backslash_0\}) = 1$; so $h'_{m(0)}(\backslash\neg A\backslash) = 1$; so for any δ, $h'_{m(0)}(\backslash\neg A\backslash)/\!/\delta = 1$; so by HF($\neg$), $h'_{m(0)}(/A/)/\!/\delta = 0$; so $h'_{m(0)}(/A/) = 0$; so $h'_m(\Gamma_0) = 1$ and $h'_{m(0)}(A) = 0$; so by VFNα*, $\Gamma_0 \not\models^*_{Fn\alpha} /A/_0$. This is impossible; reject the assumption: if $\Gamma \models_{Fn\alpha} /A/$, then $\Gamma \vdash_{NFn\alpha} /A/$.

REFERENCES

[1] Bergmann, Moor and Nelson. *The Logic Book 4th Ed.* McGraw-Hill, New York, 2004.

[2] Fitting and Mendelsohn. *First-Order Modal Logic.* Kluwer Academic Publishers, Boston, 1999.

[3] James Garson. "Quantification in Modal Logic." In *Handbook of Philosophical Logic II.* Ed. Gabbay and Guenthner. Kluwer Academic Publishers, Boston, 1984.

[4] Graham Priest. *An Introduction to Non-Classical Logic*. Cambridge University Press, Cambridge, 2001.

[5] Alvin Plantinga. "Actualism and Possible Worlds." Reprint in *Essays in the Metaphysics of Modality*. Ed. Davidson. Oxford University Press, New York, 2003.

[6] Alvin Plantinga. "World and Essence." Reprint in *Essays in the Metaphysics of Modality*. Ed. Davidson. Oxford University Press, New York, 2003.

[7] W.V. Quine. "Reference and Modality." Revised version in his, *From a Logical Point of View, 2nd edition*. Harvard University Press, Cambridge, 1980.

[8] Greg Restall. "Four-Valued Semantics for Relevant Logics (And Some of Their Rivals)." *Journal of Philosophical Logic* 24 (1995): 139-160.

[9] Richard Routley. "The American Plan Completed: Alternative Classical-Style Semantics, Without Stars, for Relevant and Paraconsistent Logics." *Studia Logica* 43 (1984): 131-158.

[10] Tony Roy. "Things and *De Re* Modality." *Noûs* 34 (2000): 56-84.

[11] Tony Roy. "Making Sense of Relevant Semantics." Unpublished, but available at http://cal.csusb.edu/Faculty/Philosophy/roy/msrs-paper.pdf.

[12] Tony Roy. *Symbolic Logic: An Accessible Introduction to Serious Mathematical Logic* Unpublished, but available at http://cal.csusb.edu/Faculty/Philosophy/roy/int-ml.htm.

[13] Peter Woodruff. "Logic and Truth Value Gaps." In *Philosophical Problems in Logic.* Ed. Lambert. D. Reidel, Dordrecht, 1970.

The *Australasian Journal of Logic* (ISSN 1448-5052) disseminates articles that significantly advance the study of logic, in its mathematical, philosophical or computational guises. The scope of the journal includes all areas of logic, both pure and applied to topics in philosophy, mathematics, computation, linguistics and the other sciences.

Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

The journal is freely available at the journal website at

http://www.philosophy.unimelb.edu.au/ajl/.

All issues of the journal are archived electronically at the journal website.

SUBSCRIPTIONS Individuals may subscribe to the journal by sending an email, including a full name, an institutional affiliation and an email address to the managing editor at ajl-editors@unimelb.edu.au. Subscribers will receive email abstracts of accepted papers to an address of their choice. For institutional subscription, please email the managing editor at ajl-editors@unimelb.edu.au.

Complete published papers may be downloaded at the journal's website at http://www.philosophy.unimelb.edu.au/ajl/. The journal currently publishes in pdf format.

SUBMISSION The journal accepts submissions of papers electronically. To submit an article for publication, send the LaTeX source of a submission to a member of the editorial board. For a current list of the editorial board, consult the website.