Inexpressiveness of First-Order Fragments

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abstract: It is well-known that first-order logic is semi-decidable. Therefore, first-order logic is less than ideal for computational purposes (computer science, knowledge engineering). Certain fragments of first-order logic are of interest because they are decidable. But decidability is gained at the cost of expressiveness. The objective of this paper is to investigate inexpressiveness of fragments that have received much attention.

1 introduction

It is well-known that first-order logic (fo) is semi-decidable. This makes fo less than ideal for computational purposes. Therefore there is much interest in restrictions of fo that are decidable. Restrictions can be imposed by restricting the use of quantifiers. This usually takes the form of restriction on the quantifier prefix of formulas in prefix normal form. Restrictions can also be imposed by restricting the use of variables. It is this kind of restriction that this paper is concerned with.

Such restricted logics (and their languages) are called *fragments*. The fragments investigated here are the following.

1. The fluted fragment (fl)
2. The two-variable fragment (fo^2)
3. The modal fragment (mf)
4. The guarded fragment (gf)

The fluted fragment allows formulas in which the order of the arguments of each predicate is precisely the order of the enclosing quantifier scopes. The two-variable fragment allows formulas with occurrences of only two variables. (Without loss of generality, these variables can be taken to be $\{x, y\}$.) The modal fragment allows formulas in the image of basic modal logic under the standard translation (see below). The guarded fragment allows formulas that conform to the schema

$$\exists y(R(x, y) \wedge \phi(x, y))$$

where $R(x, y)$ is an atom with occurrences of x and y only, and $\phi(x, y)$ is a subformula with occurrences of x and y only. They are defined formally in the next section.

2 PRELIMINARIES

The definition of syntax and semantics of, and inference in first-order predicate logic can be found in Andrews [2]. In this paper, the connectives are $\neg$ and $\wedge$, and the quantifier is $\exists$. Connectives $\vee$ and $\rightarrow$ and quantifier $\forall$ are introduced as abbreviations as usual. The set of predicate symbols typically are those that occur in some given finite set of formulas called *premises*. The finite set of predicate symbols is referred to as the *lexicon*. If L is a lexicon and $R \in L$, then $ar(R)$ denotes the arity of R. $ar(L) := \max\{ar(R) : R \in L\}$.

To provide examples (and counterexamples), the following lexicon will be used. $L_0 = \{P, R\}$, where $ar(P) = 1$ and $ar(R) = 2$. The syntax, of course, will depend on the logic under consideration.

A subformula is *prime* if it is atomic or of the form $\exists x\zeta$ or $\forall x\zeta$.

A first-order L-*structure* $\mathcal{A}$ consists of a set A, the *domain*, and a mapping that assigns to each $R \in L$ a subset $R^{\mathcal{A}} \subseteq A^{ar(R)}$. The class of L-structures is denoted $\mathbf{M}_L$. The notions of satisfaction and truth are standard. If ψ is a formula over L with free variables among $\{x_1, \dots, x_k\}$, and ψ is satisfied in $\mathcal{A}$ by the assignment of values to variables $\{x_i \mapsto a_i\}_{1 \leqslant i \leqslant k}$, we write $\mathcal{A}, a_1 \cdots a_k \models \psi$. If ψ is a sentence and ψ is true in $\mathcal{A}$, we write $\mathcal{A}, \varepsilon \models \psi$ or simply $\mathcal{A} \models \psi$. For convenience, α is taken to be a string in A^{ω}, called a *state*, and $\mathcal{A}, \alpha \models \psi$ has the meaning: $\psi[\alpha_i/x_i]$ is true in $\mathcal{A}$, where $\phi[\alpha_i/x_i]$ is ϕ with x_i replaced with the ith element of α. $\varepsilon \in A^*$ denotes the empty string, also a state.

Let ϕ be a formula of the logic under consideration. ϕ is *satisfiable* if there exists a structure $\mathcal{A}$ and a state α of $\mathcal{A}$ such that $\mathcal{A}, \alpha \models \phi$. ϕ is *satisfiable in* $\mathcal{A}$ if there exists a state α of $\mathcal{A}$ such that $\mathcal{A}, \alpha \models \phi$. ϕ is *true in* $\mathcal{A}$ if for every state α of $\mathcal{A}$: $\mathcal{A}, \alpha \models \phi$, written $\mathcal{A} \models \phi$. Observe that if ϕ is a sentence (no free variables), then for any state α: $\mathcal{A}, \alpha \models \phi$ iff $\mathcal{A} \models \phi$.

Let θ be a particular occurrence of a subformula of formula ϕ.

DEFINITION 1 The *polarity* (positive or negative) of θ is defined as follows.

(BASIS) If ϕ is one of $\theta \wedge \psi, \theta \vee \psi, \psi \rightarrow \theta, \exists x\theta$, or $\forall x\theta$ ($\phi = \theta \rightarrow \psi$), then θ is positive (negative) in ϕ.

(INDUCTION)

1. If $\phi = \psi$ and θ is positive (negative) in ψ, then θ is positive (negative) in ϕ.
2. If $\phi = \neg\psi$ and θ is positive (negative) in ψ, then θ is negative (positive) in ϕ.
3. If $\phi = \psi \wedge \rho$ or $\phi = \psi \vee \rho$, then if θ is a subformula of ψ and θ is positive (negative) in ψ, then θ is positive (negative) in ϕ; if θ is a subformula of ρ and θ is positive (negative) in ρ, then θ is positive (negative) in ϕ.
4. If $\phi = \psi \rightarrow \rho$, then if θ is a subformula of ψ, and θ is positive (negative) in ψ, then θ is negative (positive) in ϕ; if θ is a subformula of ρ and θ is positive (negative) in ρ, then θ is positive (negative) in ϕ.
5. If $\phi = \exists x\psi$ or $\phi = \forall x\psi$ and θ is positive (negative) in ψ, then θ is positive (negative) in ϕ. □

An important inference rule in first-order logic is the *Principle of Monotonicity*, embodied in the following theorem.

THEOREM 2 (THE PRINCIPLE OF MONOTONICITY) *Let* θ *be a positive (respectively, negative) occurrence of a subformula of formula* ϕ*, let* $\theta \rightarrow \rho$ *(respectively,* $\rho \rightarrow \theta$*) be a formula, and let* ϕ' *be obtained from* ϕ *by substituting* ρ *for that occurrence* θ *in* ϕ*. Then from* $\theta \rightarrow \rho$ *(respectively,* $\rho \rightarrow \theta$*),* $\phi \rightarrow \phi'$ *can be inferred.* □

Proof: The proof is a straightforward induction on the complexity of ϕ. ■

COROLLARY 3 *Let* θ *be a positive* (respectively, negative) occurrence of a subformula of formula ϕ, and let ϕ' be obtained from ϕ by substituting $\top$ (respectively, $\bot$) for that occurrence θ in ϕ. Then from ϕ, ϕ' can be inferred. □

To make this paper self-contained, the fragments to be investigated are defined in this section. In the following definitions, let L be a lexicon, $X_n = \{x_1, \ldots, x_n\}$ be an ordered set of variables, and $Y = \{x, y\}$.

DEFINITION 4 The grammar of the fluted fragment is defined recursively as follows.

1. If $R \in L$ with arity $m < n$ and $\mathbf{x} = x_{n-m+1} \cdots x_n$, then $R\mathbf{x}$ is a FL-formula over X_n.
2. If ϕ is a FL-formula over X_n, then $\neg\phi$ is a FL-formula over X_n.
3. If ϕ and ψ are FL-formulas over X_n, then $\phi \wedge \psi$ is a FL-formula over X_n.

4. If ϕ is a FL-formula over X_n where $n \geqslant 1$, then $\exists x_n \phi$ is a FL-formula over X_{n-1}. $\square$

Any alphabetic variant of a FL-formula is a FL-formula as well. Examples of FL-formulas with lexicon L_0 are $\forall x_1(Px_1 \to \exists x_2(Px_2 \wedge Rx_1x_2))$ and $\exists x(Px \wedge \forall y(Py \to \neg Rxy))$.

DEFINITION 5 The grammar of the two-variable fragment is

1. If $R \in L$ with arity m and $\mathbf{x} \in Y^m$, then $R\mathbf{x}$ is a FO^2-formula over Y.
2. If ϕ is a FO^2-formula over Y, then $\neg\phi$ is a FO^2-formula over Y.
3. If ϕ and ψ are FO^2-formulas over Y, then $\phi \wedge \psi$ is a FO^2-formula over Y.
4. If ϕ is a FO^2-formula over Y, then $\exists x\phi$ is a FO^2-formula over $Y - \{x\}$. $\square$

Some examples of FO^2-formulas with lexicon L_0 are $\forall x \forall y((Rxy \to Ryx) \wedge (Ryx \to Rxy))$, $\forall x(Rxx)$.

The *standard translation* st from basic modal logic to first-order logic is defined as follows.

1. $st(p) = Px$
2. $st(\neg\phi) = \neg st(\phi)$
3. $st(\phi \wedge \psi) = st(\phi) \wedge st(\psi)$
4. $st(\Diamond\phi) = \exists y(Rxy \wedge st(\phi)[y/x])$

The domain of st is *Basic Modal Logic* (i.e., unimodal logic). The range of st is the first-order modal fragment (MF). R is the accessibility relation.

DEFINITION 6 The grammar of the modal fragment is recursively defined

1. If $P \in L$ is a unary predicate, Px is a MF-formula.
2. If ϕ is a MF-formula, $\neg\phi$ is a MF-formula.
3. If ϕ and ψ are MF-formulas, then $\phi \wedge \psi$ is a MF-formula.
4. If ϕ is a MF-formula, then $\exists y(Rxy \wedge \phi[y/x])$ is a MF-formula. $\square$

Some examples of MF-formulas having lexicon L_0 are: $\neg Px$, $\exists y(Rxy \wedge Py)$.

DEFINITION 7 The grammar of the guarded fragment is given as follows.

1. If $R \in L$ with arity m and $\mathbf{x} \in Y^m$, then $R\mathbf{x}$ is a GF-formula over Y.
2. If ϕ is a GF-formula over Y, then $\neg\phi$ is a GF-formula over Y.

3. If ϕ and ψ are GF-formulas over Y, then $\phi \wedge \psi$ is a GF-formula over Y.

4. If ϕ is a GF-formula over Y, then $\exists x(G(x, y) \wedge \phi(x, y))$ and $\exists y(G(x, y) \wedge \phi(x, y))$ are GF-formulas over Y. $G(x, y)$ is an atomic formula with occurrences of x and y only and $\phi(x, y)$ is a GF-subformula. $G(x, y)$ is called the *guard* of the quantification. □

(This fragment is called *fragment 2* in Andréka et al. [1].) Some GF-formulas with the lexicon L_0 are: $\exists x(Ryx \wedge \exists y(Rxy \wedge Px))$ and $\exists y(Rxy \wedge \exists x(Px \wedge Rxy) \wedge \exists x(Py \wedge Rxy))$.

It should be noted that a fragment of a logic does not necessarily inherit the expressiveness of its parent. However, a fragment of a logic does inherit the inexpressiveness of the parent.

Since it is the objective of this paper to investigate the lack of expressiveness of first-order fragments, and since ML is a fragment of both FL and FO^2 and GF is a fragment of FO^2, in focusing on FL and FO^2 any result regarding inexpressiveness carries over to ML and GF as well.

3 HINTIKKA CONSTITUENTS

Hintikka devised constituents to be a generalization to FO of the *minterm* of Boolean logic [3]. In Boolean logic one proves that any Boolean formula is equivalent to a disjunction of minterms. Hintikka proved that any first-order formula is equivalent to a disjunction of constituents. For a review of constituent theory, see Rantala [5]. This section reviews the main results of Hintikka's constituent theory.

Let Φ be any set of prime formulas. A conjunction in which for each $\rho \in \Phi$ either ρ or $\neg\rho$ (but not both) occurs as a conjunct is a minimal conjunction over Φ. The set of minimal conjunctions over Φ will be denoted $\Delta\Phi$. It is well-known from Boolean logic that if $\Delta\Phi = \{\theta_1, \dots, \theta_l\}$, and ψ is any Boolean combination of formulas of Φ, then the following are tautologies.

1. $\neg(\theta_i \wedge \theta_j)$, for $i \neq j$
2. $\theta_1 \vee \cdots \vee \theta_l$
3. either $\theta_i \rightarrow \psi$ or $\theta_i \rightarrow \neg\psi$, for $1 \leqslant i \leqslant l$

Of particular interest is $\Phi = At_L(\mathbf{x})$, the set of atomic formulas of L over the variables $\mathbf{x}$ in the logic under consideration.

This is extended to FO. First define the following operations on sets of formulas. Let Θ be a set of formulas.

$$\neg\Theta := \{\neg\theta : \theta \in \Theta\}$$

$$\exists x\Theta := \{\exists x\theta : \theta \in \Theta\}$$

$\forall x\Theta := \{\forall x\theta : \theta \in \Theta\}$

DEFINITION 8 Constituents are defined inductively as follows. Let $\mathbf{x}$ be a set of variables and $y \notin \mathbf{x}$.

BASIS: $\Gamma_L^{(0)}(\mathbf{x}) := \Delta At_L(\mathbf{x})$

INDUCTION: $\Gamma_L^{(i+1)}(\mathbf{x}) := \{\theta \wedge \bigwedge \exists y\Theta \wedge \forall y \bigvee \Theta : (\theta \in \Delta At_L(\mathbf{x})) \wedge (\emptyset \neq \Theta \subseteq \Gamma_L^{(i)}(\mathbf{x}, y)\}$ □

A formula $\phi \in \Gamma_L^{(h)}(\mathbf{x})$ is a *constituent of* L *of height* h *over the variables* $\mathbf{x}$. If $\mathbf{x} = \emptyset$, then ϕ is a *constituent sentence*. As defined here, the height of ϕ is synonymous with the quantifier rank of ϕ ($qr(\phi)$). Height is used to suggest a tree representation of the constituent. (Note that in the literature relating to constituent theory, the term 'depth' is so used.)

Now the main results of constituent theory can be given.

THEOREM 9 (INCOMPATIBILITY PROPERTY) *If* ϕ *and* ψ *are constituents of* L *of height* h *over the variables* $\mathbf{x}$*, and* $\phi \neq \psi$*, then* $\phi \wedge \psi$ *is inconsistent.*

(EXHAUSTIVENESS PROPERTY) *The disjunction of all constituents of* L *of height* h *over the variables* $\mathbf{x}$ *is logically valid.* □

Proof: It suffices to observe first that $\theta \wedge \bigwedge \exists y\Theta \wedge \forall y \bigvee \Theta$ in the definition of $\Gamma_L^{(i+1)}(\mathbf{x})$ is equivalent to $\theta \wedge \bigwedge \exists y\Theta \wedge \bigwedge \neg\exists y(\Gamma_L^{(i)}(\mathbf{x}, y) - \Theta)$, and then to observe that this is a minimal conjunction over a set of prime formulas. ■

THEOREM 10 *Let* ψ *be a formula of* L *over* $\mathbf{x}$*, where* $qr(\psi) = r$*. Then for every* $n \geqslant r$*,* ψ *is logically equivalent to the disjunction of constituents* $\Gamma_\psi \subseteq \Gamma_L^{(n)}(\mathbf{x})$.

Proof: The proof is by induction on the complexity of ψ. ■

COROLLARY 11 *Let* ψ *be a formula of* L *over* $\mathbf{x}$*, where* $qr(\psi) = r$*. Let* ϕ *be a constituent of* L *of height* $n \geqslant r$ *over* $\mathbf{x}$*. Then either* $\phi \rightarrow \psi$ *or* $\phi \rightarrow \neg\psi$ *is logically valid.* □

Proof: The corollary follows immediately from Theorem 10 and the Incompatibility Property of constituents. ■

The constituent satisfied by a model can be used in conjunction with Padoa's Principle to obtain inexpressiveness results.

THEOREM 12 *If* $M_1, M_2 \in \mathbf{M}_L$ *such that* $M_1 \models \gamma_1 \in \Gamma_L^{(n)}()$ *and* $M_2 \models \gamma_2 \in \Gamma_L^{(n)}()$*, then there exists a sentence* ϕ *of quantifier rank* $\leqslant n$ *which can distinguish between* M_1 *and* M_2 *iff* $\gamma_1 \neq \gamma_2$. □

Proof: Let $n = \mathrm{ar}(L)$. Suppose $M_1, M_2 \in \mathbf{M}_L$ and ϕ is any sentence of quantifier rank $\leqslant n$. Let $M_1 \models \gamma_1 \in \Gamma_L^{(n)}()$ and $M_2 \models \gamma_2 \in \Gamma_L^{(n)}()$. Then ϕ cannot distinguish between M_1 and M_2 if $\gamma_1 = \gamma_2$, since if $M_1 \models \phi$, $M_1 \models \phi \wedge \gamma_1$ which implies $\models \gamma_1 \to \phi$, and so if $M_2 \models \gamma_1$ then $M_2 \models \gamma_1 \wedge (\gamma_1 \to \phi)$ whence $M_2 \models \phi$. If $\gamma_1 \neq \gamma_2$, then either γ_1 or γ_2 can distinguish between M_1 and M_2, The theorem follows. ■

Definition 8 shows that a constituent has a tree structure $\mathcal{T}$. The nodes θ_σ may be taken to be elements of $\Delta\mathrm{At}_L(\mathbf{x})$, where σ is an index such that $h(\sigma) = \mathrm{card}(\mathbf{x})$. With the understanding that the subtree $(\theta_\sigma]$ is interpreted

1. if σ is terminal, then $(\theta_\sigma]$ denotes θ_σ, and
2. if σ is nonterminal with height k, then $(\theta_\sigma]$ denotes $\theta_\sigma \wedge \exists x_{k+1}(\theta_{\sigma 1}] \wedge \cdots \wedge \exists x_{k+1}(\theta_{\sigma w(\sigma)}] \wedge \forall x_{k+1}((\theta_{\sigma 1}] \vee \cdots \vee (\theta_{\sigma w(\sigma)}])$,

the formula denoted by $(\theta_\sigma]$ is a constituent of L of height $h(\mathcal{T}) - h(\sigma)$ over the variables $\mathbf{x}$. If $h(\sigma) = 0$, the formula denoted by $(\theta_\sigma]$ is a constituent sentence.

Some concepts relating to trees and the indices of their elements are reviewed next. Let $\mathbb{P}^*$ be the set of finite strings over $\mathbb{P}$, the positive integers. String concatenation is denoted by juxtaposition. The empty string is ε.

A subset $\Sigma \subseteq \mathbb{P}^*$ provides indices for the elements of $\mathcal{T}$. Define the ***height of*** $\sigma \in \mathcal{T}$, $h(\sigma) :=$ the length of string σ. For all $\sigma, \tau \in \mathbb{P}^*, i \in \mathbb{P}$, if $\sigma i \tau \in \mathcal{T}$ then $\sigma i \tau$ is a *descendant of* σ and σi is an *immediate descendant of* σ. Define $w(\sigma) :=$ the number of immediate descendants of σ. Thus $\sigma 1, \sigma 2, \ldots, \sigma w(\sigma)$ are the immediate descendants of σ. If $w(\sigma) = 0$, then σ is *terminal* in $\mathcal{T}$. If all terminal elements of $\mathcal{T}$ have the same height, then $\mathcal{T}$ is *balanced*. In this case, $h(\mathcal{T}) := h(\sigma)$, where σ is any terminal element in $\mathcal{T}$. Define the ***depth of*** $\sigma \in \mathcal{T}$, $d(\sigma) := h(\mathcal{T}) - h(\sigma)$. If $0 < h(\sigma) < h(\mathcal{T})$, then σ is *internal* in $\mathcal{T}$. An element σ together with all of its descendants is defined to be the *subtree rooted on* σ, and is denoted $(\sigma]$.

Hintikka gives an easy test for inconsistency of constituents, based on omission of variables. If ϕ is a constituent, then $\phi^{[-k]}$ is defined to be ϕ with the last k variables omitted, and $\phi_{[-k]}$ is defined to be ϕ with the first k variables omitted. Here omission of a variable is accomplished by removing all atomic formulas in which that variable occurs, as well as the quantifier, if any, associated with that variable, and any connectives that thereby become idle. Semantically, the L- structures that satisfy a consistent constituent ϕ form a subclass of the L-structures that satisfy $\phi^{[-1]}$. The same result holds for $\phi_{[-1]}$. But as the following theorem shows, the superclass is the same in both cases.

THEOREM 13 *Either the constituent sentence ϕ is inconsistent or $\phi^{[-1]}$ and $\phi_{[-1]}$ are equivalent.* □

Proof: Since ϕ is a constituent sentence, $\phi \to \phi^{[-1]}$ and $\phi \to \phi_{[-1]}$ by the Principle of Monotonicity. Hence $\phi \to (\phi^{[-1]} \wedge \phi_{[-1]})$. Moreover, $\phi^{[-1]}$ and

$\phi_{[-1]}$ are constituent sentences of the same height. It follows from the Incompatibility Property that either $\phi^{[-1]}$ and $\phi_{[-1]}$ are equivalent (i.e., identical up to possible repetition of constituents, order of conjunction and disjunction, and change of variable), or ϕ is inconsistent. Hence the theorem follows. ▪

4 CHARACTERISTIC CONSTITUENT

It follows from the Exhaustiveness Property that if $\mathcal{A}$ is any L-structure, $\mathcal{A} \models \bigvee \Gamma_L^{(n)}()$. Further, because of the Incompatibility Property, $\mathcal{A} \models \phi$ for exactly one constituent $\phi \in \Gamma_L^{(n)}()$. Thus there exists a unique constituent ϕ of L of height $h = ar(L)$ for each L-structure $\mathcal{A}$ such that $\mathcal{A} \models \phi$. This unique constituent is called the *characteristic constituent* of the L-structure.

LEMMA 14 *Let* $\mathcal{A} = (A, F)$ *be an* L*-structure. Then a constituent* $\gamma \in \Gamma_L^{(ar(L))}()$*, unique to* $\mathcal{A}$*, can be constructed from* $\mathcal{A}$*.* □

Proof: γ is constructed as follows. Directed graph $G = (N, E)$, where nodes $N = \bigcup_{0 \leqslant i \leqslant ar(L)} A^i$ and edges $E = \{(\sigma, \sigma a) : \sigma \in (N - A^{ar(L)}) \wedge a \in A\}$. Obviously, G is a tree of height $ar(L)$. For $\sigma \in N$, define $\lambda(\sigma) := \theta_\sigma \in \Delta At_L(\mathbf{x})$, where $card(\mathbf{x}) = h(\sigma)$ and $\mathcal{A}, \sigma \models \theta_\sigma$. $\lambda(\varepsilon) := \top$.

Directed graph $G' = (N', E')$, where $N' = \{\lambda(\sigma) : \sigma \in N\}$ and $E'/Now = \{(\lambda(\sigma_1), \lambda(\sigma_2)) : \sigma_1, \sigma_2 \in N \wedge (\sigma_1, \sigma_2) \in E\}$. Obviously G' also is a tree.

Finally, merge nodes $\theta_{\sigma a}$ and $\theta_{\sigma b}$ if $(\theta_{\sigma a}] = (\theta_{\sigma b}]$ to form G''. G'' is a tree, and moreover G'' is a tree representation of a constituent γ of height $ar(L)$. This construction is deterministic, so γ is a constituent in $\Gamma_L^{(ar(L))}()$ unique to $\mathcal{A}$. ▪

THEOREM 15 *Let* $\mathcal{A}$ *be a* L*-structure, and let* γ *be the constituent unique to* $\mathcal{A}$ *constructed according to Lemma 14. Then* $\mathcal{A} \models \gamma$*.* □

Proof: The proof is by induction on the depth of θ_σ, with the induction hypothesis: for θ_σ at depth $k > 0$, σa is at depth $k' < k$ and $\mathcal{A}, \sigma a \models (\theta_{\sigma a}]$.

BASIS: $d(\sigma) = 1$, so $d(\sigma a) = 0$. By definition, $(\theta_{\sigma a}] = \theta_{\sigma a}$, and so $\mathcal{A}, \sigma \models \exists x_{h(\sigma a)} (\theta_{\sigma a}]$. Since this holds for all immediate descendants of σ, $\mathcal{A}, \sigma \models (\theta_\sigma]$.

INDUCTION: Let θ_σ be at depth k and so $\theta_{\sigma a}$ is at depth $k - 1$. By the induction hypothesis, $\mathcal{A}, \sigma a \models (\theta_{\sigma a}]$. By definition of $\exists$, for $1 \leqslant i \leqslant w(\theta_\sigma)$, $\mathcal{A}, \sigma \models \exists x_{k+1} (\theta_{\sigma a_i}]$. By definition of a constituent tree, $\mathcal{A}, \sigma \models (\theta_\sigma]$.

At depth $h(\mathcal{T})$, $\mathcal{A}, \varepsilon \models (\top]$, i.e., $\mathcal{A} \models (\top]$, i.e., $\mathcal{A} \models \gamma$. ▪

The characteristic constituent has further significance. To facilitate this discussion, let us introduce some notation introduced. Let $\tau = a_1 \cdots a_n$. Then $\tau_{\ominus i} := a_1 \cdots a_{i-1} a_{i+1} \cdots a_n$, i.e., τ with the ith element eliminated. Let

$\theta \in At_L(x_1 \cdots x_n)$ be a minimal conjunction. Then it follows that $\theta^{\ominus i} := \theta[x_{i+1}/x_i, \ldots, x_{n+1}/x_n]$.

Now let G be the graph defined in Lemma 14: $G = (N.E)$, where $N = \bigcup_{0 \leqslant i \leqslant ar(L)} A^i$ and $E = \{(\sigma, \sigma a) : \sigma \in (N - A^{ar(L)}) \wedge a \in A\}$. Define $H^{(n+1)}$ like G, except the nodes $M = \bigcup_{0 \leqslant i \leqslant n+1} A^i$ and the edges $F = \{(\tau, \tau a) : \tau \in (M - A^{n+1}) \wedge a \in A\}$. Define $\theta_\tau \in M' : \theta_\tau \in At_L(\mathbf{x}_{h(\tau)})$ and $\mathcal{A}, \tau \models \theta_\tau$. Similarly to E', F' is defined as $\{(\theta_\tau, \theta_{\tau a}) : \tau \in (M - A^{n+1}) \wedge a \in A\}$. Then $H'^{(n+1)} = (M', F')$. Suppose $n = ar(L)$. Then for $\theta_\tau \in H'^{(n+1)}$ at height k, $\theta_\tau := \bigwedge_{1 \leqslant i \leqslant k} \theta^{\ominus i}_{\tau_{\ominus i}}$. It is easy to see that if $\mathcal{A}, \tau_{\ominus i} \models \theta_\rho$, then $\mathcal{A}, \tau \models \theta^{\ominus i}_\rho$. Thus G' completely determines $H'^{(n+1)}$ (and G'' completely determines $H''^{(n+1)}$). Inductively, the characteristic constituent for which G'' is the tree representation completely determines every constituent $\gamma^{(n)}$ of height $n \geqslant ar(L)$ such that $\mathcal{A} \models \gamma^{(n)}$. The construction of $\gamma^{(n)}$ exactly follows the construction of the characteristic constituent. The proof that $\mathcal{A} \models \gamma^{(n)}$ follows the proof that $\mathcal{A} \models \gamma$, the characteristic constituent $\gamma \in \Gamma^{(ar(L))}_L()$, with the following additional observation. Any atomic formula ρ must contain $\leqslant ar(L)$ free variables. Therefore at height $ar(L) + 1$, if $\mathcal{A}, \tau \models \rho$, then $\mathcal{A}, \tau_{\ominus i} \models \rho$ for some i and hence ρ occurs in some $\theta^{\ominus i}_{\tau_{\ominus i}}$ that is used to construct θ_τ.

To illustrate, let the lexicon $L = L_0 = \{P, R\}$, where $n = ar(L) = 2$ and $P^{\mathcal{A}} = \{a, b, c\}, R^{\mathcal{A}} = \{aa, ab, ba\}$. Then $\theta_{aba} \in H'^{(3)}$. By construction, $\mathcal{A}, ab \models \theta_{ab} = Px_1 \wedge Px_2 \wedge Rx_1x_1 \wedge Rx_1x_2 \wedge Rx_2x_1 \wedge \neg Rx_2x_2$, $\mathcal{A}, aa \models \theta_{aa} = Px_1 \wedge Px_2 \wedge Rx_1x_1 \wedge Rx_1x_2 \wedge Rx_2x_1 \wedge Rx_2x_2$, and $\mathcal{A}, ba \models \theta_{ba} = Px_1 \wedge Px_2 \wedge \neg Rx_1x_1 \wedge Rx_1x_2 \wedge Rx_2x_1 \wedge Rx_2x_2$. In this event, $\theta_{aba} = \theta_{ab} \wedge \theta^{\ominus 2}_{aa} \wedge \theta^{\ominus 1}_{ba} = (Px_1 \wedge Px_2 \wedge Rx_1x_1 \wedge Rx_1x_2 \wedge Rx_2x_1 \wedge \neg Rx_2x_2) \wedge (Px_1 \wedge Px_3 \wedge Rx_1x_1 \wedge Rx_1x_3 \wedge Rx_3x_1 \wedge Rx_3x_3) \wedge (Px_2 \wedge Px_3 \wedge \neg Rx_2x_2 \wedge Rx_2x_3 \wedge Rx_3x_2 \wedge Rx_3x_3) = Px_1 \wedge Px_2 \wedge Px_3 \wedge Rx_1x_1 \wedge Rx_1x_2 \wedge Rx_1x_3 \wedge Rx_2x_1 \wedge \neg Rx_2x_2 \wedge Rx_2x_3 \wedge Rx_3x_1 \wedge Rx_3x_2 \wedge Rx_3x_3$.

Moreover, if $n < ar(L)$, say $n = ar(L) - k$, then the characteristic constituent $\gamma \to \gamma^{(n)}$ by the Principle of Monotonicity since $\gamma^{(n)} = \gamma^{[-k]}$. Thus the characteristic constituent indeed characterizes its L-structure $\mathcal{A}$.

5 EXPRESSIVENESS AND INEXPRESSIVENESS

Based on Theorem 12, several inexpressiveness results are obtained.

THEOREM 16 *The fluted fragment cannot express symmetry.* □

Proof: A counterexample is presented. Define model M_1: domain $A_1 := \{a, b\}$, $F_1(P) := A_1$, $F_1(Q) := \{ab, ba\}$. Relation Q^{M_1} is symmetric. Define model M_2: domain $A_2 := \{a, b\}$, $F_2(P) := A_2$, $F_2(Q) := \{ab, bb\}$. Relation Q^{M_2} is not symmetric. But M_1 and M_2 satisfy the same constituent, which can easily be seen by constructing the characteristic constituent. For both M_1

and M_2 the characteristic constituent is

$$\exists x_1(Px_1 \land \exists x_2(Px_2 \land Qx_1x_2) \land \exists x_2(Px_2 \land \neg Qx_1x_2) \land \\ \forall x_2((Px_2 \land Qx_1x_2) \lor (Px_2 \land \neg Qx_1x_2)))$$

From this counterexample, it follows that no fluted formula can express symmetry. ■

THEOREM 17 *The fluted fragment cannot express reflexivity.*

Proof: A counterexample is the following. Define model M_1: domain $A_1 := \{a, b\}$, $F_1(P) := A_1$, $F_1(Q) := \{aa, bb\}$. Relation Q^{M_1} is reflexive. Define model M_2: domain $A_2 := \{a, b\}$, $F_2(P) := A_2$, $F_2(Q) := \{ab, ba\}$. Relation Q^{M_2} is not reflexive. But M_1 and M_2 satisfy the same constituent, which again can easily be seen by constructing the characteristic constituent to obtain

$$\exists x_1(Px_1 \land \exists x_2(Px_2 \land Qx_1x_2) \land \exists x_2(Px_2 \land \neg Qx_1x_2) \land \\ \forall x_2((Px_2 \land Qx_1x_2) \lor (Px_2 \land \neg Qx_1x_2)))$$

From this counterexample, it follows that no fluted formula can express reflexivity. ■

THEOREM 18 *The fluted fragment cannot express transitivity.*

Proof: The proof is by counterexample. Define model M_1: domain $A_1 := \{a, b, c\}$, $F_1(P) := A_1$, $F_1(Q) := \{ab, bc, ac\}$. Relation Q^{M_1} is transitive. Define model M_2: domain $A_2 := \{a, b, c\}$, $F_2(P) := A_2$, $F_2(Q) := \{aa, ab, bc\}$. Relation Q^{M_2} is not transitive. But M_1 and M_2 satisfy the same constituent. Again, this can easily be seen by constructing the characteristic constituent, which construction yields

$$\exists x_1(Px_1 \land \exists x_2(Px_2 \land Qx_1x_2) \land \exists x_2(Px_2 \land \neg Qx_1x_2) \land \\ \forall x_2((Px_2 \land Qx_1x_2) \lor (Px_2 \land \neg Qx_1x_2))) \land \\ \exists x_1(Px_1 \land \exists x_2(Px_2 \land \neg Qx_1x_2) \land \forall x_2(Px_2 \land \neg Qx_1x_2))$$

From this counterexample, it follows that no fluted formula can express transitivity. ■

COROLLARY 19 *The modal fragment cannot express symmetry, reflexivity, or transitivity.* □

THEOREM 20 *The fragment* FO^2 *cannot express transitivity.* □

Proof: A counterexample is given. Define model M_1 as follows: domain $A_1 := \{a, b, c, d\}$, $F_1(P) := A_1$, $F_1(Q) := \{aa, ab, ba, bb, cc, dd\}$. Relation Q^{M_1} is transitive. Define model M_2 as follows: domain $A_2 := \{a, b, c, d\}$, $F_2(P) := A_2$, $F_2(Q) := \{aa, ab, ba, bb, bc, cb, cc, dd\}$. Relation Q^{M_2} is not transitive. But M_1 and M_2 satisfy the same constituent. This can easily be seen by constructing the characteristic constituent. The constituent is constructed as before, but θ_σ is an element of $\Delta At2_L$, the set of minimal conjunctions of atomic FO^2 formulas. The characteristic constituent is

$$\begin{aligned}&\exists x(Px \wedge \exists y(Py \wedge Qxx \wedge Qyy \wedge Qxy \wedge Qyx) \wedge \\ &\quad \exists y(Py \wedge Qxx \wedge Qyy \wedge \neg Qxy \wedge \neg Qyx) \wedge \\ &\quad\quad \forall y((Py \wedge Qxx \wedge Qyy \wedge Qxy \wedge Qyx) \vee \\ &\quad\quad\quad (Py \wedge Qxx \wedge Qyy \wedge \neg Qxy \wedge \neg Qyx)))\end{aligned}$$

From this counterexample, it follows that no FO^2 formula can express transitivity. ∎

COROLLARY 21 *The guarded fragment cannot express transitivity.* □

In a similar manner, it can be shown that antisymmetry is beyond the expressiveness of fluted logic.

6 DISCUSSION

The principal result of this paper is that each first-order structure determines a natural transition system. This transition system, appropriately labeled, in turn determines a constituent sentence characteristic of that structure. Two structures can be distinguished by a formula iff the constituents characteristic of the two structures differ. It easily follows from this result that as fragments of FO, FL and FO^2 suffer significant deficits in expressiveness. It is not possible to specify a partial order relation or an equivalence relation with either fluted logic or FO^2, and therefore these fragments present some difficulty in reasoning about mathematical entities that use partial orders or equivalences in an essential way. But fluted logic does excel in construal of syllogistic, and its extension to polyadic relations. That is, fluted logic is an excellent natural language reasoning environment. In syllogistic, the *is-a* relation is basic. For example: Every maple is-a tree is construed $\forall x(maple(x) \to tree(x))$, or in variable-free form, $\forall(maple \to tree)$. Again, No man is-a island (i.e., not: some man is-a island) is interpreted $\neg\exists x(man(x) \wedge island(x))$, or $\neg\exists(man \wedge island)$. This is extended to polyadic relations as exemplified by the *of-a* relation: Every cat is-a companion of-a human, which becomes $\forall x(cat(x) \to \exists y(companion(x, y) \wedge human(y)))$ The *of-a* relation is typical of relations that take a subject and an object, such as prepositions and transitive verbs. Relations of higher arity, such as verbs that take a subject, a direct object, and an indirect object (for example, give) are treated similarly. Example: A teacher gives a quiz to a student. becomes $\exists(teacher \wedge \exists(quiz \wedge \exists(student \wedge give)))$.

Thus, FL does have the expressiveness of syllogistic and can easily extend syllogistic to polyadic relations. Construal of natural language in FL has a naturalness and an intuitive nature that FO and most of its fragments lack. Moreover, reasoning in FL is decidable (Purdy [4]). At least one decision algorithm exists for FL (Schmidt and Hustadt [6]). Consequently, FL is particularly well-suited for construing natural language reasoning.

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