Tonk Strikes Back[*]

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1 INFERENTIAL ROLE SEMANTICS AND THE GHOST OF TONK

What is a logical constant? In which terms should we characterize the meaning of logical words like "and", "or", "implies"? An attractive answer is: in terms of their inferential roles, i.e. in terms of the role they play in building inferences. Logical words are first and foremost the basis for reasoning, so it seems reasonable that their meaning should be at least determined by the way they can be used in reasoning. Peculiarly, their meaning is fixed by a small class of special inferences: the rules that deal with each of them. We shall label this thesis the Inferential Role Thesis (IRT).

The following implementation of (IRT) is put forward by Došen [1989]: given a purely structural sequent system[1], each connective $\oslash$ is introduced through a specific double rule that translates a sequent with $\oslash$ as main connective in a formula into purely structural sequents. These double rules provide local analyses of logical constants of an object language, by contextually defining them in the metalanguage in which the deductive system is expressed. Considering conjunction again, the following double rule would do:

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[1]We give an example of such a calculus on page 36.

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge)$$

From top to bottom, it says that the conjunction can *link* two formulas derivable from the same multiset of sentences: the logical sequent obtained at the bottom of the rule abbreviates the two structural sequents on top; and bottom up, the rule analyses the meaning of $\wedge$: it indicates the way to go back to the structural metalanguage, once you have introduced the conjunction symbol. In this sense, a criterion for *logicality* of expressions is given: a constant is a logical constant if and only if it can be analysed in purely structural terms. This way of seeing things leads to:

CLAIM 1 *Logical constants are characterized by double rules translating logical expressions into structural expressions.*

Does the double rule analysis constitute a *normative* criterion, that is, a criterion that would ban pseudo logical constants, whose adjunction to a previously consistent deduction system S creates inconsistency? At first sight, one might think that the double-line phrasing encapsulates some form of harmony between upward and downward direction.[2] Unfortunately, this is not true. Prior gave rules for a fake logical connective, tonk, the addition of which to a consistent deduction system resulted in inconsistency [1960].[3] One can devise blonk, a close relative to Prior's tonk which is defined in terms of a double rule:

$$\frac{\Gamma \vdash A \quad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \text{ blonk } B}\ (\text{blonk})$$

Indeed, applying the upward direction of the rule to the axiom A blonk B $\vdash$ A blonk B, and taking $\Gamma = \{A \text{ blonk } B\}$ and Γ' empty, one can derive both sequents A blonk B $\vdash$ A and $\vdash$ B. The possibility to derive the second one amounts of course to the inconsistency of the system.[4] This shows that

[2]We take harmony in a non-technical sense which should nevertheless be reminiscent of dummettian ideas in a natural deduction setting.

[3]Tonk sequent rules are

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \text{ tonk } B}\ (\text{tonk-R}) \qquad \frac{B \vdash \Delta}{A \text{ tonk } B \vdash \Delta}\ (\text{tonk-L})$$

Their notorious effect is to make $A \vdash B$ derivable for any A and B (the proof is trivial). But if we just pick out one of them as a double rule introducing the constant tonk, no inconsistency will arise. To do the job and create an inconsistency in a double rule framework, we appeal to blonk. The inconsistency created by blonk is due to the arbitrary splitting of contexts in the upward direction. Technically, this amounts to artificially forcing the invertibility of a non invertible rule.

[4]Cook stresses the significance of the fact that tonk being pathological depends on the background consequence relation being transitive, by exhibiting a nearly reasonable non-transitive consequence relation to which tonk can be safely added [2005]. By contrast, the pathology of blonk depends only on the consequence relation being reflexive. It is clear that in order to derive inconsistency from some given rules, one has to rely on some properties of the consequence relation. Reflexivity is perhaps the most modest we can think of.

Došen's strategy does not tackle successfully the normativity issue. Even if it can block some fake connectives like tonk the rules of which are not given by a double rule, Blonk will not be banned. For Došen, this does not conflict with the idea of an analysis of *logicality*. Došen might argue there is indeed nothing wrong with the blonk rule, just because an analysis of what logical constants are does not have to provide a normative theory discriminating good logical constants from bad ones. Of course, this falls short of explaining what is the difference between blonk and the logical connectives we actually use.

On the other side of the picture, a very well-known solution to tonk-like difficulties has been given by Nuel Belnap in his 1961 paper. Belnap formulates a general criterion for the admissibility of new logical constants in a system of syntactic rules. To put it roughly, Belnap's restriction is to add only connectives that preserve good properties of the deducibility relation $\vdash$, *i.e.* connectives such that adding them to a previously given system yields a conservative extension of that system. Hence, under the hypothesis that the original system is a consistent one, tonk-like connectives are not admissible anymore.

Belnap's criterion does not address the logicality issue, but might help. Since we want to add a normative component to Došen's analysis, we could take conservativity to be just that component and say that double rules that actually define a logical constant are those that yield conservative extensions. Unfortunately, this is not fully satisfactory. Why not? One good thing with Došen's double-line idea is that it is expressed as a local property of the form of rules, so it would be nicer to have also a local phrasing of the normativity constraint. Compare the requirement of conservativity that simultaneoulsy applies to a rule introducing a new item and to a given background system of rules, to a local normative criterion that would just apply to a rule.

There is more than that though: Belnap's criterion only tells us that *in certain contexts*, certain rules are not good rules. But first, it is not true that in every contexts, logical rules have to be conservative. Dummett has shown that this can also happen in some particular cases with perfectly well-behaved rules.[5] And second, pathological constants like tonk can on the other hand be conservative in certain contexts. Though tonk will not be conservative over a consistent system, if it is added to a system in which $A \vdash B$ is already derivable for any A and B, it will be conservative. We are looking for a context-*in*sensitive criterion, unlike conservativity. Therefore, what we need to single out is a local property of the double rules that rules like the one for conjunction have and that the one for blonk do not have. In other words, we have to solve the following problem:

[5]Dummett makes the point in a natural deduction setting [1991, page 288–290]. If L has a disjunction with a restricted elimination rule, allowing no collateral assumptions in the subordinate deductions of the minor premises, adding an unrestricted disjunction will result in an extension L' which is not conservative over L.

PROBLEM I *Find a property* P *such that genuine logical constants are characterised by double rules enjoying* P.

2 A SOLUTION INSIDE BASIC LOGIC

2.1 THE BASIC LOGIC SETTING

To phrase this property, we will rely on the setting of Basic Logic (BL) as introduced by Sambin *et alii* [2000]: the interesting fact about BL for us now is that it uses a double rule analysis of logical constants in the setting of a sequent calculus. *Prima facie*, sequent systems are not part of the double rule picture, because sequent rules for a logical constant are left and right introduction rules. The point is that it is possible to see left and right introduction rules as derived from a double rule.

More precisely, (BL) is a very elementary sequent system that is conceived as the product of an interaction between an already given purely structural metalanguage $\mathfrak{L}$ and a logical language $\mathcal{L}$ that is built up step-by-step. We start from a structural sequent calculus $\mathfrak{S}$: its language is $\mathfrak{L}$, made of atomic formulae $A, B, \ldots$, multisets of formulae $\Gamma, \Delta, \ldots$, and one binary punctuation symbol, the comma. A *structure* is defined as a finite multiset of formulas, each formula separated by commas from the others. $\mathfrak{L}^\star$ is the set of structures defined over $\mathfrak{L}$-expressions. A derivability relation is defined over $\mathfrak{L}^{\star 2}$ and is denoted as usual by $\vdash$. Two *links* are expressible in $\mathfrak{S}$: *and*, and *yields*. For example the following derivation:

$$\frac{\Gamma, A \vdash B \qquad \Delta \vdash B}{\Gamma, A, \Delta \vdash B}$$

should be read:

(((Γ and A) yields B) and (Δ yields B)) yields ((Γ and A and Δ) yields B).

The rules of $\mathfrak{S}$ are identity axioms, exchange rules, and a split form of cut, right and left compositions. Right and left compositions (abbreviated as CUTL and CUTR) are immediately derived from the meaning attached to the link "yields":

$$\frac{\Gamma \vdash A \qquad \Gamma' \vdash \Delta}{\Gamma'(\Gamma/A) \vdash \Delta}\ \textsc{cut}L \qquad \frac{\Gamma \vdash \Delta' \qquad A \vdash \Delta}{\Gamma \vdash \Delta'(\Delta/A)}\ \textsc{cut}R$$

The left composition rule allows one, provided that $A \in \Gamma'$, to replace one occurrence of A by Γ in Γ'. The right composition rule allows one, provided that $A \in \Delta'$ to replace one occurrence of A by Δ in Δ'. To sum up the justification of these half-cut rules, one can say that when A appears on the left of "yields", it can be replaced by the set of hypotheses used to deduce A (the formulas that *yield* A), and when it appears on the right, it can be replaced by the set of its conclusions (the formulas that *are yielded* by A).

In BL, logical connectives are introduced according to a so-called reflection principle: each of them must reflect a specific structural link. This requirement is close to Došen's metaphor of logical constants as punctuation marks, but the picture will be given a much more precise meaning. In the terms of Sambin, introduction rules are the solution of a double rule construed as a definitional equation for a logical constant.

Let's see on an example how the solving process works. Take the following definitional equation:

$$\frac{A, B \vdash \Gamma}{A \otimes B \vdash \Gamma}\ (\otimes)$$

We first notice that the context aside the introduced formula is empty: this is called visibility principle, according to which BL-rules have to make their principal formulas *visible*, standing alone on their side of the turnstile[6]. Solving such an equation consists in giving the sequent calculus rules that govern the use of complex statements with $\otimes$ as principal connective on the left and on the right of $\vdash$. The aim is that the rules which solve the equation correspond to the two directions, upwards and downwards, of the equation. The downward sense of the double rule will always be conserved as a rule of Formation. Thus, in our case, we will have:

$$\frac{A, B \vdash \Gamma}{A \otimes B \vdash \Gamma}\ (\otimes\text{-Formation})$$

But in the other direction, it is impossible to keep

$$\frac{A \otimes B \vdash \Gamma}{A, B \vdash \Gamma}\ (\otimes\text{-Implicit Reflection})$$

because it is not an introduction rule, and therefore not a sequent calculus rule. It describes what is deducible from $A \otimes B$. But it constrains only *implicitly* the meaning of $\otimes$: it is a proper "Reflection rule" which indicates the way back to the structural metalanguage, but at the cost of presupposing a knowledge of the meaning of $A \otimes B$. We have to make explicit what remains implicit here, *i.e.* to find an introduction rule which is provably equivalent to the implicit rule.

How do we find such a $\otimes$-explicit reflection ($\otimes$-ER) rule? We can start from a $\otimes$-identity axiom: $A \otimes B \vdash A \otimes B$. Then, we apply IR to it:

$$\frac{A \otimes B \vdash A \otimes B}{A, B \vdash A \otimes B}\ (\otimes\text{-IR})$$

in order to have the $\otimes$-compound assertion appear on the other side of the sequent. The obtaining of a $\otimes$-reflection axiom ($A, B \vdash A \otimes B$) is the first step towards an ER right introduction rule equivalent to the IR elimination rule.

[6]The visibility principle also accounts for the splitting of cut just mentioned: in a sequent calculus enjoying visibility, full cut is not an admissible rule. See Sambin *et alii* [2000].

The ER rule just generalizes the reflection axiom: the idea is to put also the antecedent formulas A and B on the right side to get a full right rule. This is accomplished by two simple cuts:

$$\dfrac{\dfrac{\Gamma \vdash A \qquad A, B \vdash A \otimes B}{\Gamma, B \vdash A \otimes B} \qquad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \otimes B}$$

We are now in position to show that the two rules ⊗-IR and ⊗-ER:

$$\dfrac{A \otimes B \vdash \Gamma}{A, B \vdash \Gamma}\ (\otimes\text{-IR}) \qquad \dfrac{\Gamma \vdash A \qquad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \otimes B}\ (\otimes\text{-ER})$$

are (equivalent. We have just proved one direction. Now, suppose ⊗-ER a primitive rule of a system S. The following proof guarantees that ⊗-IR is admissible:

$$\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \otimes B}\ (\otimes\text{-ER}) \qquad A \otimes B \vdash \Delta}{A, B \vdash \Delta}$$

As one may notice, according to the definitional equations picture, conjunction (as well as disjunction) will split into multiplicative and additive connectives. Actually, the following definitional equation stands for the additive conjunction:

$$\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \& B}\ (\&)$$

The distinction between additive and multiplicative conjunction depends on the difference between a link "and" inside the scope of a link "yields" (multiplicative) or outside (additive). Of course, disjunction is subject to the same phenomenon.

Definitional equations can be formulated (see Sambin *et alii* [2000] for details) for the 0-ary connectives $\bot$, $\top$, 1, 0, but difficulties arise when it comes to implication. Remember that Basic Logic definitional equations and rules are *visible*: contexts aside principal and active formulas must be empty. The definitional equation for implication has the following form:

$$\dfrac{A \vdash B}{\vdash A \to B}\ (\to)$$

But this equation is not solvable: it is impossible to apply →-IR to the identity axiom $A \to B \vdash A \to B$ because →-IR only applies to sequents with empty antecedents. This raises the interesting issue of the good framework in which one is supposed to *formulate* definitional equations (keeping in mind that equations are always *solved* using only elementary means available in the structural calculus —see Definition 2 below)[7].

What is distinctive of the BL setting? First, the interplay between $\mathfrak{L}$ and $\mathcal{L}$, which implies that each logical constant mirrors a structural *link*, and which is guaranteed by the reflection principle. Then, the fact that what is performed on one side of the turnstile must also always be performed on the other side. A symmetry principle is implied in the definition of the class of the definitional equations (if an equation is solvable on the left, the corresponding equation on the right is also solvable). And finally, the emptiness of the contexts aside principal formulas in BL rules, coming from the visibility principle.

We can now state a precise description of the property P we are after. Belnap's criterion, conservativity, is fine, except that it is language relative and cannot be directly expressed as a property of definitional equations *per se*. We need to pinpoint some property of definitional equations that is responsible for the holding of conservativity in normal cases, and that blonk will not enjoy though it can be conservative over an already pathological language. Now, as long as one does not go into higher-order languages, it is well-known that conservativity results from the subformula property, which stems itself from cut-elimination. The problem of finding P reduces then to the problem of finding a "local" property, *i.e.* a property of definitional equations *per se* yielding cut elimination. To be fully precise, given the following definition,

DEFINITION 1 (CORRECTNESS) *A set of introduction rules for a logical constant is correct if and only if when they are added to a basic logical sequent calculus, the new system still enjoys cut elimination.*

P should guarantee that introduction rules derived from a P definitional equation are correct.

Fortunately, the proof of cut elimination is modular enough; to check that cut elimination is preserved through basic logical system, it is enough to check that essential cuts, *i.e.* cuts on formulas which have just been introduced, can be eliminated.[8] Therefore, P just has to guarantee that this can be done. Now,

[7]One can extend Basic Logic by adding structural rules or by liberalizing contexts. Liberalizing contexts yields Linear Logic, which would give the following *solvable* definitional equation for $\rightarrow$:

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta}\ (\rightarrow)$$

One would then have to show that formulating definitional equations with less constraints is harmless.

[8]Here, we simplify matters a bit. In particular, one has also to check that commutative steps, which allows one to raise the cut until introduction rules are met, can be performed. See Sambin *et alii* for details about cut elimination in BL and the family of calculus that can be obtained from BL [2000]. In BL, visibility, which prevents contexts from occurring aside a principal formula simplifies greatly the usual proof. The proof of cut elimination can be extended from BL to

the logical insight is that properly solving a definitional equation corresponds precisely to performing an essential reduction step. P can thus be phrased as the property of being solvable in the following sense:

DEFINITION 2 (SOLVABLE EQUATION) *An equation is solvable iff there exists an* ER *rule such that*

1. *there is an elementary proof of* ER *from* IR.
2. *there is an elementary proof of* IR *from* ER.

where an elementary proof is a proof which uses only elementary means (axioms, exchange rules, CUTL *and* CUTR)[9].

The following theorem holds:

THEOREM 1 *If a definitional equation is solvable, then the corresponding formation and explicit reflection rules correctly define a logical constant.*

Proof: We shall consider an arbitrary constant $\oslash$ whose definitional equation (E) has been properly solved. Without loss of generality, we assume that $\oslash$ is a binary connective and appears in (E) on the left of the sequent. (E) is of the form

$$\frac{\mathrm{Seq}_1(A, B/\Delta, \Delta')}{A \oslash B \vdash \Delta, \Delta'} \; (E)$$

where $\mathrm{Seq}_1(A, B/\Delta, \Delta')$ stands for one or two sequents whose premisses are among A and B and whose conclusions are structures among Δ and Δ'.

Now, suppose we have to deal with a principal cut on a formula $A \oslash B$, it looks like that:

$$\dfrac{\dfrac{\begin{matrix}\Pi_1 \\ \vdots \\ \mathrm{Seq}_1(A, B/\Delta, \Delta')\end{matrix}}{A \oslash B \vdash \Delta, \Delta'} \oslash F \qquad \dfrac{\begin{matrix}\Pi_2 \\ \vdots \\ \mathrm{Seq}_2(\Gamma, \Gamma'/A, B)\end{matrix}}{\Gamma, \Gamma' \vdash A \oslash B} \oslash \mathrm{ER}}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \; \mathrm{CUT}$$

$\mathrm{Seq}_2(\Gamma, \Gamma'/A, B)$ representing the premise(s) of the explicit reflection rule.[10]

Now, by hypothesis, (E) is properly solvable. Therefore, there is an elementary proof schema (Sol) which derives $\oslash$ER from $\oslash$IR.

logics with stronger structural rules.

[9] Here "elementary" is synonymous with "available in the structural calculus": definitional equations are solved within $\mathfrak{C}$.

[10] Actually, there might be more than one ER rule and therefore more than one form of essential cut. In this case, there will be several proof schemas giving the ER rules from the IR rules to deal with the different essential cuts.

Two observations on (Sol) are in order. First, since it is elementary and there is no introduction rule on the right for $\oslash$, every occurrence of a $\oslash$ on the right can be traced back to an axiom.

Second, we can assume that (Sol) contain only cuts on formula of lower complexity than $A \oslash B$. A cut like

$$\dfrac{\begin{matrix}\Pi_1 \\ \vdots \\ \Gamma \vdash A \oslash B\end{matrix} \qquad \begin{matrix}\Pi_2 \\ \vdots \\ A \oslash B \vdash \Delta\end{matrix}}{\Gamma \vdash \Delta}\ \textsc{cut}$$

can be eliminated in the following way. We know that the occurrence of $A \oslash B$ on the right comes from an axiom AX, therefore, we can uniformly substitute Δ to the occurrences of $A \oslash B$ coming from that axiom (we shall speak of linked occurrences)[11]. The axiom AX has be turned into $A \oslash B \vdash \Delta$, which is given by Π_2. Therefore, (Sol) is of the following form (boldface font marks linked occurrences):

$$\dfrac{\begin{matrix}\dfrac{A \oslash B \vdash \mathbf{A \oslash B}}{Seq_{1'}(A, B/\mathbf{A \oslash B})}\ \oslash\ \textsc{ir} \\ \vdots\end{matrix} \qquad Seq_2(\Gamma, \Gamma'/A, B)}{\Gamma, \Gamma' \vdash \mathbf{A \oslash B}}\ (Sol)$$

where $Seq_{1'}(A, B/\mathbf{A \oslash B})$ is an instantiation of $Seq_1(A, B/\Delta, \Delta')$ that is the (or one of) the sequent(s) $Seq_1(A, B/\Delta, \Delta')$ with $\mathbf{A \oslash B}$ standing for Δ (one of Δ and Δ').

Let's go back to our the essential CUT. Explicit reflection has been obtained through solving (E). Thanks to (Sol), we can obtain the following proof of the same sequent using implicit reflection instead of explicit reflection:

$$\dfrac{\dfrac{\begin{matrix}\Pi_1 \\ \vdots \\ Seq_1(A, B/\Delta, \Delta')\end{matrix}}{A \oslash B \vdash \Delta, \Delta'}\ \oslash F \qquad \dfrac{\begin{matrix}\dfrac{A \oslash B \vdash \mathbf{A \oslash B}}{Seq_{1'}(A, B/\mathbf{A \oslash B})}\ \oslash\ \textsc{ir}^{12} \\ \vdots\end{matrix} \qquad \begin{matrix}\Pi_2 \\ \vdots \\ Seq_2(\Gamma, \Gamma'/A, B)\end{matrix}}{\Gamma, \Gamma' \vdash \mathbf{A \oslash B}}\ (Sol)}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \textsc{cut}$$

Then we can glue the leftmost part of the proof on top of the axiom in order to obtain a proof of the final sequent without the last cut: we use the same trick as before and just substitute Δ, Δ' for $\mathbf{A \oslash B}$ in the consequence of the axiom and in linked occurrences downward:

[11] This kind of substitution on proof-trunks preserves the property of being a proof-trunk, as shown by the Substitution Lemma 4.1 in Sambin *et alii* [2000].

$$\dfrac{\begin{matrix} \dfrac{\dfrac{\begin{matrix}\Pi_1\\ \vdots\\ \mathrm{Seq}_1(A,B/\Delta,\Delta')\end{matrix}}{A \oslash B \vdash \Delta,\Delta'}\,\oslash\,\mathrm{F}}{\mathrm{Seq}_{1'}(A,B/\Delta,\Delta')}\,\oslash\,\text{IR} \\ \vdots \end{matrix} \qquad \begin{matrix}\Pi_2\\ \vdots\\ \mathrm{Seq}_2(\Gamma,\Gamma'/A,B)\end{matrix}}{\Gamma,\Gamma' \vdash \Delta,\Delta'}\ (\mathrm{Sol})$$

What we have is not a real proof, because $\oslash$ IR is not a sequent calculus rule. But the implicit reflection rule just follows a formation rule, so we can eliminate this roundabout step.

$$\dfrac{\begin{matrix}\Pi_1\\ \vdots\\ \mathrm{Seq}_{1'}(A,B/\Delta,\Delta')\\ \vdots\end{matrix} \qquad \begin{matrix}\Pi_2\\ \vdots\\ \mathrm{Seq}_2(\Gamma,\Gamma'/A,B)\end{matrix}}{\Gamma,\Gamma' \vdash \Delta,\Delta'}\ (\mathrm{Sol})$$

(Sol) contains only axioms and cuts on formulas of lower complexity than $A \oslash B$. Therefore, we have completed the desired reduction step. QED

The proof of Theorem 1 supports

CLAIM 2 *Solvable definitional equations do characterize genuine logical constants.*

because solvability yields the reduction step for essential cuts, the reduction step yields correctness, and correctness yields conservativity. Thus, solvability is clearly established as a sufficient condition for definitional equations to characterize a genuine logical constant. One could also ask if solvability is a necessary condition. This question can be raised in two different ways. First, granting that introduction rule should come from double rules, *i.e.* granting Claim 1, one could ask if our notion of solvability can be relaxed so that introduction rules could be more easily derived from double rules. Here, we shall note that without the equivalence of ER and IR, introduction rules cannot be said to be rules for the very constant characterized by the double rule. Now one could suggest to ease the elementarity requirement. But, as will be shown by the analysis of blonk, this is not compatible with the holding of Theorem 1. Second, one could challenge Claim 1 at the same time, and ask whether all introduction rules for which there principal cuts can be eliminated can be analyzed through a double rule. This amounts to asking for the converse of Theorem 1. It can indeed be proven using both a uniform substitution trick mimicking the one we used and a combinatorial analysis of the form of introduction rules.

3 THE PROBLEM WITH BLONK

Theorem 1 and Claim 2 provide us with a better understanding of what is wrong with blonk and tonk. Let's try to solve the blonk definitional equation. One can try either to give a strong or a weak ER rule. A strong ER rule will yield the IR rule, but it would make the system inconsistent, so Theorem 1 tells us that there is no elementary proof from IR to this ER rule. This strong ER would be something like the axiom

$$\frac{}{A \text{ blonk } B \vdash C}\ (\text{blonk ER})$$

It can be derived from IR,

$$\dfrac{\dfrac{A \text{ blonk } C \vdash A \text{ blonk } C}{\vdash C}\ (\text{IR})}{A \text{ blonk } B \vdash C}\ \text{Weak}$$

but this proof is not elementary, since weakening is used. On the contrary, a weak ER rule would not result in an inconsistent system. In that case, it will not be possible to get IR from ER. For example, one could try to think of blonk as some kind of additive conjunction:

$$\frac{A \vdash \Delta}{A \text{ blonk } B \vdash \Delta}\ \text{ER}$$

There is an elementary proof from IR to this ER rule. But, again, this is not a solution because there is no way back from it to the IR rule.

Claim 2 bans pathological definitional equations. But Theorem 1 tells us also something about pathological introductions rules, like the ones for tonk, namely that there is no definitional equation of which they are a solution. Claim 2 bans therefore pathological introduction rules as well.

As a conclusion, though there is no double rule for tonk, tonk strikes back in a double rule setting, because pathological double rules like the blonk rule do exist. But a definitional equation defines a genuine constant only if it can be solved. Fortunately, pathological rules cannot be solved. Thus the dark side of logical inference does not prevail.

One might ask further if our Claim 2 makes a point in favor of basic logic or linear logic as the one true logic. Would the special role of weakening we have just underlined constitute an argument against say, classical or intuitionistic logic? The idea would be to argue that logical constants are essentially "basic" objects on the ground that the structural parts of classical or intuitionistic logic is not an accurate setting for resolving definitional equations. But we do think there is a *non sequitur* here. The fact that weakening should not count as an elementary means is not an argument against weakening *per se*, it just shows that it is not part of what logicality is about, namely reflecting links in a structural calculus. Following Sambin [2000], the picture we have in mind is a pluralistic one in which basic logic as a common ancestor of a wide range of logics, which can be obtained from it by adding structural rules. Definitional equations show in which sense a logical constant remains the same in different structural contexts.

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