Semantic Decision Procedures for Some Relevant Logics

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Abstract: This paper proves decidability of a range of weak relevant logics using decision procedures based on the Routley-Meyer semantics. Logics are categorized as F-logics, for those proved decidable using a filtration method, and U-logics, for those proved decidable using a direct (unfiltered) method. Both of these methods are set out as reductio methods, in the style of Hughes and Cresswell. We also examine some extensions of the U-logics where the method fails and infinite sequences of worlds can be generated.

In *Relevant Logics and their Rivals* [12, pages 399–406], Richard Routley (as he was then known) purported to have established the decidability of a range of weak relevant logics. He first claimed to have proved decidability for the system B, on pp. 401-2, using a filtration method upon the Routley–Meyer semantics, which yielded the finite model property for B. After warning that the method would by no means extend to all the systems with semantic postulates given in Chapter 4 and 5 of [12], he extended the result to the postulate $Raaa$, for the axiom $A \,\&\, (A \to B) \to B$, the postulate Raa^*a, for $A \to {\sim}A \to {\sim}A$, and also to the postulate, if $Rabc$ then Rac^*b^*, for $A \to {\sim}B \to .B \to {\sim}A$. There are also a couple of less significant postulates given. Routley then examined the use of a second filtration, but realized its shortcomings in establishing postulate P2 — if $a \leq b$ and $Rbcd$ then $Racd$ — for the finite model. He briefly examined a third filtration, realized that it would not yield a decidability result without deductive closures on the worlds of the finite models, and hence suggested using operational semantics instead of the Routley–Meyer semantics.

However, Routley had erred in two places in his first filtration. Fortunately for those of us interested in weak relevant logics and Routley–Meyer-style semantics, his second filtration can still be used and his results can be revived and extended using the simplified semantics of Priest and Sylvan [10] and Restall

[11], which has subsequently become available. The object of §3 of this paper is to show how this is to be done. In the process, I do not want to detract from what has otherwise been an outstanding volume, of which all of us in the field make heavy use.

It must also be said that Fine had first proved the decidability of weak relevant logics without $A \to B \to .B \to C \to .A \to C$ and $A \to B \to .C \to A \to .C \to B$ in [6], pp. 365–8, using his own semantics which embraces both theories and prime theories. The above postulate P2 simplifies to — (ii) if $t \leq u$ then $(t \cdot v) \leq (u \cdot v)$ [6, page 348] — which enables the decidability proof to go through without anything resembling the simplified semantics of Priest, Sylvan and Restall.

Moreover, a different range of systems have been shown decidable using proof-theory in Brady [3,4,5], viz. the contraction-less logics, DW, TW and RW, by using Gentzen systems based on the work of Dunn [1, pages 381–391] and Giambrone [7]. This suggests that there may be a semantic method, based on a Routley-Meyer semantics, for establishing the same result, which might then extend to other systems. We will show this in §5 of the paper, indicating what goes wrong with some of these extensions.

1 The Logics

We present axiomatizations for the main logics referred to in this paper: B, BX, BC, DW, DWX, DWC, DJ, DK, DL, D, DC, RBC, TW, TWX, TWC and RW, together with their disjunctive rule extensions. We take as primitives: $\sim$, &, $\vee$, $\to$, and we consider the following axioms, rules and meta-rule, and the class of logics constructed using them, in Figures 1 and 2.[1] (Each of the subtracted axioms and rules are redundant in the respective system.)

For any logic L, let L^d be L + MR1. For each of the listed logics L without A12, $L = L^d$, i. e., they have the same set of theorems. (See Slaney [13] and [14] for this result.)

We distinguish two kinds of logics for the purposes of this paper : An F-logic is one of the following: B^d (or B), DW^d (or DW), DJ^d (or DJ), DK^d, DL^d, D^d, DC^d, RBC^d. These will used in §3 for filtrations and in §4 for the reductio method based on these.

A U-logic is one of the following : B^d (or B), BX^d, BC^d, DW^d (or DW), DWX^d, DWC^d, TW^d (or TW), TWX^d, TWC^d, RW^d (or RW). These will be used in §5 for the reductio method without filtrations. Since simplified semantics are used throughout, disjunctive rules are always required.

[1] We use 'RBC' to represent the logic R, without the two B-axioms, A16 and A17, but with the other half of classicality, R5.

A1	$A \to A$
A2	$A \& B \to A$
A3	$A \& B \to B$
A4	$(A \to B) \& (A \to C) \to .A \to B \& C$
A5	$A \to A \vee B$
A6	$B \to A \vee B$
A7	$(A \to C) \& (B \to C) \to .A \vee B \to C$
A8	$A \& (B \vee C) \to (A \& B) \vee (A \& C)$
A9	$\sim\sim A \to A$
A10	$A \to \sim B \to .B \to \sim A$
A11	$(A \to B) \& (B \to C) \to .A \to C$
A12	$A \vee \sim A$
A13	$A \to \sim A \to \sim A$
A14	$(A \to .A \to B) \to .A \to B$
A15	$A \& (A \to B) \to B$
A16	$A \to B \to .B \to C \to .A \to C$
A17	$A \to B \to .C \to A \to .C \to B$
A18	$A \to .A \to B \to B$
R1	$A, A \to B \Rightarrow B$
R2	$A, B \Rightarrow A \& B$
R3	$A \to B, C \to D \Rightarrow B \to C \to .A \to D$
R4	$A \to \sim B \Rightarrow B \to \sim A$
R5	$\sim A, A \vee B \Rightarrow B$
MR1	If $A \Rightarrow B$ then $C \vee A \Rightarrow C \vee B$

Figure 1: Axioms, Rules and Meta-Rule

2 The Simplified Semantics

Priest and Sylvan [10] introduced a simplified version of the Routley–Meyer semantics for the logic B^+ and then B, the essential feature being the removal of the defined ordering relation '$\leq$' for the reduced semantics and the use of the following special truth-condition for evaluating entailments $A \to B$ at the base world T, viz.

- $I(A \to B, T) = T$ iff, for all $b \in K$, if $I(A, b) = T$ then $I(B, b) = T$.

The usual Routley–Meyer truth-condition applies for $A \to B$ for worlds $a \neq T$.

Then, Restall [11] extended this simplified semantics to a wide range of relevant and other logics. However, he did not use the special truth-condition for $A \to B$, but instead added the semantic postulate, $RTab \Leftrightarrow a = b$, which simplifies some of the other postulates, together with the corresponding soundness and completeness arguments. For our purposes, it is Restall's form of the semantics that works best and so we set it out for our logics of interest. We start with the logic B^d, which has the same theorems as B.

B	A1–9 + R1–4
BX	B + A12
BC	BX + R5
DW	B + A10 − R4
DWX	DW + A12
DWC	DWX + R5
DJ	DW + A11
DK	DJ + A12
DL	DK + A13 − A12
D	DL + A14 + A15
DC	D + R5
RBC	DC + A18
TW	DW + A16 + A17 − R3
TWX	TW + A12
TWC	TWX + R5
RW	TW + A18 − A17

Figure 2: Systems

A B model structure (BMS) consists of the concepts $\langle T, K, R, * \rangle$, where K is a set of worlds, the base world $T \in K$, R is a 3-place relation on K, and $*$ is a 1-place function on K, subject to the semantic postulates: for $a, b \in K$:

P1 $a^{**} = a$.

P2 $RTab \Leftrightarrow a = b$.

A B-model is a BMS with valuation v, which assigns a truth-value T or F to each sentential variable for each world. Each valuation is uniquely extended to an interpretation I by induction on formulae, as follows: for $a \in K$, sentential variable p, and formulae A, B:

(i) $I(p, a) = v(p, a)$.

(ii) $I({\sim}A, a) = T$ iff $I(A, a^*) = F$.

(iii) $I(A \mathbin{\&} B, a) = T$ iff $I(A, a) = T$ and $I(B, a) = T$.

(iv) $I(A \vee B, a) = T$ iff $I(A, a) = T$ or $I(B, a) = T$.

(v) $I(A \rightarrow B, a) = T$ iff, for all $b, c \in K$, if $Rabc$ and $I(A, b) = T$ then $I(B, c) = T$.

A formula A is *true* in a B-model iff $I(A, T) = T$, and A is *valid* in a BMS iff A is true in every B-model on that BMS. A is *valid in the simplified semantics for* B iff A is valid in every BMS.

THEOREM 1 For formulae A, if A is a theorem of B^d (or B) then A is valid in the simplified semantics for B.

PROOF The usual soundness theorem follows as in [12], but with P2 simplifying the proof of the *Entailment Lemma* (Lemma 4.2 on page 302) for reduced modelling. For MR1, see *Relevant Logics and their Rivals* [12, pages 336-7], or Priest and Sylvan [10, pages 220-1]. ♯

THEOREM 2 For formulae A, if A is valid in the simplified semantics for B then A is a theorem of B^d (or B).

PROOF The completeness theorem is generally standard [12], but with two essential differences. First, the canonical base world T_B is constructed so that each rule of B^d is closed in it, with help from MR1. (This construction was used in Brady [2, pages 252–256], with quantifiers.) Second, to prove $R_B T_B ab \Leftrightarrow a = b$ in the canonical model structure, Restall introduces (for B^+) [11, page 490], a copy of T_B, called T'_B, which has the same members as T_B, but satisfies the above equivalence for $a, b \in K_B$. $R_B abc$ is then defined normally for $a \neq T'_B$. Then, in proving $I_c(A \rightarrow B, a) = T$ iff $A \rightarrow B \in a$, for the canonical interpretation I_c, we need to consider the special case, $a = T'_B$. However, for B, with negation, we also need a copy of T^*_B, viz. $T^{*\prime}_B$, so as to satisfy P1.[2] $R_B T^{*\prime}_B ab$ is just defined as $R_B T^*_B ab$, $T^{*\prime}_B$ and T^*_B being regarded as distinct elements of K_B. ♯

We extend these soundness and completeness results to the F-logics and U-logics, defined in §1. Figure 3 lists the semantic postulate corresponding to each additional axiom and rule.

THEOREM 3 For formulae A, if A is a theorem of one of the F-logics or U-logics L then A is valid in the simplified semantics for L.

PROOF Restall [11, pages 484–6] showed soundness for A11, A14, A15, A16, A17 and A18. After introducing an extra semantic primitive, '$\leq$', with postulates and valuation condition below, Restall [11, pages 506–7], showed soundness for A10, A12 and A13. Soundness for A5 follows easily. '$\leq$' is a 2-place relation on K, satisfying the following: For $a, b \in K$, and sentential variables p,

P13 $a \leq b \Rightarrow b^* \leq a^*$

P14 $a \leq b \;\&\; Rbcd \Rightarrow Racd$.[3]

VC If $a \leq b$ and $v(p, a) = T$ then $v(p, b) = T$.

The theorem, 'if $a \leq b$ and $I(A, a) = T$ then $I(A, b) = T$', then follows by induction on formulae A [12, page 302].

[2] Such an addition as $T^{*\prime}_B$ was used by Giambrone and Meyer [8, pages 11–12].

[3] Restall breaks this up into two cases: $a = T$ and $a \neq T$, but this does not appear to be needed here.

Additional Axioms and Rule		Corresponding Postulate	
A10	$A \to {\sim}B \to .B \to {\sim}A$	P3	$Rabc \Rightarrow Rac^*b^*$
A11	$(A \to B) \& (B \to C) \to .A \to C$	P4	$Rabc \Rightarrow \exists x \in K, Rabx \& Raxc$
A12	$A \vee {\sim}A$	P5	$T^* \leq T$
A13	$A \to {\sim}A \to {\sim}A$	P6	(i) Raa^*a, for $a \neq T$, and (ii) $T^* \leq T$
A14	$(A \to .A \to B) \to .A \to B$	P7	$Rabc \Rightarrow \exists x \in K, Rabx \& Rxbc$
A15	$A \& (A \to B) \to B$	P8	$Raaa$
A16	$A \to B \to .B \to C \to .A \to C$	P9	$Rabz \& Rzcd \Rightarrow \exists x \in K, Racx \& Rbxd$
A17	$A \to B \to .C \to A \to .C \to B$	P10	$Rabz \& Rzcd \Rightarrow \exists x \in K, Rbcx \& Raxd$
A18	$A \to .A \to B \to B$	P11	$Rabc \Rightarrow Rbac$
R5	${\sim}A, A \vee B \Rightarrow B$	P12	$T \leq T^*$

[When A12 and R5 are included, replace P6 (ii) and P12 by $T = T^*$ and simplify P6 to Raa^*a.]

Figure 3: Semantic Postulates Corresponding to Axioms and Rules

However, we only require this process for $T^* \leq T$ and $T \leq T^*$. In which case, P13 is automatically satisfied, given P1, and P14 reduces to the two respective forms, given P2:

P14A $T^* \leq T \Rightarrow RT^*aa$

P14B $T \leq T^* \& RT^*ab \Rightarrow a = b$

So, in the event that $T^* \leq T$ is required, i. e. for A12 and A13, we add:

P15 RT^*aa

and the valuation condition:

VC1 If $v(p, T^*) = T$ then $v(p, T) = T$, for all p

and, in the event that $T \leq T^*$ is required, i. e. for R5, we add the postulate:

P16 $RT^*ab \Rightarrow a = b$

and the valuation condition:

VC2 If $v(p, T) = T$ then $v(p, T^*) = T$, for all p

though these are not required on their own, for the indicated logics. ♯

THEOREM 4 For formulae A, if A is valid in the simplified semantics for one of the F-logics or U-logics L then A is a theorem of L.

PROOF Completeness is somewhat more complicated because of the split definition of the canonical relation $R_L abc$, for the logics L. We also define $a \leq b$, for $a, b \in K_L$, as $a \subseteq b$.

For the logics, B^d, BX^d, BC^d, DW^d, DWX^d, DWC^d, DJ^d, DK^d, DL^d, D^d, DC^d, TW^d, $TWX^d$and TWC^d, this is relatively straight-forward with postulates P3, P4, P6, P7, P8, P9 and P10 requiring additional checking when $a = T_L'$, as in Restall [11, pages 492–494, 507–8]. For P9 and P10, we also need to check $b = T_L'$ and $z = T_L'$, but $b = T_L$ and $z = T_L$ do the required work in each case. (Note that P8 is required for P7, but A15 is derivable from A14 anyway.) Also, we can see that $R_L T_L^{*\prime} aa$ holds for logics with A12, since $T_L^* \subseteq T_L$ and $R_L T_L aa$. Further, for logics with A12 and A5, we put $T_L^{*\prime} = T_L'$ since $T^* = T$.

However, there is a special definition for R_L for logics L containing A18. We will assume, for the purposes of the logic RBC^d, that A10–15 and R5 are included in L, as there is much interlinkage between the various axioms and the rule in re-establishing the postulates. Also, for the purposes of RW^d, we will assume that A10 is included in L. We define R_L as:

$$R_L T_L' bc \Leftrightarrow b = c, \qquad R_L a T_L' c \Leftrightarrow a = c, \qquad R_L ab T_L' \Leftrightarrow a = b^*$$

$R_L abc$ is then defined normally for $a \neq T_L'$, $b \neq T_L'$ and $c \neq T_L'$. In checking the postulates, P3 and P11 are fine, P8 and the unrestricted P6 use $T_L^{*\prime} = T_L'$, P4 uses P3 and P8, P7 uses P6 and P8, and P9 and P10 both use P3 and P11.

In checking the proof of '$I_c(A \rightarrow B, a) = T$ iff $A \rightarrow B \in a$', we show that $A \rightarrow B \in a$ iff, for all $b, c \in K_L$, if $R_L abc$ and $A \in b$ then $B \in c$. In the case $a \neq T_L'$, $A \rightarrow B \in T_L'$ iff, for all $b \in K_L$, if $A \in b$ then $B \in b$, as shown in [10,11] for B^+. For the case $b = T_L'$, if $A \rightarrow B \in a$, $a = c$, $A \in T_L'$ then $B \in c$, since $A \rightarrow .A \rightarrow B \rightarrow B$ is a theorem of L and a is a $T_L - L$-theory.[4] The converse direction is no problem since T_L will always suffice for T_L'. (Also see Restall [11, page 485].) In case $c = T_L'$, if $A \rightarrow B \in a$, $a = b^*$, $A \in b$ then $B \in T_L'$, since $\sim B \rightarrow .A \rightarrow B \rightarrow \sim A$ is a theorem. ♯

3 THE FILTRATION METHOD

We follow *Relevant Logics and their Rivals* [12, pages 399–403] in setting up Routley's first filtration and show up the two defects in the argument. We then move onto the second filtration, but employing the simplified semantics of §2 for the systems indicated. Generally, the filtration method finitizes the set of worlds of a model by setting as equivalent worlds ones that evaluate a certain finite set X of formulae the same way. This set X is generally the set of subformulae of a given formula under test, closed under the negation of unnegated formulae. The idea here is to show that the logic concerned has the finite

[4]For the definition of a $T - L$-theory, see [12, page 306].

model property, i. e. that each non-theorem can be falsified in a finite model. This yields decidability for the logic by a standard argument [12, pages 401-2].

For an F-logic L, we define an appropriate closure X of a formula A as the set of all subformulae of A closed under the condition: if B is a member of X with its principal connective not being a negation then ~B is a member of X. We consider a Routley-Meyer model $\langle T, O, K, R, *, \leq, \nu \rangle$, where K is a set of worlds, $T \in O$, $O \subseteq K$, R is a 3-place relation on K, $*$ is a 1-place function on K, $\leq$ is a 2-place relation on K, subject to the set of postulates below for the logic L, for $a, b \in K$,

P1 $a \leq a$,

P2 $a \leq b$ and $Rbcd \Rightarrow Racd$,

P3 $a \leq b \Rightarrow b^* \leq a^*$,

P4 $a = a^{**}$,

P5 $a \in O, Rabc \Rightarrow b \leq c$,

P6 $(\exists x \in O)Rxaa$,

(plus any extra postulates appropriate to the F-logic L) and ν is a valuation assigning T or F to sentential variables p at each world $a \in K$, subject to the condition: for sentential variables p, if $a \leq b$ and $\nu(p, a) = T$ then $\nu(p, b) = T$.

We define, for worlds $a, b \in K$ and appropriate closure X, $a =_X b$ for $(\forall B \in X)(I(B, a) = I(B, b))$. We call such a and b X-equivalent. We note that $a =_X b$ iff $a^* =_X b^*$. We define X-equivalence classes $\check{a}(X)$ as $\{b \in K : a =_X b\}$, generated by the element a. We call these classes X-worlds and form the set $\check{K}$ of such worlds, i. e. $\check{K}(X) = \{\check{a}(X) : a \in K\}$. We will standardly leave off '(X)' to simplify terminology. Further, $\check{K}$ is finite, as each X-world represents a valuation of the set of formulae in X, and so if X has n formulae then there are at most 2^n valuations of X and hence at most 2^n members of $\check{K}$.

We go on to define a (finite) X-model $\check{M}$ in $\check{K}$ which is Routley's first filtration: $\langle \check{T}, \check{O}, \check{K}, \check{R}, \check{*}, \check{\leq}, \check{\nu} \rangle$. $\check{T}$ is already given, $\check{a}^{\check{*}} =_{df} (a^*)\check{}$, and $\check{O}$, $\check{R}$, $\check{\leq}$, and $\check{\nu}$ are given as follows:

- $\check{a} \in \check{O}$ iff $(\forall B \in X)I(B \to B, a) = T$,
- $\check{R}\check{a}\check{b}\check{c}$ iff $(\forall B, C \in X)(B \to C \in X \;\&\; I(B \to C, a) = T \;\&\; I(B, b) = T \Rightarrow I(C, c) = T)$,
- $\check{a}\check{\leq}\check{b}$ iff $(\forall B \in X)(I(B, a) = T \Rightarrow I(B, b) = T)$,
- $\check{\nu}(p, \check{a}) = T$ iff $\nu(p, a) = T$ and $p \in X$.

The first problem is that the definition of $\check{a} \in \check{O}$ does not seem independent of the choice of the element a in $\check{a}$. One could have $a =_X b$, $I(B \to B, a) = T$ and $I(B \to B, b) = F$, for some $B \in X$ such that $B \to B \notin X$, since the formula B can be evaluated at worlds outside of $\check{a}$.

The second problem is in checking P5, we again seem to need $B \to B \in X$, when $B \in X$. We let $B \in X$ and $I(B, b) = T$, and, since $\check{a} \in \check{O}$, $I(B \to B, a) = T$. Since $\check{R}\check{a}\check{b}\check{c}$ we would need $B \to B \in X$ to derive $I(B, c) = T$, as required for $\check{b}\check{\leq}\check{c}$.

One could try replacing the definition of $\check{a} \in \check{O}$ by $(\forall B \to B \in X)I(B \to B, a) = T$. This solves the first problem but P5 still does not follow. One could additionally try putting $(\forall B \in X)(B \to B \in X \;\&\; I(B, a) = T \Rightarrow I(B, b) = T)$ for $\check{a}\check{\leq}\check{b}$. This then solves the two problems but creates a new one for P2. Clearly, we cannot add the condition 'if $B \in X$ then $B \to B \in X$' without losing the finitude of X.

We could try to remove $\check{O}$ by using reduced models, i. e. by putting $\check{O} = \check{T}$, but P5 is still a problem for the original definition of $\check{\leq}$ and P2 is still a problem for the revised definition of $\check{\leq}$. So, we consider the simplified semantics, without $\check{O}$ and $\check{\leq}$, but retaining the defintion of $\check{R}$. This does overcome the problems with the initial postulates, and does provide finite modelling for such systems as B^d and DW^d, as set out in [12, pages 400–3], but does not cope with any of the existential postulates, such as P4 and P7 of §2, which would yield systems such as DJ^d and D^d.

So, we move on to the second filtration [12, pages 403-4], which is defined as for the first filtration, but with the following definitions of $\check{O}$, $\check{R}$ and $\check{\leq}$:

- $\check{a} \in \check{O}$ iff $(\exists a \in \check{a})a \in O$,
- $R\check{a}\check{b}\check{c}$ iff $(\exists a \in \check{a})(\exists b \in \check{b})(\exists c \in \check{c})Rabc$,
- $\check{a}\check{\leq}\check{b}$ iff $(\exists a \in \check{a})(\exists b \in \check{b})a \leq b$.

However, as pointed out in [12, page 404], there is a problem with ensuring that the two bs in P2: $a \leq b \;\&\; Rbcd \Rightarrow Racd$, are the same after the definitions of $\check{a} \leq \check{b}$ and $\check{R}\check{b}\check{c}\check{d}$ are applied. We overcome this problem with the adoption of the simplified semantics, thus dropping 'O' and '$\leq$' (except for logics containing A12 but not R5). Moreover, there is a similar problem with the two postulates '$Rabz \;\&\; Rzcd \Rightarrow (\exists x \in K)(Rbcx \;\&\; Raxd)$' and '$Rabz \;\&\; Rzcd \Rightarrow (\exists x \in K)(Racx \;\&\; Rbxd)$' in ensuring that the two zs are the same after the definitions of $\check{R}\check{a}\check{b}\check{z}$ and $\check{R}\check{z}\check{c}\check{d}$ have been applied. So, we proceed to apply the second filtration to the F-logics, B^d, DW^d, DJ^d, DK^d, DL^d, D^d, DC^d and RBC^d, which do have simplified semantics and do not use these two postulates.

There is another similar problem for the postulate P2 of §2, viz. $RTab \Leftrightarrow a = b$, in that if $\check{R}\check{T}\check{a}\check{b}$ is assumed then, for some $a, b \in K$, $RTab$ may not hold for the specific base world T in $\check{T}$. We overcome this by distinguishing in $\check{K}$

between $\check{T}$ and $\dot{T}$, where $\check{T}$ is $\{T\}$ and $\dot{T}$ is $\{b \in K : T =_X b\} - \{T\}$. We also add $\check{T}^{\check{*}}$ which is $\{T^*\}$, to be distinguished in $\check{K}$ from $\dot{T}^{\check{*}}$ which is $\{b \in K : T^* =_X b\} - \{T^*\}$. [If $\dot{T}$ is null, then delete $\dot{T}$ and $\dot{T}^{\check{*}}$ throughout the following argument.]

The base world in the second filtration is then taken to be $\check{T}$. We put $\check{T}^{\check{*}\check{*}}$ as $\check{T}$, and $\dot{T}^{\check{*}\check{*}}$ as $\dot{T}$. We will continue to use $\check{a}$ to represent elements of $\check{K}$ and $a \in \check{a}$ to represent elements of K which are members of $\check{a}$. For the definitions of $\check{R}\check{T}\check{a}\check{b}$, $\check{a} \check{\leq} \check{b}$ and $\check{v}(p, \check{a})$, we let the forms $a \in \check{a}$, etc. hold when $T \in \check{T}$ and $T \in \check{T}^{\check{*}}$, and for $\dot{T}$ and $\dot{T}^{\check{*}}$ in accordance with their set-theoretic memberships.

Lemma 1 Whenever $b \in \check{a}$, $b^* \in \check{a}^{\check{*}}$, for any element $\check{a}$ in $\check{K}$.

Proof For $\check{a}$, which is neither $\check{T}$, $\dot{T}$, $\check{T}^{\check{*}}$, nor $\dot{T}^{\check{*}}$, $a =_X b$ and hence $a^* =_X b^*$. Then $b^* \in a^{\check{*}}$ and, since $\check{a}^{\check{*}} =_{df} a^{\check{*}}$, $b^* \in \check{a}^{\check{*}}$. If $b \in \check{T}$ then $b = T$, $b^* = T^*$ and $b^* \in \check{T}^{\check{*}}$. If $b \in \dot{T}$ then $b =_X T$ and $b \neq T$. Hence $b^* =_X T^*$ and $b \neq T^*$, and thus $b^* \in \dot{T}^{\check{*}}$. If $b \in \check{T}^{\check{*}}$ then $b = T^*$, $b^* = T$, $b^* \in \check{T}$ and $b^* \in \check{T}^{\check{*}\check{*}}$. If $b \in \dot{T}^{\check{*}}$ then $b =_X T^*$ and $b \neq T^*$, $b^* =_X T$ and $b^* \neq T$, $b^* \in \dot{T}$ and $b^* \in \dot{T}^{\check{*}\check{*}}$. ♯

Theorem 5 Given a model $\mathfrak{M}$, $\langle T, K, R, *, \leq, v\rangle$, of the simplified semantics of one of the F-logics L, the second filtration X-model $\check{\mathfrak{M}}$: $\langle \check{T}, \check{K}, \check{R}, \check{*}, \check{\leq}, \check{v}\rangle$, as defined above and with appropriate semantic postulates for L, is also a model of L. ('$\check{\leq}$' is included only for logics with A12 but not R5.)

Proof We check the semantic postulates, P1–8, P11 and P12, listed in §2 for the logics L. For the most part, [12, pages 400–3], is followed.

P1. For $\check{a}$, which is neither $\check{T}$, $\dot{T}$, $\check{T}^{\check{*}}$, nor $\dot{T}^{\check{*}}$, $\check{a}^{\check{*}\check{*}} = (a^{**})\check{} = \check{a}$. The cases for $\check{T}$, $\dot{T}$, $\check{T}^{\check{*}}$, and $\dot{T}^{\check{*}}$are clear from the above definitions.

P2. Let $\check{R}\check{T}\check{a}\check{b}$. Then, by definition, $RTab$, for some $a \in \check{a}$ and $b \in \check{b}$, and hence $a = b$. Then $\check{a} = \check{b}$. Conversely, let $\check{a} = \check{b}$. Since $RTaa$, $\check{R}\check{T}\check{a}\check{a}$ and hence $\check{R}\check{T}\check{a}\check{b}$.

P3. For $\check{a}$, $\check{b}$ and $\check{c}$, which are neither $\check{T}$, $\dot{T}$, $\check{T}^{\check{*}}$, nor $\dot{T}^{\check{*}}$, let $\check{R}\check{a}\check{b}\check{c}$. Then $Rabc$, for some $a \in \check{a}$, $b \in \check{b}$, $c \in \check{c}$, and hence Rac^*b^*. Then $\check{R}\check{a}(c^*)\check{}(b^*)\check{}$ and $\check{R}\check{a}\check{c}^{\check{*}}\check{b}^{\check{*}}$. By Lemma 1, this result extends to the four additional cases.

P4. Let $\check{R}\check{a}\check{b}\check{c}$. Then $Rabc$, for some $a \in \check{a}$, $b \in \check{b}$, $c \in \check{c}$ and hence, for some $x \in K$, $Rabx$ and $Raxc$. Let $\check{x}$ be the member of $\check{K}$ such that $x \in \check{x}$. Then $\check{R}\check{a}\check{b}\check{x}$ and $\check{R}\check{a}\check{x}\check{c}$ for this $\check{x}$.

P5. Since $T^* \leq T$, $\check{T}^{\check{*}} \check{\leq} \check{T}$, by definition of '$\check{\leq}$'.

P6. For part (i), let $\check{a} \neq \check{T}$, then any such element a of K, such that $a \in \check{a}$, is not identical with T. Hence, for this a, Raa^*a and, by Lemma 1, $\check{R}\check{a}\check{a}^{\check{*}}\check{a}$. Part (ii) is clear from P5, when required.

P7. Similar to P4.

P8. Let $\check{a} \in \check{K}$. For any $a \in \check{a}$, $Raaa$ and hence $\check{R}\check{a}\check{a}\check{a}$.

P11. Let $\check{R}\check{a}\check{b}\check{c}$. Then $Rabc$, for some $a \in \check{a}$, $b \in \check{b}$, $c \in \check{c}$, and hence $Rbac$ and $\check{R}\check{b}\check{a}\check{c}$.

P12. We just consider $\check{T} = \check{T}^{\check{*}}$, in view of the logics L. Since $T = T^*$, $\{T\} = \{T^*\}$ and $\check{T} = \check{T}^{\check{*}}$. For logics with A12 but not R5, we check the additional postulate, P15, and the valuation condition VC1.

P15. Since RT^*aa, by definition, $R\check{T}^{\check{*}}\check{a}\check{a}$.

VC1. Let $\check{v}(p, \check{T}^{\check{*}}) = T$ and $\check{T}^{\check{*}} \check{\leq} \check{T}$. Then $v(p, T^*) = T$, $p \in X$ and $T^* \leq T$. Hence $v(p, T) = T$ and $\check{v}(p, \check{T}) = T$. ♯

Let $\check{I}(A, \check{a})$ be the interpretation which extends the valuation $\check{v}(p, \check{a})$ to all formulae A in accordance with the semantics.

COROLLARY 1 For logics with A12 but not R5, with P15 and VC1, for all formulae A, if $\check{I}(A, \check{T}^{\check{*}}) = T$ then $\check{I}(A, \check{T}) = T$.

The following theorem shows that the second filtration interpretation $\check{I}(A, \check{a})$ over members of X takes the same value as that of all of the interpretations $I(A, a)$, for $a \in \check{a}$.

THEOREM 6 For the model $\mathfrak{M}$ and the X-model $\check{\mathfrak{M}}$ of Theorem 5, for $A \in X$, $\check{I}(A, \check{a}) = I(A, a)$, for each $a \in \check{a}$.

PROOF We check this result inductively on formulae in X, for $a \in \check{a}$. The argument is mostly as in [12, page 401].

p: $\check{v}(p, \check{a}) = T$ iff $v(p, a) = T$, for $p \in X$.

~: $\check{I}(\sim A, \check{a}) = T$ iff $\check{I}(A, \check{a}^{\check{*}}) = F$, iff $I(A, a^*) = F$, by the induction hypothesis, iff $I(\sim A, a) = T$. We use Lemma 1 to ensure that $a^* \in \check{a}^{\check{*}}$ for the given $a \in \check{a}$.

&, ∨: Straightforward.

→: Let $\check{I}(A \to B, \check{a}) = T$. Then, for all $\check{b}, \check{c} \in \check{K}$, if $\check{R}\check{a}\check{b}\check{c}$ and $\check{I}(A, \check{b}) = T$ then $\check{I}(B, \check{c}) = T$. Let $Rabc$ and $I(A, b) = T$, for any $a \in \check{a}$ and $b, c \in K$. Then, by definition, $\check{R}\check{a}\check{b}\check{c}$, where $b \in \check{b}$, $c \in \check{c}$, and, by induction hypothesis, $\check{I}(A, \check{b}) = T$ and hence $\check{I}(B, \check{c}) = T$ and $I(B, c) = T$. So, $I(A \to B, a) = T$. Let $\check{I}(A \to B, \check{a}) = F$. Then, for some $\check{b}, \check{c} \in \check{K}$, $\check{R}\check{a}\check{b}\check{c}$, $\check{I}(A, \check{b}) = T$ and $\check{I}(B, \check{c}) = F$. By definition, $Rabc$, for some $a \in \check{a}$, $b \in \check{b}$, $c \in \check{c}$, and, by induction hypothesis, $I(A, b) = T$ and $I(B, c) = F$. Hence, $I(A \to B, a) = F$ for this $a \in \check{a}$. However, since $A \to B \in X$, $I(A \to B, a) = F$ for all $a \in \check{a}$. ♯

The corollary shows that the splitting of the original X-worlds $\check{T}$ and $\dot{T}$ does not affect their interpretations of formulae in X.

COROLLARY 2 For $A \in X$, $\check{I}(A, \check{T}) = \check{I}(A, \dot{T})$ and $\check{I}(A, \check{T}^{\check{*}}) = \check{I}(A, \dot{T}^{\check{*}})$, given that $\dot{T}$ and $\dot{T}^{\check{*}}$ are both non-empty and are thus proper elements of $\check{K}$.

PROOF By Theorem 6, $\check{I}(A, \check{T}) = I(A, T)$. There is an element $a \in \dot{T}$ such that $a =_X T$ and $a \neq T$, and thus $I(A, T) = I(A, a)$, and, by Theorem 6, $\check{I}(A, \dot{T}) = I(A, a)$. So, $\check{I}(A, \check{T}) = \check{I}(A, \dot{T})$. $\check{I}(A, \check{T}^{\check{*}}) = \check{I}(A, \dot{T}^{\check{*}})$ follows similarly, since, by Lemma 1, $a^* \in \dot{T}^{\check{*}}$. ♯

THEOREM 7 Each F-logic B^d, DW^d, DJ^d, DK^d, DL^d, D^d, $DC^d$and $RBC^d$is decidable.

PROOF Each of the logics have the finite model property and are hence decidable. To show this, let A be a non-theorem of one of the F-logics L. Then, by Theorems 2 and 4, A is invalid in the simplified semantics for L and $I(A, T) = F$, for some model $\mathfrak{M}$. Let X be the appropriate closure of the formula A. By Theorems 5 and 6, there is a second filtration X-model $\check{\mathfrak{M}}$ such that $\check{I}(A, \check{T}) = F$, since $A \in X$. So, A is falsified in the model $\check{\mathfrak{M}}$, which is finite since the set $\check{K}$ of X-worlds is finite. Decidability then follows [12, pages 401–2]. ♯

The next theorem tightens this decidability argument further by showing that the theoremhood or non-theoremhood of a formula A in an F-logic L can be determined just by its X-models.

THEOREM 8 For a given formula A, A is a theorem of an F-logic L iff A is true in all the X-models of L, where X is the appropriate closure of A.

PROOF If A is a theorem of an F-logic L then A must be true in any of the X-models, as they are models by Theorem 5 and soundness was shown by Theorems 1 and 3. If A is a non-theorem of L then, by the argument in the proof of Theorem 7, $\check{I}(A, \check{T}) = F$ in an X-model $\check{\mathfrak{M}}$, where X is the appropriate closure of the formula A. ♯

The finite size of the X-models is determinable in terms of features of the formula A under test. The size of X is no more than twice the number of subformulae of A, taking into account the closure condition, and the number of X-worlds is limited to 2^n, where n is the number of formulae in X. In fact, starting with a formula A, we can identify each possible "X-world" by its set of interpretations (T or F) on the members of X, then choose $\check{T}$ and $\check{T}^{\check{*}}$ (and $\dot{T}$ and $\dot{T}^{\check{*}}$) and define all the relation $\check{R}$ and function $\check{*}$ (and also $\check{\leq}$, if necessary) on them to determine all the "X-models". These "X-models" and "X-worlds" within them are then pared down if semantic postulates or valuation or interpretation conditions fail, producing X-models that are appropriate for the F-logic L. Thus, all the possible X-models for L can be constructed for the formula A, which, given Theorem 8, can then be shown to be invalid, if $\check{I}(A, \check{T}) = F$ for some X-model, or valid, if $\check{I}(A, \check{T}) = T$ for all X-models. Though a decision procedure, this is very tedious and so we adapt this to the reductio method, commonly used in other areas of logic such as modal logic.

4 The Ensuing Reductio Method

In the REDUCTIO METHOD, we start with the formula A under test being assigned F at $\check{T}$, and only construct what parts of the X-model(s) of the simplified semantics that are needed to try to falsify it. If an X-model can be determined, consistent with the semantic postulates and valuation and interpretation conditions of the F-logic L, A is invalid and hence a non-theorem of L. If no such

X-models can be found, and this can only be registered in the derivation of contradictions in each of an exhaustive set of X-model attempts to falsify A, then A is valid in all X-models and is hence a theorem of L.

We follow Hughes and Cresswell's reductio procedure for the modal logic T [9, pages 82-96] in setting out our reductio procedures for the F-logics L, but we need to include a rule for ∗-worlds and a rule for identifying worlds. We also need to distinguish between the two worlds b and c when Rabc holds, as they behave differently. We also need to add some further rules to ensure that the various semantic postulates for the logics L apply.

We set out the rules for the reductio procedure for our F-logics L, as follows: (Rules 1–6 are common to all logics L, whilst rule 7 is special to any of the particular logics L stronger than B.) (For simplicity, we drop the '′'-notation.)

1. Put the formula A under test in a rectangle, labelled with world T, and put the truth-value F under its main connective.

We continue to use rectangles around members of X, generated from A and taking truth-values T or F (put under the main connectives), and we append to each rectangle a world label. In fact, when we refer to worlds in the sequel we will assume an associated rectangle containing some members of X with truth-values.

2. We apply the usual truth-table rules to the connectives &, with T under it, and ∨, with F under it, to get definite values for the respective conjuncts and disjuncts. We continue to apply these truth-table rules to & and ∨, where applicable, in any of the subsequent worlds.

3. For each un-starred world a, we introduce a ∗-world a^* so that if a formula ∼B takes T (F) in world a then we put B with value F (T) in a^*, and if a formula B takes T (F) in world a and $\sim B \in X$ then we put ∼B with F (T) in a^*. Whether a^* is a new world or not will depend on rule 5 and, in the case of T and T^*, rule 7D or 7H. (Note however that, for all logics L, $a^{**} = a$, and that we constructed a^* so that all the above assignments still apply when applied from a^* into a, as well as applying from a into a^*.)

4A. In a world a, if there is an '→'-formula taking F then there must be worlds b and c, where we put the antecedent of the '→', with a T under it, into the rectangle for b, and we put the consequent of the '→', with an F under it, into the rectangle for c. We pick a point on the outside of the rectangle for a, draw arrows from this point to the rectangles for b and for c, and draw a line (slash) through the stem of the arrow towards b, indicating that b is to be treated differently from c in relation to both this and the next clause. We call this an R-relation between a, b and c, or just Rabc. We use a different starting point to draw such arrows for each '→'-formula taking F in a. Whether either or both of b and c are new worlds will depend on rule 5. (The rule does not have to be applied if the requisite worlds b and c already exist with the respective valuations.)

4B. In a world a, if there is an '→'-formula taking T, then for each existing

R-relationship between a and worlds b and c, shown by arrows from a to b with a slash and from a to c without a slash, both from the same point on a, if the antecedent of the '→'-formula has a T under it in the rectangle for b, then we put the consequent of the '→', with a T under it, into the rectangle for c.

4C. Since the postulate $RTab \Leftrightarrow a = b$ is common to all logics L, we introduce the single world a (with the two arrows from the T-rectangle to the a-rectangle) when evaluating an '→'-formula, with an F under it, in world T, putting both the antecedent and consequent of the '→'-formula into a, with appropriate values. Whether a is a new world will depend on rule 5. (The rule does not have to be applied if the requisite world a already exists with the respective valuations.) Further, for every world b, including T, RTbb holds and so, when evaluating an '→'-formula, with T under it, we draw in the arrows so that whenever the antecedent of the '→'-formula takes T in b the consequent takes T in b.

5C. Any two worlds with the same value assignments for all the members of X must be identified. This means that the same rectangles are used for both these worlds and thus the same ∗- and R-relationships hold for these worlds. However, there are exceptions for T and T^*. All the worlds having the same values over X as for T are put identical to $\dot{T}$, and all the worlds having the same values over X as for T^* are put identical to $\dot{T}^*$. Moreover, one can get to situations where there are no further distinct X-worlds and in the application of rule 4A or even rule 3 one is left to consider all the possible identifications between the worlds needed for these rules and the existing worlds.

5B. For the purpose of trying to establish a consistent X-model, we may choose to make an identification between worlds so that there is a consistent assignment of values to the formulae in the identified world. If this indeed yields an X-model, this will result in a restricted form of the X-model one is working with. We will call this a restricted X-model. If a contradiction in another world is derived from this identification then this contradiction only applies to this particular identification and other identifications would have to be tried to see whether X-models or contradictions result.

6. In applying the usual truth-table rules to the connectives &, with F under it, and ∨, with T under it, we get three different value combinations for the respective conjuncts and disjuncts. These yield what we call alternative X-models. Whilst any of these alternatives would suffice as an X-model, contradictions would need to be established in all three to produce a general contradiction for the procedure at this point.

Rules 1–6 apply for the logic B. For stronger logics L, we now add special rules applicable for the particular logic L we are dealing with. We examine the various semantic postulates of §2 in turn. Note that different starting points are used for each pair of arrows representing an R-relation, drawn from any single rectangle.

7A. For P3. $Rabc \Rightarrow Rac^*b^*$, we add the R-relation Rac^*b^* to each existing R-relation $Rabc$, inserting formulae and values according to rule 4B. We do not need to re-apply this rule to Rac^*b^* as $Rab^{**}c^{**}$ already holds.

7B. For P4. $Rabc \Rightarrow x \in K$, $Rabx$ and $Raxc$, we add a world x and the R-relations $Rabx$ and $Raxc$ to each existing R-relation $Rabc$. We insert formulae and values according to rule 4B for both $Rabx$ and $Raxc$. (The rule does not have to be applied if the requisite world x already exists with the respective R-relations.)

Realising that this process could run on until all X-worlds are exhausted, I would suggest identifying x with b, if possible, in accordance with rule 5B. For then 7B need not be re-applied to $Rabb$.

7C. For P5. $T^* \leq T$, we introduce the special $\leq$-relationship, indicated by a single arrow from T^* to T, in which every formula taking T at world T^* is given the value T in world T and every formula taking F at T^* is given the value F at T.

7D. For P6. Raa^*a, we add this R-relation for each world a other than T and apply rule 4B. $T^* \leq T$ is then added as for rule 7C. If $T = T^*$ applies, then identify these worlds as well. By rule 5A, this will imply the identification of $\dot{T}$ and $\dot{T}^*$, as formulae would have the same values in $\dot{T}$ and $\dot{T}^*$.

7E. For P7. $Rabc \Rightarrow (\exists x \in K)Rabx \;\&\; Rxbc$, we proceed in a similar manner to rule 7B, but I suggest identifying x with a, for then 7E need not be re-applied to $Raba$.

7F. For P8, $Raaa$, we add this R-relation for each world a and apply rule 4B.

7G. For P11. $Rabc \Rightarrow Rbac$, we proceed in a similar manner to rule 7A, and we do not need to re-apply this rule to $Rbac$.

7H. For P12. $T = T^*$, we identify these worlds. As for rule 7D, $\dot{T}$ and $\dot{T}^*$ are also identified. (We note that $T \leq T^*$ does not arise for the logics L.)

The following theorem states the relationship between the reductio method and theoremhood for the F-logics L.

THEOREM 9 A formula A is a theorem of an F-logic L if a contradiction is obtained in each of an exhaustive set of derived X-models for L that assign the value F to A at world $\check{T}$. A is a non-theorem of the logic L if a consistent X-model for L is obtained which assigns the value F to A at world $\check{T}$.

PROOF Clearly, if a contradiction is obtained in each of an exhaustive set of derived X-models for L that assign the value F to A at world $\check{T}$ then there are no consistent such X-models for A and $\check{I}(A, \check{T}) = T$ for all X-models of L. Then, by Theorem 8, A is a theorem of L. In the process of exhausting a set of derived X-models for L one must include all alternative X-models under rule 6 and all restricted X-models under rules 5A and 5B. On the other hand, if a consistent X-model for L is obtained which assigns the value F to A at world $\check{T}$ then $\check{I}(A, \check{T}) = F$ for this X-model and, by Theorem 8, A is a non-theorem of L. ♯

5 The Reductio Method for Contraction-less Logics

We proceed to develop a similar reductio method for the contraction-less logics B^{d}, DW^{d}, TW^{d} and RW^{d}, and, except for RW^{d}, systems obtained by the addition of A12. $A \vee \ A$ (add 'X' to name, prior to '^{d}') and systems obtained by the addition of both A12 and R5. $\sim A, A \vee B \Rightarrow B$ (add 'C' to name, prior to '^{d}'). These are called the U-logics in §2. As indicated in §3, the first and second filtrations will not work for the (new) axioms A16 and A17 of TW, and so, we need another approach. (Recall that B = B^{d}, DW = DW^{d}, TW = TW^{d} and RW = RW^{d}. We will drop the '^{d}' when naming these systems for simplicity.)

We will not use a filtration method and tackle the semantics directly. Since B and DW have already been treated in §4, we start by focussing on TW. If we consider the two semantic postulates for A16 and A17 of TW, viz.

P9 $Rabz \,\&\, Rzcd \Rightarrow \exists x \in K,\ Racx \,\&\, Rbxd$, and

P10 $Rabz \,\&\, Rzcd \Rightarrow \exists x \in K,\ Rbcx \,\&\, Raxd$

there seems to be a limit to the number of re-applications of these postulates if the introduced worlds x are new on each occasion and there are no postulates, further than those for TW, to concern us, at least for the moment. In order to apply these postulates there must be two R-relations of the form $Rabz$ and $Rzcd$, and the introduced x cannot play the role of the z as they stand, which means that we have to look around at our original set of R-relations for other worlds for z to equal. This means that z must ultimately come from the finite set of worlds and R-relations that we would already have from the reductio method to this point and, if we insist that the introduced worlds are all new, these two postulates will not have the propensity to generate an infinite number of worlds. We set this argument out more fully below.

We will continue to use the simplified semantics as it will give a simpler and more straight-forward decision procedure. Assuming the introduction of such new worlds for the logic TW, we set out the rules for the reductio procedure, much as in §4:

1. Put the formula A under test in a rectangle, labelled with world T, and put the truth-value F under its main connective.

2. We apply the usual truth-table rules, in this and subsequent worlds, to the connectives &, with T under it, and $\vee$, with F under it.

3. For each un-starred world a, we introduce a $*$-world a^* so that if a formula $\sim B$ takes T (F) in world a then we put B with value F (T) in a^*, and if a formula B takes T (F) in world a then we put $\sim B$ with F (T) in a^*. a^* is to be a new world, unless there is already a world with all these properties. (Note as before that $a^{**} = a$.)

4A. In a world a, if there is an '$\rightarrow$'-formula taking F then there must be worlds b and c, where we put the antecedent of the '$\rightarrow$'-formula, with a T under it, into the rectangle for b, and we put the consequent of the '→', with an F under it, into the rectangle for c. As before, we draw arrows from a point on a to the rectangles for b and for c, with a slash through the arrow towards b. We call this an R-relation between a, b and c, or just Rabc. b and c are both taken to be new worlds, except as for rule 4C below. These worlds are new with respect to existing worlds and their ∗-worlds. (The rule is not applied if the requisite worlds b and c already exist with the respective valuations.)

4B. In a world a, if there is an '→'-formula taking T then, for each existing R-relationship between a and worlds b and c, shown by arrows from a to b with a slash and from a to c without a slash, both from the same point on a, if the antecedent of the '→'-formula has a T under it in the rectangle for b, then we put the consequent of the '→', with a T under it, into the rectangle for c.

4C. Since, for the simplified semantics, $RTab \Leftrightarrow a = b$, we introduce the single world a (with the two arrows from the T-rectangle to the a-rectangle) when evaluating an '→'-formula, with an F under it, in world T, putting both the antecedent and consequent of the '→'-formula into a, with appropriate values. Similarly to 4A, the world a is new, unless such a world already exists. Further, for every world b, including T, RTbb holds and so, when evaluating an '→'-formula, with T under it, we draw in the arrows so that whenever the antecedent of the '→'-formula takes T in b the consequent takes T in b.

6. In applying the usual truth-table rules to the connectives &, with F under it, and ∨, with T under it, we get three different value combinations for the respective conjuncts and disjuncts. These yield what we call alternative models.

7A. We add the R-relation Rac^*b^* to each existing R-relation Rabc, inserting formulae and values according to rule 4B.

7B. For each existing pair of R-relations Rabz and Rzcd, we add the R-relations Racx and Rbxd, and Rbcy and Rayd, where both worlds x and y are new (i. e. new to existing worlds and their ∗-worlds), inserting formulae and values according to rule 4B. This rule is not applied when such x or y already exist with the requisite R-relations.

For the purposes of comparing this reductio procedure with one based precisely on the simplified semantics for TW, we consider, in the proof below, the addition of the following rule:

5. We can identify worlds a^*, introduced in rule 3, with any existing world, as we can for the worlds b and c, introduced in rule 4A, the world a, introduced in rule 4C, and also the worlds x and y, introduced in rule 7B, whether such an existing world initially had the requisite properties or not.

We now state the relationship between the above reductio method and theoremhood for TW, the proof of which involves a general argument concerning the use of new worlds.

THEOREM 10 A formula A is a theorem of TW if a contradiction is obtained in each of an exhaustive set of derived models that assign the value F to A at world T. A is a non-theorem of TW if a consistent model is obtained which assigns the value F to A at world T.

PROOF If a consistent model is obtained which assigns the value F to A at world T then $I(A, T) = F$ for this model and A is invalid in the simplified semantics for TW. By Theorem 3 of §2, A is a non-theorem of TW. (Note that $\text{TW} = \text{TW}^d$.)

Let a contradiction be obtained in each of an exhaustive set of derived models that assign the value F to A at world T. These contradictions would exhaust a set of alternative models, but would be based on the above rules 3, 4A, 4C and 7B, where new worlds were chosen for a^*, b, c, a, x and y. If we were to add rule 5 and allow identifications of these worlds with pre-existing worlds then the argument to the contradiction(s) would still apply as there would be no changes to the existing assignments of values in the now identified worlds. Indeed, the identifications may induce further contradictions. There may also be further applications of rule 7B due to the identifications, but this is not going to remove any derived contradictions. Also, any contradiction based on the lack of application of the rules 3, 4A, 4C and 7B, due to the presence of pre-existing worlds with the requisite properties, would still apply if such rules were nevertheless applied to introduce new worlds or other worlds in conjunction with rule 5. Thus, we have established the contradiction(s) within the simplified semantics for TW, and hence there can be no consistent models for A such that $I(A, T) = F$. So, $I(A, T) = T$ for all models and A is valid in the simplified semantics for TW. Hence, by Theorem 4 of §2, A is a theorem of TW. ♯

We now proceed with the decidability argument based on the reductio method for TW.

THEOREM 11 TW is decidable, using the above reductio method.

PROOF The reductio method for TW builds up a set of worlds, starting with T, existentially introducing new worlds using the rules 3, 4A, 4C and 7B, the latter three with new R-relations. We also introduce new R-relations using the rules 4B, 4C and 7A. We examine the various types of world and R-relation introduction. Rules 3 and most of 4 are formula-based, in that the presence of a formula induces the rule, whilst rule 7 and the second part of 4C (4C–2) involving the R-relation $RTbb$ are not, in that the rule applies to any existing world or when an R-relation or certain configuration of R-relations occurs. Rules 3 and 7A are idempotent, i. e. the original world or R-relation is reached after two applications of the respective rule. The rule 4C–2 applies only to the pre-existing worlds b.

So, the rules 3, 4A–C and 7A are either formula-based or idempotent or rely on pre-existing worlds. (We also use this terminology for the associated

R-relations.) A reductio system, all of whose rules are of one of these three types, would be decidable as only finitely many worlds could be generated, the number of which would be dependent upon the number of subformulae in the formula under test, and so the number of generated R-relations would also be finite.

The only rule that is likely to cause trouble as far as decidability is concerned is rule 7B, as it may be capable of generating, through successive applications, an infinite sequence of worlds. We examine the possibilities for this.

First consider the interaction of rule 4C-2, which yields $RTaa$. Let this supply one of the two R-relations, $Rabz$ and $Rzcd$. If $RTaa$ and $Racd$ then $RTcx$ and $Raxd$, for some x, and $Racy$ and $RTyd$, for some y. We do not apply rule 7B here as can be seen by putting $x = c$ and $y = d$. Let us consider $RabT$ and $RTcc$. The only way $RabT$ can be introduced is by rule 4C-2, in which case, $a = b = T$ and this becomes an instance of the first case. Clearly, $RabT$ is not formula-based. The problem with introducing $RabT$ using rule 7B with 'T' in the d-position is that ultimately one must start with $RTTT$ and $RTTT$, which is also an instance of the first case above. If we try to derive RaT^*b^* using rule 7B, one must start with something of the same form, which is again not formula-based. As a result we can assume that both the $Rabz$ and the $Rzcd$ are generated by rule 4 and/or rule 7B. Rule 7A does not have a significant impact due to its idempotence. Consider an application of rule 7B in the forms, $Rabz$ and $Rzcd \Rightarrow Racx$ and $Rbxd$, for some world x; and $Rabz$ and $Rzcd \Rightarrow Rbcy$ and $Rayd$, for some world y. Since x and y are new, there is no immediate generation of further worlds by application of rule 7B. Also, the R-relations that start such applications are all formula-based and so are finite in number. Since each world introduced by rule 4 is new, we would have finite strings of these R-relations all linked together like $Rabz$ and $Rzcd$ are, or like $Rabz$ and $Rbcd$, depending on the shape of the formulae involved. Other than the link z, the other worlds are all different, and when rule 7B is applied this link is replaced by other linking worlds x and y.

We consider a string of linked formula-based R-relations of length n, linked so that rule 7B can apply between successive R-relations. We assume that a finite number of worlds are generated by an exhaustive set of applications of rule 7B to this string. We then add another linked formula-based R-relation, $Rdef$, to the end of the string, making it a string of length $n+1$, and show that there are only finitely many extra worlds introduced by exhaustive applications of rule 7B.

Consider R-relations of the type $Rayd$ and $Rbxd$, obtained after any (finite) number of applications of 7B to the initial string of length n, where the 'x' and 'y' are worlds introduced by the last such application of 7B and 'a' and 'b' can either be worlds from a formula-based R-relation Rabz or introduced by 7B like the 'x' or the 'y' worlds but earlier. The R-relations $Rayd$ and $Rbxd$ are the only types that can interact with $Rdef$ using rule 7B. So, in applying 7B here, we obtain $Raex'$ and $Ryx'f$, $Ryey'$ and $Ray'f$, $Rbex''$ and $Rxx''f$, and $Rxey''$

and $Rby''f$, where x', y', x'' and y'' are all new. To take this further, each of the worlds a, b, x and y in the first positions may link up with an R-relation with these worlds in third position, such an R-relation being obtained from a number of applications of 7B to the initial string of length n. Such worlds a, b, x and y would have been introduced by an application of 7B, involving for example something of form $R\grave{x}cx$ for the 'x', where '$\grave{x}$' and 'c' are worlds from a formula-based R-relation or introduced earlier (than 'x') by 7B. Applying 7B to $R\grave{x}cx$ and $Rxey''$ yields $R\grave{x}ex'''$ and $Rcx'''y''$, and $Rcey'''$ and $R\grave{x}y'''y''$, for new x''' and y'''. There may be other subsequent occurrences of such forms $R\grave{x}cx$ obtained by subsequent applications of 7B, but the same structure applies. One can again take this further by linking up with the worlds $\grave{x}$ and c. What is apparent however is that on each such linkage the worlds are introduced earlier and earlier until, by successive application, there is no such linkage at all due to the worlds being part of a formula-based R-relation. So the whole process is finite, introducing only finitely many new worlds and R-relations. Note that there is no generation of further worlds by linking any of the newly introduced worlds x', y', x'', y'', x''', y''', etc. as they cannot appear in first position without an R-relation of form $Rfgh$. Also note that application of 7A would just give a $*$-version of this whole process.

So, by an induction argument, only finitely many worlds can be introduced by rule 7B from any string of linked formula-based R-relations of any finite length. This addresses our only remaining concern and thus TW is decidable, using the reductio method. ♯

COROLLARY 3 B and DW are decidable.

PROOF The reductio method for DW is obtained by removing rule 7B from that for TW and the reductio method for B is obtained by removing rule 7A from that for DW. Hence DW and B are both decidable by these methods as, with the absence of 7B, there is no impediment to the finiteness of the set of introduced worlds, as mentioned in the above proof. ♯

We now extend the decidability argument for TW to RW and then to the remaining U-logics.

THEOREM 12 RW is decidable.

PROOF We first simplify the formulae that we need to consider so that the proof can go through. As shown by Slaney [13], it suffices to consider just entailments of form $A \rightarrow B$ when determining theorems of RW, due to the primeness of theorems, viz. $\vdash A \vee B$ iff $\vdash A$ or $\vdash B$, and the break up of negated entailments, viz. $\vdash {\sim}(A \rightarrow B)$ iff $\vdash A$ and $\vdash {\sim}B$. So, for the reductio method, we do not have to consider negated entailments being true at world T nor entailments false at T^*, and so the formula-based R-relation RT^*ab, and its equivalents RaT^*b and Rab^*T (see rule 7C below), would never arise. However, RT^*TT^* can be derived, as can be seen from rule 7C below.

With the addition of the semantic postulate PII. $Rabc \Rightarrow Rbac$, we add the following rule to those in the reductio method for TW:

7C. We add the R-relation $Rbac$ to each existing R-relation $Rabc$, inserting formulae and values according to rule 4B.

As for rule 7A, this rule is idempotent. However, unlike 7A which is fairly harmless, rule 7C leads to some complications which will need to be examined. In conjunction with 7C, 7B will allow the introduced worlds x and y to be positioned in such a way as to allow 7B to be re-applied with x and y as links. Also, given 4C, rule 7C will yield $RaTa$, which with 7B will generate an infinite number of worlds, as we will shortly see for EW^d. However, in this case, 7C will prevent that from happening. Further, using 7A and 7C, we get Raa^*T^*, which also needs examination.

What we will show here is that there are no more new worlds introduced by the addition of rule 7C. New worlds can only be generated in conjunction with 7B where, given $Rabz$ and $Rzcd$, we add $Racx$ and $Rbxd$, for some world x, and $Rbcy$ and $Rayd$, for some world y. We assume that 7C is applied before 7B, wherever possible, as a matter of procedure. We address the three above-mentioned concerns in turn.

(i) We expand the conclusions of 7B by using 7C, as follows: $Racx$ and $Rxbd$, $Rcax$ and $Rxbd$, $Rbcy$ and $Ryad$, $Rcby$ and $Ryad$. To each of these forms, we can re-apply 7B, as follows: $Rabz$ and $Rczd$, $Rcby$ and $Rayd$, $Rcby$ and $Rayd$, $Rabz$ and $Rczd$, $Rbaz$ and $Rczd$, $Rcax$ and $Rbxd$, $Rcax$ and $Rbxd$, $Rbaz$ and $Rczd$. We fill in the variable places in accordance with existing R-relations, taking into account applications of 7C in the process. So, any re-application of 7B, as a result of 7C, does not yield any new worlds. Also note that application of 7A would just give equivalent $*$-versions of this process.

(ii) We consider the use of $RaTa$ with 7B. Similarly to the case where $RTaa$ was in the $Rabz$ position, from $RaTa$ and $Rabc$ we get $Racx$ and $RTxd$, and $RTcy$ and $Rayd$, where by putting $x = d$ and $y = c$ we obtain pre-existing R-relations. Also, when we consider $Rabc$ and $RcTc$, we get $RaTx$ and $Rbxc$, and bTy and $Rayc$. Here, we put $x = a$ and $y = b$ to obtain pre-existing R-relations, with help from 7C.

(iii) In considering Raa^*T^* with 7B, from $Rabc$ and Rcc^*T^* we get Rac^*x and $RbxT^*$, and Rbc^*y and $RayT^*$, where by putting $x = b^*$ and $y = a^*$ we obtain pre-existing R-relations. However, with Raa^*T^* in the $Rabz$ position, we would need an R-relation of the form RT^*cd, which is not formula-based, as seen above. Unless $c = T$ and $d = T^*$, RT^*cd is not generated by using 7B, since ultimately one of the forms RT^*cd, RcT^*d or $RcdT$ would need to be formula-based. If we apply 7B to Raa^*T^* and RT^*TT^* we get $RaTx$ and Ra^*xT^*, and Ra^*Ty and $RayT^*$, where we can put $x = a$ and $y = a^*$.

There are no other ways that new worlds might be introduced by 7C that would not be introduced otherwise, and so RW is decidable, given the argument for TW. ♯

THEOREM 13 All the U-logics are decidable.

PROOF We just need to examine the addition of A12. $A \vee \sim A$ and of A12 together with R5. $\sim A, A \vee B \Rightarrow B$, to B, DW and TW. For A12, we add the semantic postulate P5, together with the ordering relation '$\leq$', the valuation condition VC1 and P15. RT^*aa. We add the following rule 7D to the reductio method.

7D. We introduce the special $\leq$-relationship, indicated by a single arrow from T^* to T, in which every formula taking T at world T^* is put with value T in world T and every formula taking F at T^* is put with value F at T. We also add the R-relation RT^*aa, for all existing worlds a.

For BX^d and DWX^d, 7D does not change the finitude of the worlds. For TWX^d, we need to examine the effect of RT^*aa on 7B. Firstly, RT^*aa and $Rabc$ yield RT^*bx and $Raxc$, and $Raby$ and RT^*yc, where we can put $x = b$ and $y = c$. Secondly, consider $RabT^*$ and RT^*cc, whereupon $RabT^*$ (and $RaTb^*$) is not formula-based and cannot be obtained through applications of 7B, unless it is one of RTT^*T^* or $RT^*T^*T^*$. The first of these is covered by $RTaa$ and the second by RT^*aa above.

For R5, we add P12. $T \leq T^*$, which becomes $T = T^*$ in the presence of P5. We modify rule 3 and add rule 7E below.

3. a^* is a new world for $a \neq T$.

7E. We put T^* identical with T.

For BC^d and DWC^d and TWC^d, there is little change from their respective systems, B^d, DW^d and TW^d. ♯

We next examine some of the major extensions of the U-logics to see what goes wrong with the above decidability arguments and why they might thus be undecidable, though this argument itself will not constitute a proof as it deals with only one method, viz. this particular reductio method. The logics R, E and T have already been proved to be undecidable by Urquhart [15]. In fact, he proved undecidability for all logics between TW+A15 and R [15, pages 1069–70]. Perhaps his methods can be extended to other systems where undecidability is suggested in this paper but not already shown.

We start with EW^d, which is TW^d+R6, where R6 is the rule, $A \Rightarrow A \to B \to B$. Using simplified semantics, its corresponding semantic postulate is P17: $RaTa$, which in the absence of 7C does lead to the following infinite sequence of worlds. Let $Rabc$. Then, since $RcTc$, by applying 7B, we get $RaTx$ and $Rbxc$, for some new x. Then consider $Rbxc$ and $RcTc$, in which case, $RbTx'$ and $Rxx'c$, for some x'. Further, $Rxx'c$ and $RcTc$ yields $RxTx''$ and $Rx'x''c$, for some x''. Thus, an infinite sequence of worlds can be generated.

We next consider RWX^d and RWC^d, obtained from RW^d by the successive addition of A12 and R5. With the addition of A12, we add the rule 7D, which includes the R-relation RT^*bb, for any b. However, we can combine this with Raa^*T^* using 7B to get $Rabx$ and Ra^*xb, for some x, and Ra^*by and $Rayb$, for some y. Using these new worlds x and y, Rxx^*T^* and RT^*yy, and hence $Rxyx'$ and $Rx^*x'y$, and Rx^*yy' and $Rxy'y$, for new x' and y'. We can thus generate an infinite sequence of worlds employing the rules 4C, 7A, 7C and 7D as they stand. For RWC^d, we similarly use Raa^*T and $RTbb$.

We next consider any extension of TW^d, EW^d or RW^d consisting of one or more of the following axioms: A11, A13, A14, A15, with corresponding semantic postulates: P4, P6, P7, P8. (However, for any such system L without R6 and either without A12 or with A15, $L^d = L$.) These systems include TJ, T, E and R, where TJ $=$ TW $+$ A11, T $=$ TW $+$ A13 $+$ A14, E $=$ EW $+$ A13 $+$ A14, R $=$ RW $+$ A14. P4 and P7 clearly lead to an infinite sequence of worlds by successive re-application. P6 and P8 are both of the form $Raba$, for some b. This form generally leads to an infinite progression by applying 7B as follows. $Raba$ and $Racd$ yields $Racx$ and $Rbxd$, and $Rbcy$ and $Rayd$. In turn, $Raba$ and $Racx$ yields $Racx'$ and $Rbx'x$, and $Raba$ and $Racx'$ yields $Racx''$ and $Rbx''x'$. The avoidance of the form $Raba$ is one of the reasons why newness is important in rule 4A.

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