Propositions, Properties, and Paradox

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Abstract

Contrary to views that diagnose the paradoxes of truth and related notions in terms of sentences not expressing propositions, or expressing propositions different from what they appear to express and which aren't paradoxical, or expressing multiple propositions none of which are paradoxical, the paper argues that the basic paradoxes are paradoxes of propositions; or alternatively, of sentential quantification. Similarly for the paradoxes of satisfaction: the basic paradoxes arise for properties, or for quantification into predicate position. (In the latter case, it's argued that adopting the syntactic restrictions of Russellian type theory is not the best way to go.). The paradoxes of propositions and properties can be resolved either in classical or non-classical logic, but the paper focuses mostly on non-classical options, and develops an account of property identity in which properties defined using the notion of property identity are allowed and the naive abstraction principle holds unrestrictedly.

In his famous 1975 paper on the Liar paradox, Kripke says that Liar sentences *don't express propositions*, a view that has also been taken by many others, e.g. Rumfitt 2018 and Warren 2024.

Other philosophers (e.g. Prior 1961; Bacon 2018) have taken the view that Liar sentences *do* express propositions, but *propositions different from what they appear to express*. They appear to express propositions that assert their own untruth; if such propositions exist they threaten paradox, in the (admittedly vague) sense that standard modes of logical and truth-theoretic reasoning, when applied to them, would lead to unacceptable results. The Prior-Bacon view is that no such propositions exist, and that Liar sentences instead express other propositions that don't in any way threaten paradox (though what propositions these are is something of a mystery).

Still other philosophers (dating back to a medieval logician Bradwardine, but taken up in recent times by Read 2007 and Dorr 2020, among others) have argued that Liar sentences express *multiple propositions, none of which threaten paradox.*

The thought behind all three of these views is that truth is primarily a property not of utterances but of propositions, and that the truth-theoretic paradoxes arise solely from false assumptions about the relation between sentences and propositions; they don't arise for propositions themselves. The analogous treatment of the heterologicality paradox (less often made explicit, but presumably just as widely held) is that a predicate is true of something e only in a derivative sense—only if it expresses a property that e instantiates—and that the paradox arises only from false views about which predicates express properties, or which properties they express. The "Russell property" of not instantiating itself *would* threaten paradox, but according to these views there need be no such property: the predicate 'isn't true of itself' either doesn't express a property, or expresses a property other than the Russell property that doesn't threaten paradox, or expresses multiple properties all different from the Russell property and none of which threaten paradox.

I won't argue in detail against any of these three approaches, though I will raise some issues about them. But while there has been interesting work on all three, I will be suggesting that they take a wrong turn from the start: that the basic paradoxes arise already at the level of propositions and properties. My goals in this paper are to make this position attractive and to work out a way to deal with the paradoxes at this level, including the constraints that the paradoxes impose on the relation of property-identity.

Of the three views to which I've announced opposition, I regard the first ("don't express propositions") view as *most* defensible. That's because on the view I'll offer, one could define various notions of *paradoxicality* of propositions,[1] each a precisification or variation of the idea that the paradoxical propositions are those for which standard modes of logical and truth-theoretic reasoning lead to unacceptable results. Given this, one could take an advocate of the view that paradoxical sentences don't express propositions as simply using 'proposition' for *non-paradoxical proposition*, in one of these senses; that would leave no proposition for a paradoxical sentence to express in any natural way. (Analogously for the view that paradoxical predicates don't express properties.) No natural reinterpretation of the second or third views seems possible.

The difference between the view to be offered and the three other views, especially the first, thus comes down to whether it's better to have a restricted notion of proposition and property that excludes these "paradoxical" ones, or a broad notion that includes them. The present paper is focused more

[1] Better but more long-winded: of a proposition *threatening paradox*.

on the virtues of the broader approach than on the defects of the narrower; a detailed discussion of the shortcomings of any of these approaches would require a separate paper, especially given that even within the same approach, advocates rise to its various challenges in different ways.[2] That said, Section 6 has a brief discussion of some influential remarks of Kripke's where he advocates the narrow view; and Section 4 contains a general argument against all the views that deny the exist of properties and propositions that threaten paradox, though it does not purport to be conclusive.

It is the motivation, development, and defense of the broad approach to properties and propositions that I now turn. Much of the technical development of my preferred version isn't surprising: in Section 5 I follow others in adapting Kripke's Kleene-based theory of truth to the theory of properties and propositions. In Sections 1-4 and 6 I mainly pursue various philosophical issues that arise in motivating and interpreting this apparatus. One important quasi-technical matter not adequately addressed in the current literature is the identity conditions of properties and propositions, and Section 7 offers an account of that. Section 8 addresses another technical question that might be thought to undermine an autonomous theory of propositions.

1 Realism about properties; Indeterminacy

I'm going to use 'property' in a more inclusive than usual sense, in which for any natural number n there are n-place properties. Properties in the normal sense are 1-place; n-place properties for $n > 1$ are what are normally called n-place relations; and 0-place properties are what are normally called propositions. Associated with n-place properties is an $(n+1)$-place predicate of n-instantiation, ξ_n: '$x_1...x_n\xi_n y$' means that y is an n-place property instantiated by $x_1...x_n$ (in that order, when $n > 1$). Note that in the propositional case ($n = 0$) there are no $x_1...x_n$ involved: '$\xi_0 y$' means intuitively that y is a true proposition, and I'll sometimes write it as 'True(y)'.

(I'll be using 'property' in the abundant sense of the term, in which there are such properties as *being either grue or an axolotl or the capital of Venezuela.* The theory of e.g. *physical* properties is rather different. Treating propositions as 0-place properties would make little sense if properties weren't taken as abundant.)

Quine of course saw no need to "countenance" propositions and properties,

[2]Among the challenges of the first approach: providing a natural and precise demarcation between those utterances that express propositions and those that don't; and formulating a theory of propositions in this restricted sense (*r-propositions*) without using in that theory sentences that don't express r-propositions.

i.e. take them with any ontological seriousness. While sentences, as used on a given occasion, can be true, and predicates, as used on an occasion, can be true of things, his view was that these linguistic notions should be treated as fundamental: talk of sentences expressing (on a given occasion) true propositions, and predicates expressing properties, is at best a misleading manner of speaking. Another view that regards such talk as involving at best a useful fiction is a neo-Fregean view, according to which such talk ought to be replaced not by talk of linguistic truth or satisfaction, but by quantification into the sentential or predicate position. (See Section 3.) In contrast to both of these are views that take propositions and properties with utmost seriousness: propositions and properties are fundamental, explanatorally as well as ontologically, in that linguistic truth (if it makes sense at all) can only be explained as truth of an expressed proposition (and analogously for truth-of). In this paper I will take no official stand on these matters: if you want to read me as offering a theory of the true nature of properties (in the inclusive sense that includes propositions), you may, but you can equally read me as trying to lay out the most useful fiction for a Quinean sententialist, or for paraphrasing quantification into sentential and predicate positions. It will make no difference to the general shape of the resolution of the paradoxes.

Whether we take talk of propositions and properties with "ontological seriousness" or regard it as in one way or another fictional, it seems clear that a central part of the reality or fiction has to be principles connecting them to language and thought. Focusing on language, I'd put the central principle as follows:

(EXP) A sentence, as used on a given occasion, is meaningful only if on that occasion it expresses a unique proposition; and as used on that occasion, the sentence is true at a world w iff the proposition so expressed is true at w. (And analogously for predicates, properties, and truth of.)

(The analog for thought is that a belief, desire, occurrent thought, etc., is contentful only if it expresses a unique proposition; and the thought is true at a world w iff the proposition so expressed is true at w.)

I've built in uniqueness (which obviously flies against the multiple propositions view). Is there any reason independent of the paradoxes to oppose it? My formulation of (EXP) clearly accommodates cases of homophony, polysemy and indexicality, where the same sentence expresses different propositions on different occasions. But it might be thought to ignore the existence of indeterminacy. Some examples of indeterminacy stem from polysemy: to use an example from Dorr 2020, the word 'computer' is sometimes used broadly to include tablets, and sometimes narrowly to exclude them; and in many circumstances we use the term without bothering to distinguish the broader and narrower use. (And really there are many more than two uses—sometimes we exclude the less sophisticated tablets but not the more sophisticated ones, and again it's often the case that nothing in our minds or in external context decides the matter.) There are also examples of indeterminacy in which a speaker doesn't even have the conceptual resources to make relevant distinctions: e.g. pre-Newtonians didn't understand the difference between mass and weight, so a sentence of form 'b is heavier than c' in their mouths is best regarded as indeterminate between two different relations,[3] and one can imagine circumstances on which their sentence is true on one but not the other. These and many other sorts of examples (including, plausibly, ordinary vagueness) suggest that indeterminacy is quite rampant, affecting every or almost every sentence. (And, I'd add, every or almost every belief, desire, occurrent thought, and so forth.) So, it might be said, meaningful uses of a sentences typically don't express anything close to *unique* propositions.

One could say that, but one could equally well say that when an utterance (or a thought) is indeterminate, it *expresses an indeterminate proposition*. There's no more than a verbal issue between these ways of putting the matter.[4] But I prefer this second way of putting things, simply because the alternative would require a more complicated formulation than (EXP).

It's worth noting though that my formulation (EXP) does take a stand on how we are to use 'true' in connection with indeterminacy: its stand is that if 'There are exactly three computers in the room' as used on a given occasion is indeterminate in truth value, then 'The proposition that there are exactly three computers in the room is true' as used on that occasion is indeterminate *in just the same way*. In Kit Fine's familiar terminology (1975), there's a "penumbral connection" between 'True' and the other terms in the language that might be used in an indeterminate way, that guarantees the equivalence.[5] This is not the only possible convention for using 'True' in connection with indeterminate utterances: another is to use it for super-truth, that is, to

[3]Actually more, since for instance there is similar indeterminacy in the Newtonian notion of mass, and there probably is in our best current language of physics as well.

[4]The propositions of the multiple proposition way of talking ("precise propositions") are maximal refinements of indeterminate propositions. In the other direction, indeterminate propositions can be viewed as constructs out of precise propositions, e.g. as functions taking factors of precisification into precise propositions; in these terms, **(EXP)** would be restated to assert that each use of a sentence is associated with a unique such function.

[5]Fine used the term 'penumbral connection' to describe the connection between related vague terms like 'green' and 'blue'. It seems less natural to use it in the present context, so I'd prefer to say that the indeterminacy in 'True' is "correlative": it involves a correlation between the indeterminacy of 'True' and that of other terms. But since Fine's terminology is familiar, I'll stick with it.

take the truth of an utterance to mean that *all* the determinate propositions that it's indeterminate between are true; still another is to use it for subtruth, that is, to take the truth of an utterance to mean that *some of* the determinate propositions that it's indeterminate between are true. (And there are others, e.g. *most of* and *90% of*.) But I think the penumbral connection version incorporated into (EXP) most accords with the normal use of 'True' (especially since with rampant indeterminacy, very little is supertrue), and so I'll stick with it in this paper. (We needn't preclude expressing the others, we can just call them different things, say 'determinately true' and 'partially true'.)

Although most or all language is highly indeterminate, this matters little to the theory of sentential truth, or truth-of, on this penumbral connection understanding of it. Similarly, it matters little to the theory of truth of propositions to include indeterminate propositions. That's because the penumbral connection understanding of 'true' makes talk of truth for indeterminate propositions formally the same as talk of truth for determinate ones. Given this, we can mostly ignore indeterminacy in what follows.

2 Are determinate properties just modalized sets?

What sorts of things are determinate properties (in this broad sense that includes determinate propositions and relations)? One prima facie natural view of them (I'll call it *the set-theoretic view*) is that a determinate n-place property is just a set of $(n+1)$-tuples, where the first component of each such n-tuple is a possible world and where the remaining components (when $n > 0$) are objects in that world, and where no two members have the same first component. So propositions (the $n = 0$ case) are just sets of worlds. This formulation leaves open whether there is a set of all possible worlds. (On standard assumptions about sets, that would require an upper bound on the cardinality of the non-mathematical objects in each world, which seems unattractive; on the other hand, without a set of all possible worlds, the proposal rules out propositions true in all possible worlds.) This formulation also leaves open whether the objects existing in each world form a set. The answer, on standard set theory, has to be 'no' (at least if the "possible worlds" include the actual world): the actual world includes all the sets, and there are too many of them to form a set. Given this, the set-theoretic view requires that there is no determinate 1-place property of being a set, or of self-identity, or of not being an electron; indeed, no determinate

1-place property that even in the actual world applies to all sets or to all self-identical things or to all non-electrons.

(An indeterminate property has determinate ones as refinements. So the set theoretic conception of determinate properties requires that no *refinement of* an *in*determinate property is instantiated at the actual world by all self-identical things, or all sets, or all non-electrons. In what follows I'll mostly restrict the discussion to determinate properties for simplicity: the generalization to indeterminate ones will be obvious, and I don't think anything substantial would be changed by the constant reminder that the properties we express are mostly highly indeterminate.)

I'm skeptical of the set-theoretic view of properties, but many people exploit its limitations on set existence to rule out paradoxical properties, such as the "Russell property" of being a property that doesn't instantiate itself. Their view is that there's no Russell *set*, and for analogous reasons there's no such property. If properties are modeled on any standard hierarchical set theory, the idea is that properties come in ranks, or types, and any property is instantiated only by properties of lower type. There may be a property of being a property *of type less than* α that doesn't instantiate itself[6]—it would coincide with the property of being a property of type less than α—but it itself is of type α and so doesn't instantiate itself, with no threat of paradox.

Gödel is said to have remarked that there never was a Russell paradox for *sets* (meaning presumably that the hierarchical conception of sets is so obviously right that no one should have taken the supposed paradox for sets with any seriousness), but that the corresponding paradox for properties remains unsolved.[7] Evidently he did not much care for the set-theoretic view of properties, or the Russellian theory of types for properties. I'm with him.

What's the alternative? We could base the theory of properties on a non-standard view of sets on which sets are no longer stratified into ranks; but I'll argue in Section 4 that even with such modification, the set-theoretic view is suspect. Instead, let's assume a language (regimented to exclude ambiguities, indexicality and the like, but not ruling out indeterminacy) where for any formula $F(x_1, ..., x_n; u_1, ..., u_m)$ there is a singular term $\lambda x_1...x_n F(x_1, ..., x_n; u_1, ..., u_m)$ with the u_i free; relative to any assignment of entities e_1,...,e_m to its free variables it is taken to denote an n-ary property. I'm allowing properties themselves to count as entities (so λ-terms can be

[6]'May be' because often, advocates of the view are still more restrictive, allowing a property of type $n+1$ to be instantiated only by properties of type exactly n (rather than, $\leq n$), thereby mimicking the Russellian theory of types rather than a modern set theory like Zermelo-Frankel.

[7]See the opening sentence of Myhill 1984. Gödel expressed the view in print, less snappily, in his 1947 (pp. 518-19).

substituted for the free variables of λ-terms), though this is far less important than the fact that I'm imposing no exclusion on the formula F: in particular, I allow even formulas that don't have sets as extensions and also that contain instantiation predicates. So if the language contains 'is self-identical' and 'isn't an electron' and 'doesn't instantiate$_1$ itself' (where the subscript indicates instantiation of a 1-place property), then we have λ-terms corresponding to each of them.[8] (An alternative to regarding λ-expressions as singular terms will be considered in Section 3.)

3 The neo-Fregean view

I've mentioned a neo-Fregean view, increasingly popular in recent years (e.g. Bacon 2024, and more qualifiedly, Trueman 2021), on which all talk of properties, propositions, and so forth is a convenient but fundamentally misleading fiction: what we really have is just quantification into non-singular-term positions, e.g. 1-place predicate quantification and quantification into sentential position. On such a view, there is no general category of "entities" that includes both physical objects and properties; predicate quantification, and sentential quantification, aren't quantification "over" anything at all. Advocates of this view almost invariably lapse into talk of quantification over propositions, properties, and so forth, because in English it's very hard to avoid this, but they usually take this as just a misleading manner of speaking. On this view we can still use the locution '$\lambda x_1...x_n F(x_1,...,x_n)$' in literal truths, but instead of regarding it as a singular term standing for an n-ary property, regard it instead as a complex n-ary predicate. (The 'ξ_n' then become mere copulas, and can be dropped from the notation.) And in the more general version $\lambda x_1...x_n F(x_1,...,x_n; u_1,...,u_m)$ we regard the u_1,...,u_m as not restricted to ranging over "entities", but as replaceable by predicates or sentences as well as singular terms. (Perhaps there should be distinct styles of variables, depending on whether the substituends are singular terms and if not, for which arity (number of places) the substituends are supposed to be.)

[8]Excluding 'instantiates$_1$' from λ-terms would give rise to only a very weak theory of properties, analogous to the bottom level of the Tarski hierarchy of truth predicates, the level where no sentence containing 'true' comes out true. One could also consider a ramified theory, in which we replace the single 'instantiates$_1$' predicate by a multiplicity of predicates 'instantiates$_{1,\alpha}$', and allow each to be instantiated only by properties defined from formulas that don't include an instantiation predicate whose second subscript is α or higher. This would be analogous to the Tarski hierarchy of truth predicates, and have disadvantages analogous to those noted for the Tarski hierarchy in the early pages of Kripke 1975. I will not pursue it here.

I'm sympathetic to this neo-Fregean viewpoint, but as I said earlier, I don't want to take a stand on it. I write as if a closed $\lambda x_1, ..., x_n$ term is a singular term that refers to a special kind of entity, viz. an n-ary property (not necessarily a determinate one); but I'm fairly confident that what I say could be rewritten in terms of the neo-Fregean picture as I've just described it, though it would be hard to avoid slipping into language that according to the view isn't literally correct.

But *it's important to note that the neo-Fregean picture as I've described it involves no further stratification of predicates beyond how many places they take.* In particular, there is no requirement that an n-place predicate is associated with a rank or type, such that for instance a predicate of type 2 can only be (meaningfully or correctly) applied to predicates and sentences of type 1, or to those plus objects in a sense that excludes propositions and properties. Such further stratification is often assumed (e.g. in Trueman 2021 and Bacon 2024), but I think is unnatural: why should it count as meaningless to say that both *being self-identical* and *not being self-identical* are self-identical, that neither *being an electron* nor *not being an electron* are electrons, and so forth? As I've said, the view that we need such further stratification makes predicate quantification too much like quantification over iterative sets.

In saying these things I'm influenced by Will Nava. His (in preparation) argues, on grounds independent of the paradoxes, that a neo-Fregeanism without such added stratification is more attractive than the usual version with the stratification. He has developed the unstratified view in some detail, and argued that the sort of talk of propositions and properties that I'll be engaged in could be recast in terms of it.

4 Classical and non-classical options

In the first-order formalism in which I've chosen to work, at least as a misleading manner of speaking, we have abstraction terms $\lambda x_1 ... x_n F(x_1, ..., x_n; u_1, ..., u_m)$ and for each n, an n-ary instantiation predicate ξ_n. I've assumed the following schema, which is required by the view that formulas and sentences always express properties or propositions:

Existence: $\vdash \forall u_1 ... u_m \exists y[Property_n(y) \wedge y = \lambda x_1 ... x_n A(x_1 ... x_n, u_1 ... u_m)]$.

It's also natural to assume the following schema:

Transparency (conditional form): $\vdash \forall z_1 ... z_n \forall u_1 ... u_m \forall y[y = \lambda x_1 ... x_n A(x_1 ... x_n, u_1 ... u_m) \rightarrow \Box(z_1 ... z_n \xi_n y \leftrightarrow A(z_1 ... z_n, u_1 ... u_m))]$.

But **Transparency** is inconsistent with **Existence** in classical logic: letting $A(x)$ be $\neg(x\xi_1 x)$, the first implies that $\exists y[y = \lambda x \neg(x\xi_1 x)]$, and the second then implies that it instantiates itself if and only if it doesn't. That's Russell's paradox for properties.

Existence and **Transparency** both seem extremely attractive, and my own view is that it is probably worth restricting classical logic (as applied to truth and property instantiation) so that we can keep them both. Obviously restricting classical logic has its costs, so I should say something about the classical options.

The most popular classical option has been to restrict the **Existence** assumption, to exclude (at least) formulas A which are "paradoxical", however exactly 'paradoxical' might be defined. The view that paradoxical sentences don't express propositions obviously takes that option. But it's awkward to reconcile this with (EXP) and its analog for thought (or with a suitable modification of them without the uniqueness assumption, if one prefers to deal with indeterminacy in terms of multiple propositions). And (EXP) (and its analog for thought, and possibly so modified) seems central to the rationale for talking of properties and propositions at all.

To see the problem, start with the heterologicality paradox, which involves the predicate $H =$ 'isn't true of itself'. If we assume the standard transparency schema

$(\forall x)$['$F(u)$' is true of $x \leftrightarrow F(x)$]

then both the assumption that H is true of itself and the assumption that it isn't lead to contradiction. (Which in classical logic means that they entail anything; moreover, in classical logic H either is true of itself or it isn't.) Obviously there's a predicate 'isn't true of itself' in our language, so we can't resolve this by invoking a non-existence claim for *predicates*, but (EXP) allows us to do something related: we can say that the predicate doesn't denote a property, and therefore by (EXP) isn't true of anything.

So far so good, but what about other examples? Consider the predicate 'is either a blue balloon or a predicate that isn't true of itself'. Presumably it's true of blue balloons?[9] But then, by (EXP), there must be a property it expresses. What property could that be? The natural suggestion would be: the property of *being either a blue balloon or a property that doesn't instantiate itself*. But can there be such a property, if there's no such property as *being a property that doesn't instantiate itself*? It's hard to see how. (Certainly there can't be if we understand the disjunctive property in

[9]The presumption is even stronger for 'is either a blue balloon or a predicate true of some predicate that John mentioned yesterday', where unbeknownst to the speaker John discussed the heterologicality paradox yesterday. The discussion that follows works for it too.

the most natural way, as the least inclusive property that extends both the property of being a blue balloon and the property of being a property that doesn't instantiate itself.) I don't deny that there are moves one could make here, but they seem rather desperate.

Moreover, those who adopt the non-existence of properties solution (or its analog for predicate quantification) usually do so by modeling properties (or the theory of predicate quantification) on iterative set theory. And doing that makes things much worse: as noted before, there is then no property of being self-identical, or of not being an electron, and if we go with (EXP) that means that the predicates 'is self identical' and 'isn't an electron' aren't meaningful. These last examples could be handled by modeling properties (or the theory of predicate quantifiers) not on iterative set theory but on a non-standard set theory like Quine's *New Foundations*. But that wouldn't be enough to handle the problems of the previous paragraph.

Given this, I think the best *classical* approaches involve restricting **Transparency** (despite its whiff of analyticity) rather than **Existence**. There are certainly classical theories that do this and manage to avoid paradox: for instance, Bealer 1982, Feferman 1984, and property analogs of some of the theories in Friedman and Sheard 1987. I don't rule out that one of these theories is ultimately the way to go. But their unnaturalness makes it worth exploring non-classical options on which we can keep **Existence** and **Transparency** together.

Actually to get **Transparency** in the form I've written it, one needs a well-behaved conditional, and adding such to a non-classical logic raises complications that I don't want to get into in this paper. But the core of transparency doesn't really involve the conditional, it's the following:

Basic Transparency: For any formulas C and D, if D is like C except with "$t_1, ..., t_n \xi_n \lambda x_1...x_n A(x_1, ..., x_n, e_1, ..., e_m)$" substituted for some occurrences of "$A(t_1, ..., t_n, e_1, ..., e_m)$", then C and D are equivalent (in the sense that each implies the other).

In a logic where there's a contraposable conditional (and associated biconditional) for which sentences of form $A \to A$ are derivable, this is equivalent to **Transparency** in its conditional form; but here I want to avoid discussion of such a conditional, so will focus on **Basic Transparency**. (There are in the literature a number of treatments of such conditionals. I think the ideas here can be extended to incorporate any of them.)[10]

[10]Different kinds of conditionals serve different purposes. For the kind of conditional associated with restricted universal quantification, my preferred approach is similar to that in Field 2020a, but some small modifications in its revision rule yield dramatic

5 Adapting Kripke's construction

Kripke 1975, despite officially advocating the view that Liar sentences don't express propositions, developed a theory that can easily be co-opted as a view of propositions, including propositions that are paradoxical in classical logic.[11] And this can be extended to a theory of properties more generally (and the generalized theory is a bit smoother even in application to propositions, as will be noted at the end of Section 8). There's nothing very original in the suggestion of using his construction in something like this way: quite a few people (e.g. Gilmore 1974, Brady 1971 and 1983, and Maddy 1983) have suggested using it for a theory of classes, where classes, unlike iterative sets, are essentially 1-place properties without the modal element.[12] (Gilmore and Brady actually preceded Kripke.) Without the modal element we can't use it for propositions, but adding that is easy, and Myhill 1984 has suggested using a version with modality for a theory of properties (with an extraordinarily fine-grained theory of property identity on which any two distinct predicates denote different properties, though he suggests that we might do better). This does naturally suggest a unified theory of properties and propositions along Kripkean lines, so there is certainly precedent for what I'll do. But it will be important (e.g. for the discussion of property-identity in Section 7) to have an explicit version that fits my needs.

Let L_0 be a first-order modal language built in the usual way from atomic predicates, variables, standard connectives, quantifiers, and a modal operator $\Box$, say an S5 modality to keep things simple. (We could include names and function symbols too, but these raise minor complications that would be distracting.) I need make no assumptions about L_0, beyond its not containing the vocabulary to be added in L: in particular, there's no need to assume that L_0 contains arithmetic or any other means for expressing its own syntax. (In the end we probably want to apply the construction to a regimented version of our fullest classical sublanguage, able to express physics, set theory, and whatever else; but the construction applies to very humble languages as well.) The construction will take predicates of L_0 not to have properties of any arity in their extensions (though this could be relaxed with a bit of trouble); in

improvements (see Field 2026b). Also, its treatment of property identity should be modified along the lines suggested in Section 7 below. Another kind of conditional is for expressing logical consequence: Myhill 1984 suggests a stratified conditional for this purpose; I take that not to be a competitor of, but to supplement, a restricted quantifier conditional.

[11] Or rather, as a partial view of them: it leaves their identity conditions unspecified.

[12] Gilmore noted that these are *non-extensional* classes. Brady didn't regard them as non-extensional; neither did Maddy, for different reasons. Both their reasons will be mentioned in Section 7.

particular, if L_0 contains an identity predicate, its extension is to be the set of pairs $<o, o>$, *for o in the domain of* M_0. To emphasize this I'll write any identity predicate of L_0 as '$=_g$', with the subscript meant to suggest "ground level". For simplicity I take the intended interpretation of L_0 to be bivalent.

Let L expand L_0 by adding:

(i) An operator symbol 'λ' which, applied to any string of 0 or more variables and any formula of L (not just of L_0!), yields a singular term whose free variables are just those that are free in the formula and not part of the string.

(ii) For each $n \geq 0$, a 1-place predicate '$Property_n$', intuitively meaning "n-place property", where properties as usually understood are 1-place, propositions are 0-place properties, and relations are n-place properties for $n \geq 2$.

(iii) If L_0 doesn't contain a ground level identity predicate '$=_g$', we add either it or a 1-place predicate 'Property' that intuitively is the "infinite disjunction" of all the 'Property$_n$'. (If the ground language contains '$=_g$', one doesn't need to add 'Property', one can just use $\neg(x =_g x)$ in its place.)

(iv) For each $n \geq 0$, an $(n+1)$-ary predicate ξ_n. (For $n = 0$, this can be read as 'True', and for $n > 0$ as 'Instantiates'.)

(v) A binary predicate '$=_p$' of property identity. (The disjunction of '$=_p$' and '$=_g$' together serves as a general identity predicate.)

(If L_0 allows for finite sequences, we could simplify all this by reducing relations to 1-place properties of n-tuples, so that we'd need only $n = 0$ and $n = 1$.)

Note that '$=_p$' is allowed to appear in λ-terms.

I'll assume that (relative to any assignment of possible objects to their free variables) these λ-terms denote something at every world (and that their denotation there is a property of the obvious arity). This guarantees **Existence**. A minor variant of Kripke's construction (similar to what Gilmore *et al* suggested, and sketched below) will show how to get **Limited Basic Transparency**, like **Basic Transparency** except with ξ-substitutions restricted to those not in the scope of '$=_p$'. In Section 7 I'll suggest two ways to give a reasonable extension to '$=_p$' on which property identity is itself transparent (in the sense that if C is an atomic formula whose predicate is '$=_p$', and D results from it by a ξ-substitution, then D is equivalent to C). **Basic Transparency** follows from **Limited Basic Transparency** plus the transparency of '$=_p$'. (It's only the absence of a well-behaved conditional from the language that keeps this from extending to the conditional form of Transparency.)

The Kripke-like construction will be a construction, within classical logic, of models for L. We start with any classical worlds model M_0 for the ground language L_0. I'll use S5 models for simplicity; such a model involves a set W of worlds, and assigns to each $w \in W$ a set U_w of things that exist at w, and assigns to each pair of a $w \in W$ and an n-ary predicate a set of n-tuples of objects in the union $|M_0|$ of these U_w (or if we want to be more restrictive, just in U_w). We want to extend this to a worlds model M for the full L, one that's appropriate to a non-classical logic, in particular to a logic with a 3-valued semantics. (There's an easy further extension to the logic FDE with 4-valued semantics, but I'll stick to the 3-valued case.)

The set W of worlds in M is the same as in M_0. For each $w \in W$, the domain $U_{M,w}$ is generated in stages: for each w, we set

$U_{M,w,0} = U_w$

$U_{M,w,k+1} = U_{M,w,k} \cup \{<\lambda x_1...x_n A(x_1...x_n, u_1...u_m), e_1, ..., e_m>\} : n \geq 0 \wedge \forall i(e_i \in |M_k|)\}$, where $|M_k|$ is the union of the $U_{M,w*,k}$ for all worlds $w*$

$U_{M,w} = \cup_{k \in N} U_{M,w,k}$.

So for each $w \in W$, $U_{M,w}$ consists of U_w together with all $< \lambda x_1...x_n A(x_1, ..., x_n, u_1, ..., u_m), e_1, ..., e_m >$ when A is a formula and each e_i is in the union $|M|$ of all the $U_{M,w}$.

There's a lot of intuitive duplication in each $U_{M,w}$, which we will ultimately eliminate by contracting the model, but that involves the treatment of property identity which will come in Section 7. For now, think of the members of $|M| - |M_0|$ as modeling not properties, but representations of properties.[13]

The Kripke part of the construction will depend on an assignment of a set of pairs EQ as a candidate for the classical extension of '$=_p$'. A particular choice of this equality relation will be made in Section 7; for now I leave it arbitrary. At each "Kripke stage" we give a 3-valued worlds model, with the worlds and their domains as just described. We take the denotation of an abstraction term $\lambda x_1...x_n A(x_1, ..., x_n, u_1, ..., u_m)$, relative to an assignment s of objects in $|M|$ to the free variables, to be $\lambda x_1...x_n A(x_1, ..., x_n, s(u_1), ..., s(u_m))$. (Or if you don't want to make the simplification in note 13, it's the object $<\lambda x_1...x_n A(x_1, ..., x_n, u_1, ..., u_m), s(u_1), ..., s(u_m)>$ that I've officially put into $|M|$.) The denotation of a variable relative to s is of course just the member of $|M|$ that s assigns it.

[13]It might be useful to make a minimal reduction in the duplication even now, so as to allow for a convenient abbreviation. List the variables of the language in some definite order. For any variables x_1,...,x_n, call a formula with m free variables other than x_1,...,x_n *standard with respect to* x_1,...,x_n if those free variables are the first ones on the list other than x_1,...,x_n, and the order of their first free occurrence in the formula is the same as their order on the list. Then in $|M|_{k+1}$ we can stick to $\lambda x_1...x_n A(x_1...x_n, u_1...u_m)$ where A is standard with respect to $x_1, ..., x_n$.

This allows us, without ambiguity, to use $\lambda x_1...x_n A(x_1, ..., x_n, e_1, ..., e_m)$ as an abbreviation of $<\lambda x_1...x_n A(x_1...x_n, u_1...u_m), e_1...e_m>$.

We then proceed to assign values in $\{0, \frac{1}{2}, 1\}$ to formulas, relative to any world w and any assignment s of objects in the domain $|M|$ to the free variables, and also relative to an ordinal stage σ. The values will also depend on the choice of EQ. For each ordinal σ we stipulate the following:

1. For any k-place predicate p of the ground language, $|p(t_1, ..., t_k)|_{M,EQ,w,s,\sigma}$ is 1 if there are entities $e_1, ..., e_k$ denoted$_s$ by $t_1, ..., t_k$ respectively such that $<e_1, ..., e_k>$ is in the extension of p in M_0; 0 otherwise.

2a. For any n, $|Property_n(t)|_{M,EQ,w,s,\sigma}$ is 1 if $den_s(t)$ is of form $\lambda x_1...x_n A(x_1...x_n, e_1...e_m)$ for that specific n; 0 otherwise.

2b. $|Property(t)|_{M,EQ,w,s,\sigma}$ is 1 if *for some* n, $den_s(t)$ is of form $\lambda x_1...x_n A(x_1...x_n, e_1...e_m)$; 0 otherwise.

3. $|t_1 =_p t_2|_{M,EQ,w,s,\sigma}$ is 1 if $<den_s(t_1), den_s(t_2)> \in EQ$; 0 otherwise.

4. $|\neg A|_{M,EQ,w,s,\sigma}$ is $1 - |A|_{M,EQ,w,s,\sigma}$.

5. $|A \wedge B|_{M,EQ,w,s,\sigma}$ is the minimum of $|A|_{M,EQ,w,s,\sigma}$ and $|B|_{M,EQ,w,s,\sigma}$.

6. $|\forall v A|_{M,EQ,w,s,\sigma}$ is the minimum of the $|A|_{M,EQ,w,s(e/v),\sigma}$ for the $e \in U_{M,w}$.

7. $|\Box A|_{M,EQ,w,s,\sigma}$ is the minimum of the $|A|_{M,EQ,w*,s,\sigma}$ for the $w* \in W$

8a. $|t_1...t_n \xi_n t_{n+1}|_{M,EQ,w,s,\sigma}$ is 0 if $den_s(t_{n+1})$ is not of form $\lambda x_1...x_n A(x_1...x_n, e_1...e_m)$ for that specific n.

8b. If $den_s(t_{n+1})$ is $\lambda x_1...x_n A(x_1...x_n, e_1...e_m)$ then $|t_1...t_n \xi_n t_{n+1}|_{M,EQ,w,s,\sigma}$ is

1 if $(\exists \tau < \sigma)(\forall \rho \in [\tau, \sigma))[|A(t_1...t_n, u_1...u_m)|_{M,EQ,w,s*,\rho} = 1]$;
0 if $(\exists \tau < \sigma)(\forall \rho \in [\tau, \sigma))[|A(t_1...t_n, u_1...u_m)|_{M,EQ,w,s*,\rho} = 0]$;
$\frac{1}{2}$ otherwise;

where s^* is like s except that it assigns $e_1...e_m$ to $u_1...u_m$.

Following Kripke, we show that if $\sigma_1 \leq \sigma_2$ then for any s and w, any formula that has value 1 relative to s and w at σ_1 retains that value at σ_2, and similarly for 0: the only changes in value as σ increases are from $\frac{1}{2}$ to one of the other values. (Given this, the $(\exists \tau < \sigma)(\forall \rho \in [\tau, \sigma))$ can be simplified to $(\exists \rho < \sigma)$ in the 1 and 0 clauses of 8b.)

Since the model has a definite cardinality c, the set of pairs of formulas and assignment functions has a definite cardinality (which is just c, since c is infinite). As a result of that and the fact that a change in value is never undone, there has to be a point at which there are no changes in value: a fixed point $\Omega_{M.EQ}$. The values at the fixed point are the only ones that matter (for the given M and EQ).[14]

[14]Other fixed points are also of interest: they are obtained by modifying 8b for $\sigma = 0$, to allow some sentences that get value $\frac{1}{2}$ in the minimal fixed point to get value 0 or 1. Considering them would bring slight improvements to the theory, but would complicate the exposition, so I'll restrict attention to the minimal fixed point in what follows. (Incidentally, the interest of a 4-valued construction that generates the logic FDE lies in its

Given the fixed point, 8b implies

(FP) $|t_1...t_n\xi_n\lambda x_1...x_n A(x_1...x_n, u_1...u_m)|_{M,EQ,w,s,\Omega} = |A(t_1...t_n, u_1...u_m)|_{M,EQ,w,s,\Omega}$.

And it's easy to see that together with the Kleene rules used in clauses 4-7, this guarantees that substitution of $t_1...t_n\xi_n\lambda x_1...x_n A(x_1...x_n, u_1...u_m)$ for $A(t_1...t_n, u_1...u_m)$ outside of identity formulas results in a formula that has the same value (at the fixed point) for all worlds and all assignments to variables.

The goal of this model-theoretic construction is to allow for a definition of validity (leaving the logic of identity unspecified for the moment). We have several choices in the definition, leading to different logics, all non-classical.[15] (We could tweak the models by "closing them off" to yield classical theories, but that would be to give up on the fixed point condition **(FP)**, and hence on **Transparency**.)

6 The significance of value $\frac{1}{2}$

When Kripke formulated an analogous construction for sentential truth, he regarded sentences that get assigned value $\frac{1}{2}$ as "not expressing propositions". (See his note 18.) Even in the context of sentential truth this strikes me as highly dubious. It should go without saying that it's counterintuitive: if we take propositions to be the objects of belief then by ordinary standards, utterances of these sentences certainly seem to express propositions; that seems especially clear for "contingently paradoxical" utterances. (A person who's in Room 202 but falsely believes that he's in Room 201 can believe that what the person in Room 202 is saying isn't true, so how can it be that his use of "What the person in Room 202 is now saying isn't true" doesn't express an object of belief?) I take it that the question is, are there theoretical reasons for the view that outweigh its counter-intuitiveness?

Kripke apparently thought so: he thought that by taking sentences assigned value $\frac{1}{2}$ in his construction as not expressing propositions, classical logic would be validated for sentences that do express propositions.

allowing for more *non-minimal* fixed points; the minimal fixed point such a construction generates do not utilize the fourth value.)

[15]For instance, if we take an inference from a set Γ of formulas to a formula B to be valid iff for all fixed point models, worlds, and assignments to free variables, it preserves value 1, we get a logic of properties based on the non-classical logic K_3, which invalidates excluded middle. If we change 'preserves value 1' to 'preserves the property of having value > 0', we get a logic based instead on LP, which invalidates modus ponens for '$\supset$'. If we take it to be valid iff for all ground models, worlds and assignments, the value of B is at least the value of each member of Γ, we get a logic that invalidates both excluded middle and modus ponens.

But this thought is highly dubious. In the first place, his construction assigns semantic values only relative to a ground model. Kripke's presentation tends to obscure this fact, but it is beyond doubt. There are two reasons.

First, by Tarski's Theorem on the undefinability of truth, there's no hope of explicitly defining an absolute notion of having value 1 (at a stage of the construction) for arbitrary sentences of one's language. We can only define it relative to a model whose domain is a set, so that the quantifiers don't range over absolutely everything. That's important because even for sentences not involving 'true', the model will give "incorrect values", i.e. value 1 for some false sentences and value 0 for some true ones. (At least this is so if the model is definable in the language.)[16] For instance, if one uses a model whose set theoretic domain is the set of sets whose rank is less than the first inaccessible cardinal, then the claim that there are inaccessible cardinals gets value 0 in the model even though true in reality,[17] and its negation gets value 1 even though false in reality.

The second reason why the model-dependence is crucial to Kripke's construction for satisfaction (or the analog for property-instantiation) is that the argument that the construction reaches a fixed point depends on the domain having a cardinality, which won't be true unless the quantifiers are restricted to a set. (Of course if you imagine that the language L_0 is just intended for a restricted domain, like the natural numbers as opposed to the sets of the set theory in which the construction is given, then these issues don't arise; but so restricting the discussion would compromise Kripke's claims to be dealing with languages that contain their own truth predicates.)

This model-dependence of the values doesn't defeat their purpose, which (as mentioned at the end of Section 5) is to define validity.

But it seems clear that we better not use "has value $\frac{1}{2}$ in such and such a model" to define 'doesn't express a proposition': for instance, if the model is as above then the sentence 'Either there are inaccessible cardinals or the Liar sentence is true' will get value $\frac{1}{2}$ even though true in reality. The obvious alternative is to define 'expresses a proposition' as something like "Is either true or has a true negation". But on Kripke's theory of truth, that's a non-classical notion: the claim that the Liar sentence is neither true nor has a true negation itself gets value $\frac{1}{2}$.[18] So if we use it to define 'doesn't express

[16] And it's unlikely that undefinable models would help: Hamkins 2003 proved that the claim that there are undefinable models M for which truth in M coincides with truth simpliciter (for sentences in the language of set theory) isn't provable in standard set theory. (He shows that this claim is *consistent with* standard set theory, but the only model he provides is highly unnatural.)

[17] There must actually be inaccessible cardinals for such a model to exist.

[18] Much later in the paper Kripke discusses "closed off" constructions, which add a

a proposition' then it's hard to see how we've saved classical logic.[19]

(It is possible to extend the fixed point construction to develop a classical notion of *well-behaved* proposition on which all well-behaved propositions obey classical logic: see Field 2022, or better, 2026a. But it's still a non-classical theory, and many sentences that even Kripke would count as expressing propositions express only ill-behaved ones.)

The last few paragraphs have criticized the idea that we can use Kripke's construction in the sentential case for the conclusion that some utterances "don't count against classical logic" because they don't express propositions. But even aside from this, there's an obvious further problem for extending the idea to the theory of propositions. Do Liar *propositions* not count against classical logic because *they don't express propositions*? It's hard to make sense of that. I guess one could say that they don't count against classical logic because they don't *exist*; that would be to adopt a selective fictionalism about the Kripke-based construction for propositions, according to which some of the propositions it postulates exist and others don't. But there's no motivation for that within the construction. Better, I guess, would be to say that the construction is for *propositions and pseudo-propositions together*, and that only those that get value 1 or 0 (relative to a chosen ground model) are *genuine propositions*. That's coherent (as is saying that all dogs are black, with apparent counterexamples being mere pseudo-dogs); but it's awkward to have to appeal to pseudo-propositions in your account of genuine propositions and their truth-values.[20] Whether adopting this proposal saves classical logic in any interesting sense I leave for the reader to decide.

7 Property identity

Until now I've left the treatment of property identity a black box: I have said that '$=_p$' will be assigned a classical extension, but haven't said what

separate ordinal stage after the fixed point by a different successor rule that gets rid of the value $\frac{1}{2}$. Those constructions produce uncontroversially classical theories (albeit ones that don't accord with the motivations for Kripke's fixed point construction), and aren't to the point here: on them, no sentences get value $\frac{1}{2}$ and so Kripke wouldn't deem any sentences as not expressing propositions.

[19]And if the standard of proper acceptance of a claim is its having value 1 in some appropriate model, we couldn't accept that the Liar sentence doesn't express a proposition. (Whereas if the standard of acceptance is having value greater than 0 in some appropriate model, we would have to accept that every sentence expresses a proposition.)

[20]There's a somewhat analogous awkwardness in the proposal about dogs: the laws of Mendelian genetics for genuine dogs would have to appeal to pseudo-dogs, given that genuine dogs often have pseudo-dogs for parents.

that extension should be.

There is probably no one right answer to that: there are different conceptions of property with different fineness in identity conditions. But for some purposes it's natural to want as coarse-grained a notion as possible consistent with transparency, and that's what I'll be pursuing here.[21]

Specifying such coarse identity conditions would be easy enough if we restricted the λ terms to those not including '$=_p$'. If we made that restriction, we could simply adopt a 3-valued version of the set of possible worlds approach, except relativized to the model. That is, we could run the construction without '$=_p$' and then add '$=_p$' at the end by the stipulation

(?) $|e_1 =_p e_2|$ is 1 iff there are variables $x_1, ..., x_n$ and formulas A and B and an assignment function s such that e_1 is $den_s(\lambda x_1...x_n A(x_1...x_n, u_1...u_m))$ and e_2 is $den_s(\lambda x_1...x_n B(x_1...x_n, v_1...v_k))$ and $(\forall b_1...b_n \in |M|)(\forall w \in W)(|A(b_1...b_n, u_1...u_m)|_{M,w,s.\Omega} = |B(b_1...b_n, v_1...v_k)|_{M,w,s.\Omega})$; and it's 0 otherwise.

The proposal I'll make coincides with this when A and B do not themselves include '$=_p$'. But **(?)** gives *too* coarse a criterion of identity for abstracts containing '$=_p$'. (The theory of proper classes in Maddy 1983 is a non-modal variant of the Kripkean construction in Section 5, and contains an extensionality principle that is the non-modal analog of **(?)**; but it works only by excluding class identity from its abstracts.)

To see why **(?)** must fail if we are to maintain the transparency of '$=_p$' and allow it to appear unrestrictedly in the abstracts, note that one could use '$=_p$' to define a certain biconditional $\ll\gg$: let $A \ll\gg B$ abbreviate $\lambda A =_p \lambda B$.[22] (A and B can contain free variables, which can differ from one to the other.) But biconditionals give rise to a variety of Curry-like paradoxes, so proposition-identity must too if given free reign. And allowing '$=_p$' to appear unrestrictedly in abstracts is giving it free reign.

To be more concrete (and without explicitly invoking the biconditional), let k_1 be the proposition that *the proposition that* k_1 *is true is the same as the absurd proposition*; that is, let k_1 be the proposition $\lambda[\lambda True(k_1) =_p \lambda\bot]$. (Or at least, let k_1 be *provably equivalent to* this proposition; I'll show how to formalize this sort of provable equivalence, using just the resources of L, in Section 8. 'True' here is 'ξ_0'.) Then transparency requires that k_1 get the same value as $\lambda[k_1 =_p \lambda\bot]$ at each world. But given this, there's no way to satisfy **(?)**: (i) if there are any worlds where k_1 gets value 1, **(?)** requires it

[21] A slightly less coarse-grained treatment that utilizes non-minimal fixed points is probably preferable, but more complicated to explain: see Section 8.2 of Field forthcoming.

[22] Indeed, property identity and this biconditional are interdefinable.

to instead have value 0 at them (since $\lambda\bot$ has value 0 at all worlds); (ii) if k_1 gets value 0 at all worlds, **(?)** requires it to instead have value 1 at them all; and (iii) since k_1 is an identity proposition, **(?)** rules out its having any value other than 1 and 0.

Given this, a natural thought is to modify **(?)** by keeping its 1 clause, but modifying the 0 clause by declaring a proposition-identity to have value 0 iff there are worlds where one of its terms has value 1 and the other has value 0; $\frac{1}{2}$ when one has value $\frac{1}{2}$ and the other 0 or 1. This handles k_1: this modified account $(?_{mod})$ just dictates that k_1 has value $\frac{1}{2}$ at all worlds. But assuming transparency, $(?_{mod})$ fails for other Curry propositions, e.g. a k_2 equivalent to the proposition that *the proposition that k_2 is true is the same as a Liar proposition λQ*.[23]

I think the problem with **(?)** isn't its bivalence assumption, but rather its 1 clause. The left to right direction of that clause is pretty much required if we are to have substitutivity of identity; what we need to give up is its assumption that if propositions have the same value at all worlds, they are identical. We can keep bivalence of identity.

But it remains to give a semantics that accords with this. There are multiple possibilities. Field 2020b (partially following a suggestion in Weber 2020) proposed that we adapt, directly for property-identity, a fixed point construction that Ross Brady 1983 used for conditionals. This would work, and yield fairly reasonable results. But I think we get better results by a revision construction.[24]

The idea behind both the Brady-based approach and the revision approach I prefer is to get the desired extension EQ of '$=_p$' from a "macro-construction" composed of Kripkean micro-constructions as given in Section 5. Let's use α for an ordinal for the stages of this macro-construction; the stages differ only in that the property-identity predicate EQ_α on which they are based may differ from stage to stage. At stage 0 we can let EQ_0 be the trivial relation that holds between all abstracts of the same arity. Obviously it's transparent. In subsequent stages α, we start from a different EQ_α based on the Kripke fixed point value of formulas at prior stages.

On the Brady-based approach, at each ordinal α we let EQ_α hold (at each

[23] Transparency requires that at each world k_2 get the same value as $\lambda[k_2 =_p \lambda Q]$. But given this, there's no way to satisfy $(?_{mod})$: (i) if there are any worlds where k_2 gets value 1, $(?_{mod})$ requires it to instead have another value (since λQ has value $\frac{1}{2}$ at all worlds); (ii) $(?_{mod})$ dictates that k_2 can't have value 0 at any worlds; and (iii) if k_2 gets value $\frac{1}{2}$ at all worlds, $(?_{mod})$ requires it to instead have value 1.

[24] We could easily extend either the revision construction or the Brady-based one from property-identity to a notion $\leq_p$ of property *inclusion*, that is, of one property being at least as strong as another; $x =_p y$ would just mean $(x \leq_p y) \wedge (y \leq_p x)$.

world) between $\lambda x_1...x_n A(x_1...x_n, e_1...e_m)$ and $\lambda y_1...y_n B(y_1...y_n, e*_1...e*_k)$ in the model iff for all $b_1...b_n \in |M|$ and all $w \in W$, $A(b_1...b_n, e_1...e_m)$ and $B(b_1...b_n, e*_1...e*_k)$ get the same value at the minimal Kripke fixed point based on EQ_γ *for every* $\gamma < \alpha$. (I'm using the simplified notation of note 13.) Each EQ_α is obviously a subset of the prior ones, so this leads to a fixed point (easily seen to be non-empty); that is taken as the desired EQ, on the Brady approach. By the Kripke fixed point property, each EQ_α is transparent if the prior ones are, and since we started from an obviously transparent EQ_0, all the following ones are too, including the fixed point value of ultimate interest.

This Brady-based construction leads to fairly dramatic exceptions to **(?)**. For instance, the proposition $\lambda(\lambda\top =_p \lambda\bot)$ comes out distinct from the proposition $\lambda\bot$, even though both are necessarily false. That's because $\bot$ has value 0 at every stage, while $\lambda\top =_p \lambda\bot$ has value 0 only from stage 1 on, so $\lambda(\lambda\top =_p \lambda\bot)$ can't bear EQ_1 to $\lambda(\lambda\top =_p \lambda\bot)$ and hence can't bear EQ_α to it for any greater α. I don't take this to be an obvious problem with the Brady approach, since we can't have **(?)** in full generality anyway;[25] nonetheless we might prefer a bit more coarseness in our notion of property identity.

On the revision approach that I now prefer, we choose EQ_α by looking back not at the Kripke fixed points based on *all* prior EQ_γ but only on *the most recent ones*. The general rule is that $\lambda x_1...x_n A(x_1...x_n, e_1...e_m)$ bears EQ_α to $\lambda y_1...y_n B(y_1...y_n, e*_1...e*_k)$ iff

(REV) for some $\beta < \alpha$, and every γ in the interval $[\beta, \alpha)$, $(\forall b_1...b_n \in |M|)(\forall w \in W)(|A(b_1...b_n, e_1...e_m)|_{\gamma,w} = |B(b_1...b_n, e*_1...e*_k)|_{\gamma,w})$

where $|A(b_1...b_n, e_1...e_m)|_{\gamma,w}$ means the value at w in M at the Kripke fixed point based on EQ_γ. If α is a successor ordinal, all that matters in determining EQ_α is the Kripke fixed point at its immediate predecessor.

On this approach we don't reach a fixed point, but there are *recurrent* extensions EQ, i.e. extensions that occur arbitrarily late: $\forall\beta(\exists\gamma > \beta)(EQ_\gamma = EQ)$. Indeed there comes a point α_0 (the "critical ordinal") such that for every

[25] In Brady's own presentation, in which he took as basic a conditional rather than '$=_p$', it seemed more worrisome, especially because he used that conditional to formulate what he called an Axiom of Extensionality (for his modal-free construction in which we have classes instead of 1-place properties): this said that if $\forall w(w\xi x \to w\xi y)$ and $\forall w(w\xi y \to w\xi x)$ then $\forall z(x\xi z \to y\xi z)$. That formulation of Extensionality would be appropriate if the conditional were suitable for restricting quantification, but examples analogous to $\lambda\bot$ and $\lambda(\lambda\top =_p \lambda\bot)$ show that his isn't: they show that despite his "Extensionality", there are infinitely many empty classes, infinitely many universal classes, infinitely many classes with 0 as their sole member, and so forth. In any natural understanding of Extensionality, abandoning **(?)** is abandoning even a modalized version of Extensionality.

$\alpha \geq \alpha_0$, EQ_α is recurrent. And indeed there are ordinals Δ, greater than the critical ordinal, at which EQ_Δ is the intersection of the recurrent EQ_α. ("*Reflection ordinals.*") [26] **My proposal is to use EQ_Δ for property identity, so that it's the Kripkean construction based on EQ_Δ that ultimately matters.**

This revision approach obviously doesn't lead to the dramatic failures of **(?)** that we get on the Brady-based: e.g. the necessarily false $\lambda(\lambda\top =_p \lambda\bot)$ will be declared identical to $\lambda\bot$, since they are necessarily coextensive on all *recurrent* choices of EQ_α. But our earlier discussion shows that there must nonetheless be failures of **(?)**: e.g., if k_1 is (provably equivalent to) $\lambda[\lambda True(k_1) =_p \lambda\bot]$, then k_1 and $\lambda\bot$ are distinct despite each being provably false at all worlds; analogously for k_2. (The way that works for k_1 on the revision theory is that at each world, $<k_1, \lambda\bot>$ is in EQ_α iff α is even (counting limit ordinals as even); so $<k_1, \lambda\bot>$ is not in EQ_Δ.) Of course on the revision approach, and the Brady too, when A and B don't contain '$=_p$', there's no change in the evaluation of their identity after stage 1, and so **(?)** works fine in that case.

For the approach sketched to work, it must satisfy the laws of identity, in particular, substitutivity (in a form of that appropriate to bivalent identity in the presence of possibly non-bivalent formulas). For this, it suffices to show

(*) For any formula F and assignment function s, if $<s(x), s(y)> \in EQ_\Delta$ then for any world w, $|F(u/x)|_{\Delta,w,s} =_p |F(u/y)|_{\Delta,w,s}$.

And that's easy: if for some w, $|F(u/x)|_{\Delta,w,s} \neq_p |F(u/y)|_{\Delta,w,s}$, then $<s(x), s(y)> \notin EQ_{\Delta+1}$; but then also $<s(x), s(y)> \notin EQ_\Delta$ since EQ_Δ is the intersection of all recurrent ordinals and $\Delta + 1$ is past the critical ordinal and so $EQ_{\Delta+1}$ is recurrent.[27]

Once substitutivity is established, then a well-known procedure allows us to contract the model M of Section 5, by going to equivalence classes under

[26]For a general look at revision theories see Gupta and Belnap 1993; my (REV) builds in what they call the *Herzberger Limit Rule.* Their definition of reflection ordinals on 172 is more complicated than the definition here, but in the special case of the Herzberger rule it amounts to it.

[27]An analogous proof works for the Brady approach. On that approach one can show something stronger: that substitutivity holds *at every macro-stage*, not just the fixed point. (The proof is difficult: it requires proving the "Brady Micro-extensionality Theorem" in Field, Lederman and Øgaard 2017.) That stronger claim was needed in the context of Field 2020a, but it isn't needed when (as here and in Field 2020b) one is dealing with a primitive identity predicate (or one defined from a special conditional $\gg$ not needed for other purposes), since then it's only the ultimately selected EQ (the fixed point or reflection value) that matters.

the relation $=_p$. In the contracted model, it's these equivalence classes that serve as properties; closed λ-terms no longer denote themselves in the model, they instead denote their own equivalence classes. (Obviously this doesn't mean adopting a metaphysics where properties *just are* equivalence classes of formulas; this is just a model, used to establish the consistency of the theory. Also, employing such a model doesn't rule out that there are properties not named by closed abstraction terms of the language.)

8 Diagonalization

I've been trying to make plausible that the paradoxes of sentential truth can be seen as mere reflections of paradoxes that arise for propositions. A worry you might have about that is that I haven't produced an autonomous Liar proposition. I have mentioned the proposition that the Russell property doesn't instantiate itself, and that *looks a lot like* a Liar proposition: e.g. it is equivalent to its own negation and so can only get value $\frac{1}{2}$. (Indeed on the standards of propositional identity in the previous section, it is *identical to* any proposition that is equivalent to its own negation in every recurrent Level 1 construction. Still, on a less coarse-grained notion of identity it isn't "about itself", but rather "about the Russell property".)

How about using instead the analog of sentences that contingently attribute untruth to themselves? It's plausible that "the proposition that the person in Room 202 is expressing isn't true" expresses a Liar proposition when spoken by a person alone in Room 202 (or at any rate, a proposition that is equivalent to a liar proposition given the empirical facts). But those who antecedently think that such utterances either don't express propositions, or express only non-paradoxical ones, will resist this. It would be nice if we could construct an autonomous Liar proposition, not relying on the expressing relation, and one that seems intutively more "Liar-like" than the proposition about the Russell property.

In fact, we can do just as well as in the linguistic case. Recall that even in the linguistic case, the usual non-contingent Liar sentences, produced by Gödel-Tarski diagonalization, aren't literally self-referential: they are merely *provably equivalent to* the claim that they are untrue. And the situation in the propositional case is exactly the same, because we can get an analog of Gödel-Tarski diagonalization there, without appealing to syntax. (Here it's essential that we've gone beyond propositions to include 1-place properties as well.) It precisely mimics what we do with syntax, but it's worth quickly going through the details.

Consider the case where we're diagonalizing 1-place properties to get propositions. Let the *self-application of* a 1-place property q be the proposition $\lambda(q\xi_1 q)$. Abbreviate this as $SA_1(q)$. Given the intersubstitutivity which we naively want and which Kripke's construction validates, $SA_1(\lambda x F(x, e_1, ..., e_m))$ is effectively the same as the proposition $\lambda F(\lambda x F(x, e_1, ..., e_m), e_1, ..., e_m)$.

Given a 1-place property p (think of it as a property *of propositions*), consider the property S_p: $\lambda q[Prop_1(q) \wedge SA_1(q)\xi_1 p]$. Let the *diagonalization of* p be the self-application of S_p; in effect, $\lambda[Prop_1(S_p) \wedge SA_1(S_p)\xi_1 p]$. Clearly this proposition effectively "says of itself" that it has p. It says this of itself in the same sense that a Liar sentence produced by syntactic diagonalization does.[28]

Given this, **the Liar proposition** is simply the diagonalization of the property of not being a true proposition, i.e. it's $Diag(\lambda x(x \neg\xi_0 x))$. And **the Curry proposition** k_1 from early in Section 7 is $Diag(\lambda x[\lambda True(x) = \lambda\bot])$. This is the exact analog of what we do in the linguistic case. This undercuts the worry that the propositional paradoxes only arise because of assumptions about the linguistic expressing relation.

With that worry undercut, I see no reason to doubt that the paradoxes of truth arise directly for propositions, or for quantification into the sentential position. And I see no reason not to endorse the natural view that "paradoxical" sentences express "paradoxical" propositions, that is, propositions that can only be dealt with by restricting either classical logic or Basic Transparency.[29]

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[28] We can also get a more generalized kind of diagonalization, where we diagonalize an n-place property ($n > 0$) in a particular place, to get an $(n-1)$-place property. When q is $\lambda x_1, ..., x_n F(x_1, ..., x_n, e_1, ..., e_m)$, we let $\mathrm{SA}_{n,1}(q)$ [self application of n-place in first position] be $\lambda x_2, ..., x_n F[\lambda x_1, ..., x_n F(x_1, ..., x_n, e_1, ..., e_m), x_2, ..., x_n, e_1, ..., e_m]$, and for n-place p let S_p be $\lambda q(Prop_n(q) \wedge SA_{n,1}(q)\xi_n p))$. The diagonalization of p in the first position is $SA_{n,1}(S_p)$.

[29] Thanks to Cian Dorr, Will Nava and Jared Warren for helpful comments on an earlier draft.

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