

Discriminator logics
(Research announcement)

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A discriminator logic is the $\mathbf{1}$ -assertional logic of a discriminator variety \mathcal{V} having two constant terms $\mathbf{0}$ and $\mathbf{1}$ such that $\mathcal{V} \models \mathbf{0} \approx \mathbf{1}$ iff every member of \mathcal{V} is trivial. Examples of such logics abound in the literature. The main result of this research announcement asserts that a certain non-Fregean deductive system **SBPC**, which closely resembles the classical propositional calculus, is canonical for the class of discriminator logics in the sense that any discriminator logic **S** can be presented (up to definitional equivalence) as an axiomatic extension of **SBPC** by a set of extensional logical connectives taken from the language of **S**. The results outlined in this research announcement are extended to several generalisations of the class of discriminator logics in the main work.

1 Introduction

The *ternary discriminator* on a set A is the function $t : A^3 \rightarrow A$ defined for all $a, b, c \in A$ by [13, Chapter IV§9]:

$$t(a, b, c) := \begin{cases} c & \text{if } a = b \\ a & \text{otherwise.} \end{cases}$$

A *discriminator variety* is a variety \mathcal{V} for which there exists a ternary term $t(x, y, z)$ of \mathcal{V} that realises the ternary discriminator on every subdirectly irreducible member of \mathcal{V} . The term $t(x, y, z)$ is called a *discriminator term* for \mathcal{V} . A variety \mathcal{V} with constant terms $\mathbf{0}$ and $\mathbf{1}$ is a **(0, 1)-discriminator variety** if \mathcal{V} is a discriminator variety, and moreover, $\Theta^{\mathbf{A}}(\mathbf{0}^{\mathbf{A}}, \mathbf{1}^{\mathbf{A}}) = A \times A$ for any $\mathbf{A} \in \mathcal{V}$.

The logics inherent in **(0, 1)-discriminator varieties** encompass many well-known and important deductive systems that have previously been considered in the literature, as well as numerous other interesting but less well known logics. For specific examples, see Section 2. The main result of this research announcement asserts that, up to definitional equivalence, all such ‘discriminator logics’ can be presented as axiomatic expansions of a certain non-Fregean analogue of the classical propositional calculus. See Theorem 4.1 below.

Details of all results reported in this announcement, including proofs, may be found in the monograph [37] in preparation.

2 Discriminator logics

Following [7], a *deductive system* is a pair $\langle \Lambda, \vdash \rangle$, where Λ is an algebraic language and \vdash is a finitary and substitution-invariant consequence relation over Λ . Given a deductive system $\langle \Lambda, \vdash \rangle$, the set of all Λ -formulas is denoted by Fm_{Λ} .

Let \mathcal{V} be a variety with a constant term $\mathbf{1}$ over a language Λ . The **1-assertional logic** of \mathcal{V} , in symbols $S(\mathcal{V}, \mathbf{1})$, is the deductive system $\langle \Lambda, \vdash_{S(\mathcal{V}, \mathbf{1})} \rangle$ whose consequence relation $\vdash_{S(\mathcal{V}, \mathbf{1})}$ is determined by the equivalence [10, Definition 3.1.1]

$$\Gamma \vdash_{S(\mathcal{V}, \mathbf{1})} \varphi \quad \text{iff} \quad \{\psi \approx \mathbf{1} : \psi \in \Gamma\} \models_{\mathcal{V}} \varphi \approx \mathbf{1}.$$

The ‘logics inherent in **(0, 1)-discriminator varieties**’ of the informal remarks of the introduction are exactly the **1-assertional logics** of **(0, 1)-discriminator varieties**.

A deductive system \mathbf{S} is a *discriminator logic* if $\mathbf{S} = S(\mathcal{V}, \mathbf{1})$ for a $(\mathbf{0}, \mathbf{1})$ -discriminator variety \mathcal{V} . Examples of discriminator logics abound in the literature and include:

- **CPC**, the classical propositional calculus.
- The normal modal logic **S5** (“Gödel style”) [9].
- The n -dimensional cylindric logics.
- The n -valued Post logics.
- The n -valued Łukasiewicz logics.
- Basic fuzzy logic [26] with Baaz delta [1].
- The n -valued Łukasiewicz-Moisil logics [3].
- The n -valued symmetrical modal propositional calculi of [29, 30].
- The logic **LPF** of the Vienna Development Method [2, 31].
- The tetravalent modal logic of [23].
- The *dreiwertige Logik der Sprache* of [5] at the propositional level.
- The Heyting-Wajsberg fuzzy logic of [15, 16].
- The **R5**-calculus and the logic **PC** of [21].
- The modal-type orthomodular logic of [19, 20].
- Linear tense logics [32].

This list is not exhaustive.

3 The skew Boolean propositional calculus

Let $\Lambda[\mathbf{SBPC}]$ denote the language consisting of the binary logical connectives \wedge , \vee , \Rightarrow , and \rightarrow (respectively, the conjunction, disjunction, strong implication, and weak implication signs) and the unary logical connective \neg (the negation sign). The *skew Boolean propositional calculus*, in symbols

SBPC, is the deductive system over $\Lambda[\mathbf{SBPC}]$ presented by the following set $\Sigma[\mathbf{SBPC}]$ of axioms and inference rules:

$$\begin{aligned} & \vdash \varphi \Rightarrow (\psi \Rightarrow \varphi) \\ & \vdash (\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi)) \\ & \vdash ((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi \\ & \vdash \varphi \Rightarrow (\psi \rightarrow \varphi) \end{aligned} \tag{3.1}$$

$$\begin{aligned} & \vdash (\varphi \rightarrow (\psi \rightarrow \chi)) \Rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)) \\ & \vdash ((\varphi \rightarrow \psi) \rightarrow \varphi) \Rightarrow \varphi \end{aligned} \tag{3.2}$$

$$\begin{aligned} & \vdash ((\varphi \Rightarrow \psi) \rightarrow \psi) \Rightarrow ((\psi \Rightarrow \varphi) \rightarrow \varphi) \\ & \vdash (\varphi \Rightarrow \psi) \rightarrow (\varphi \rightarrow \psi) \end{aligned} \tag{3.3}$$

$$\begin{aligned} & \vdash \varphi \rightarrow (\varphi \vee \psi) \\ & \vdash \psi \Rightarrow (\varphi \vee \psi) \\ & \vdash (\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow \chi)) \\ & \vdash (\varphi \wedge \psi) \Rightarrow \varphi \\ & \vdash (\varphi \wedge \psi) \rightarrow \psi \\ & \vdash (\varphi \Rightarrow \psi) \Rightarrow ((\varphi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow (\psi \wedge \chi))) \\ & \vdash (\varphi \rightarrow \psi) \rightarrow ((\varphi \Rightarrow \neg \psi) \Rightarrow \neg \varphi) \\ & \vdash \neg \varphi \rightarrow (\varphi \rightarrow \psi) \end{aligned}$$

$$\varphi, \varphi \rightarrow \psi \vdash \psi. \tag{MP}_{\rightarrow}$$

Observe that, by (3.3) and $(\text{MP}_{\rightarrow})$, modus ponens for \Rightarrow is a derived rule of inference of **SBPC**.

Let $\Lambda[\mathbf{CPC}]$ denote the language consisting of the binary logical connectives \cap , \cup , and \supset (respectively, the conjunction, disjunction, and implication signs) and the unary logical connective \sim (the negation sign). Further, let $\Sigma[\mathbf{CPC}]$ be a standard presentation (such as that of [9, Section 2.2.1]) of **CPC** over the language $\Lambda[\mathbf{CPC}]$.

For any $\{\supset\}$ -formula φ , let $\varphi[\Rightarrow]$ and $\varphi[\rightarrow]$ denote the formulas obtained from φ on replacing every occurrence of \supset in φ by \Rightarrow and by \rightarrow respectively. For any set of $\{\supset\}$ -formulas Γ , let $\Gamma[\Rightarrow] := \{\varphi[\Rightarrow] : \varphi \in \Gamma\}$ and $\Gamma[\rightarrow] := \{\varphi[\rightarrow] : \varphi \in \Gamma\}$. Then for any set of $\{\supset\}$ -formulas $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_{\mathbf{CPC}} \varphi$ iff $\Gamma[\Rightarrow] \vdash_{\mathbf{SBPC}} \varphi[\Rightarrow]$ iff $\Gamma[\rightarrow] \vdash_{\mathbf{SBPC}} \varphi[\rightarrow]$. In contrast, $\vdash_{\mathbf{CPC}} ((\varphi \supset \psi) \supset \psi) \supset ((\psi \supset \varphi) \supset \varphi)$ but $\not\vdash_{\mathbf{SBPC}} ((\varphi \rightarrow \psi) \rightarrow \psi) \Rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi)$. For more details on the relationship between \Rightarrow , \rightarrow , and material implication,

see [27, 28] or [37].

Let \mathbf{SBPC}^F denote the axiomatic extension of \mathbf{SBPC} by the (*implicational Fregean axiom*)

$$\vdash (\varphi \rightarrow \psi) \Rightarrow (\varphi \Rightarrow \psi).$$

The (implicational) Fregean axiom arises from the ‘unsymmetrisation’ of the *Fregean axiom*

$$\vdash (\varphi \subset \supset \psi) \equiv (\varphi \equiv \psi).$$

Here $\subset \supset$ denotes the (material) biconditional and \equiv denotes the (primitive or derived) binary logical connective of logical identity. The logical connective \equiv arises naturally in the context of Suszko’s sentential calculus with identity \mathbf{SCI} [38, 39, 40], which augments \mathbf{CPC} with \equiv in order that the distinction between logical identity and logical equivalence may be realised. The Fregean axiom is characteristic of Fregean logics, namely, those deductive systems for which the formulas $\varphi \equiv \psi$ and $\varphi \subset \supset \psi$ are synonymous in the sense of Smiley [35]. The paradigm of a Fregean logic [resp. non-Fregean logic] is \mathbf{CPC} [resp. \mathbf{SCI}]. For a discussion and references, see [17, 18].

The next result asserts that, to within definitional equivalence, \mathbf{CPC} is the axiomatic extension of \mathbf{SBPC} by the (implicational) Fregean axiom. Hence \mathbf{SBPC} is a non-Fregean generalisation of \mathbf{CPC} . For the precise relationship between \mathbf{SBPC} and \mathbf{SCI} , see [37].

Proposition 3.1.

1. The map $\delta : \Lambda[\mathbf{SBPC}] \rightarrow \mathbf{Fm}_{\Lambda}[\mathbf{CPC}]$ defined by

$$\begin{array}{ll} p \wedge q \mapsto p \cap q & p \rightarrow q \mapsto p \supset q \\ p \vee q \mapsto p \cup q & p \Rightarrow q \mapsto p \supset q \\ \neg p \mapsto \sim p & \end{array}$$

is an interpretation of \mathbf{SBPC}^F in \mathbf{CPC} .

2. The map $\epsilon : \Lambda[\mathbf{CPC}] \rightarrow \mathbf{Fm}_{\Lambda}[\mathbf{SBPC}]$ defined by

$$\begin{array}{ll} p \cap q \mapsto p \wedge q & p \supset q \mapsto p \rightarrow q \\ p \cup q \mapsto p \vee q & \sim p \mapsto \neg p \end{array}$$

is an interpretation of \mathbf{CPC} in \mathbf{SBPC}^F .

3. The interpretations δ and ϵ are mutually inverse.

Hence the deductive systems \mathbf{SBPC}^F and \mathbf{CPC} are definitionally equivalent.

Let $\mathcal{GD}_{(0,1)}$ denote the variety generated by the class of all algebras $\mathbf{A} := \langle A; t, 0, 1 \rangle$ of type $\langle 3, 0, 0 \rangle$, where t is the ternary discriminator on A and 0 and 1 are nullary operations such that $\Theta^{\mathbf{A}}(0, 1) = A \times A$. A proof of the following theorem may be extracted from results of [4] and [41].

Theorem 3.2. *The deductive system $S(\mathcal{GD}_{(0,1)}, \mathbf{1})$ is, up to definitional equivalence, precisely \mathbf{SBPC} .*

The notion of definitional equivalence exploited above and throughout this announcement is due to [25]. For deductive systems that are strongly and regularly algebraisable (in the sense of [18]), it is an analogue of the well known notion of term equivalence [34] for varieties. For details, see [36].

Necessary conditions to be satisfied by any reasonable notion of definitional equivalence are postulated in [14]. There, it is asserted that deductive systems \mathbf{S}_1 and \mathbf{S}_2 are definitionally equivalent if there exist

- mutually inverse syntactic translations, uniform in the sense of [14], mapping formulas of \mathbf{S}_1 to formulas of \mathbf{S}_2 and conversely, and
- mutually inverse lattice isomorphisms between the lattices of theories of \mathbf{S}_1 and \mathbf{S}_2 .

These stipulations are met by the notion of definitional equivalence used in this work.

4 The main result

Let \mathcal{F} be a language with $\mathcal{F} \cap \Lambda[\mathbf{SBPC}] = \emptyset$. Let $\mathbf{SBPC}[\mathcal{F}]$ denote the expansion of \mathbf{SBPC} to the language $\Lambda[\mathbf{SBPC}] \cup \mathcal{F}$ obtained by adjoining, for each n -ary logical connective $f \in \mathcal{F}$ and each $1 \leq i \leq n$, the *compatibility axioms*

$$\vdash (\varphi \Rightarrow \psi) \rightarrow ((\psi \Rightarrow \varphi) \rightarrow (f(\vec{\chi}_{i-1}, \varphi, \vec{\chi}_{i+1}) \Rightarrow f(\vec{\chi}_{i-1}, \psi, \vec{\chi}_{i+1}))) \quad (4.1)$$

to $\Sigma[\mathbf{SBPC}]$, where

$$f(\vec{\chi}_{i-1}, v, \vec{\chi}_{i+1}) := f(\chi_1, \dots, \chi_{i-1}, v, \chi_{i+1}, \dots, \chi_n)$$

for each $v \in \{\varphi, \psi\}$. $\mathbf{SBPC}[\mathcal{F}]$ is called the *canonical expansion of \mathbf{SBPC} by \mathcal{F} -extensional logical connectives*. Observe that, by (3.1)–(3.2) and $(\text{MP}_{\rightarrow})$, $\mathbf{SBPC}[\mathcal{F}]$ has the deduction-detachment theorem (in the sense of [9]) with deduction-detachment set $\{\varphi \rightarrow \psi\}$.

The main result of this announcement asserts

Theorem 4.1. *A deductive system \mathbf{S} over \mathcal{F} is a discriminator logic iff it is definitionally equivalent to an axiomatic extension of the canonical expansion of \mathbf{SBPC} by \mathcal{F} -extensional logical connectives.*

Theorem 4.1 solves the logification problem of Font [22, Section 6.5] for the class of $(\mathbf{0}, \mathbf{1})$ -discriminator varieties. A proof of the theorem may be obtained on combining results implicit in [11] with Theorem 3.2.

Let \mathcal{V} be a $(\mathbf{0}, \mathbf{1})$ -discriminator variety over \mathcal{F} . Another of the main results of [37], which complements Theorem 4.1, yields an effective procedure for axiomatising $S(\mathcal{V}, \mathbf{1})$ as an axiomatic expansion of \mathbf{SBPC} by \mathcal{F} -extensional logical connectives, given an equational basis for \mathcal{V} and a discriminator term for \mathcal{V} as input. The presentation Σ output by the algorithm has the property that its only (proper) rule of inference is $(\text{MP}_{\rightarrow})$. Moreover, Σ is finite if \mathcal{V} is finitely axiomatisable and \mathcal{F} is finite.

In [11] it is shown that each provability problem P of first-order logic with equality can be effectively reduced to an equivalent provability problem in the equational theory $T_{\mathcal{V}}(P)$ of an appropriate discriminator variety \mathcal{V} . Here, the definition of $T_{\mathcal{V}}(P)$ depends on P . The variety $\mathcal{GD}_{(\mathbf{0}, \mathbf{1})}$ always serves in [11] as a starting point for constructing a suitable \mathcal{V} , for any given P . In [37], an assertional analogue of the reduction of [11] is presented, where \mathbf{SBPC} realises the role of $\mathcal{GD}_{(\mathbf{0}, \mathbf{1})}$ in constructing a logical cognate of \mathcal{V} . In the reduction of [37], the first-order logical connectives \cap , \cup , \supset , and \sim are replaced with the propositional connectives \wedge , \vee , \rightarrow , and \neg respectively of \mathbf{SBPC} , while the equality symbol \approx is replaced with the symmetrisation of the propositional connective \Rightarrow .

5 $\mathbf{S5}$ as a discriminator logic

To illustrate Theorem 4.1, we present the normal modal logic $\mathbf{S5}$ (“Gödel style”) as an axiomatic expansion of \mathbf{SBPC} by extensional logical connectives.

Let $\Lambda[\mathbf{S5}^G] := \Lambda[\mathbf{CPC}] \cup \{\Box\}$, where \Box is a unary logical connective (the necessity sign). Let $\mathbf{S5}^G$ denote the normal modal logic $\mathbf{S5}$ in the “Gödel style” formulation of [9, Section 2.2.5] over the language $\Lambda[\mathbf{S5}^G]$. (Here, the rule of necessitation is taken in its *strong* form: $\varphi \vdash \Box\varphi$.¹ For a discussion and references, see [24, Section 2.4, p. 57].)

Let \mathbf{S} denote the deductive system over the language $\Lambda[\mathbf{S}] := \Lambda[\mathbf{SBPC}] \cup \Lambda[\mathbf{S5}^G]$ determined by the axioms and inference rules of $\Sigma[\mathbf{SBPC}]$, the ax-

¹This implies that the modal consequence relation $\vdash_{\mathbf{S5}^G}$ is *global* in the sense of [33, Section 3.2], and $\mathbf{S5}^G$ is algebraisable in the sense of [7].

ioms (only) of $\Sigma[\mathbf{CPC}]$, the compatibility axioms (4.1) for each n -ary logical connective $f \in \Lambda[\mathbf{S5}^G]$, together with the ‘link’ axioms

$$\begin{aligned} & \vdash \neg(\varphi \Rightarrow \varphi) \supset \varphi \\ & \vdash ((\varphi \supset \psi) \supset \psi) \Rightarrow ((\psi \supset \varphi) \supset \varphi) \\ & \vdash (\varphi \supset \psi) \Rightarrow (\varphi \rightarrow \psi). \end{aligned} \tag{5.1}$$

Observe that, by (5.1), modus ponens for \supset is a derived rule of inference of \mathbf{S} . Hence $\vdash_{\mathbf{CPC}} \subseteq \vdash_{\mathbf{S}}$.

Theorem 5.1.

1. The map $\delta : \Lambda[\mathbf{S}] \rightarrow \text{Fm}_{\Lambda[\mathbf{S5}^G]}$ defined by

$$\begin{aligned} p \cap q &\mapsto p \cap q & p \wedge q &\mapsto p \cap (\Box p \supset q) \\ p \cup q &\mapsto p \cup q & p \vee q &\mapsto \Box p \cup q \\ \sim p &\mapsto \sim p & \neg p &\mapsto \Box p \supset \sim(p \supset p) \\ p \supset q &\mapsto p \supset q & p \rightarrow q &\mapsto \Box p \supset q \\ & & p \Rightarrow q &\mapsto \Box((p \supset q) \cap (q \supset p)) \cup q \end{aligned}$$

is an interpretation of \mathbf{S} in $\mathbf{S5}^G$.

2. The map $\epsilon : \Lambda[\mathbf{S5}^G] \rightarrow \text{Fm}_{\Lambda[\mathbf{S}]}$ defined by

$$\begin{aligned} p \cap q &\mapsto p \cap q & \sim p &\mapsto \sim p \\ p \cup q &\mapsto p \cup q & p \supset q &\mapsto p \supset q \\ & & \Box p &\mapsto (p \rightarrow \neg(p \Rightarrow p)) \Rightarrow \neg(p \Rightarrow p) \end{aligned}$$

is an interpretation of $\mathbf{S5}^G$ in \mathbf{S} .

3. The interpretations δ and ϵ are mutually inverse.

Hence the deductive systems \mathbf{S} and $\mathbf{S5}^G$ are definitionally equivalent.

Theorem 5.1 shows that $\mathbf{S5}^G$ is in some sense obtained by ‘gluing’ \mathbf{CPC} and \mathbf{SBPC} together.

The preceding observation can be made precise on interpreting Theorem 5.1 algebraically. Let \mathcal{BA} denote the class of all Boolean algebras $\langle A; \cap, \cup, \sim, 0, 1 \rangle$, and let \mathcal{BA}^t denote the class obtained from \mathcal{BA} by adjoining a new ternary function symbol t to the language of \mathcal{BA} such that the realisation of t on any member of \mathcal{BA} is the ternary discriminator. Theorem 5.1 asserts that the algebraic counterpart of $\mathbf{S5}^G$ (*viz.*, the variety of monadic algebras) is, up to term equivalence, precisely the variety generated by \mathcal{BA}^t . This observation has apparently been known to specialists in general algebra for some time. (See, for example, [12, Corollary 2.7].)

6 Concluding remarks

The main result of this research announcement extends, *mutatis mutandis*, to several generalisations of the class of discriminator logics, the most important of which is the class of ‘positive fixedpoint discriminator logics’. A deductive system \mathbf{S} is a *positive fixedpoint discriminator logic* if $\mathbf{S} = S(\mathcal{V}, \mathbf{1})$ for a fixedpoint $\mathbf{1}$ -discriminator variety \mathcal{V} . For the concept of a fixedpoint $\mathbf{1}$ -discriminator variety, see [8].

Let $\Lambda[\mathbf{BCSK}]$ denote the language consisting of the binary logical connectives \Rightarrow (the strong implication sign) and \rightarrow (the weak implication sign), and let \mathcal{F} be a language with $\mathcal{F} \cap \Lambda[\mathbf{BCSK}] = \emptyset$. Let \mathbf{BCSK} denote the deductive system over $\Lambda[\mathbf{BCSK}]$ presented by those axioms and inference rules of $\Sigma[\mathbf{SBPC}]$ not involving any of the connectives \wedge , \vee , and \neg and let the *canonical expansion of \mathbf{BCSK} by \mathcal{F} -extensional logical connectives* be defined in the same way as for $\mathbf{SBPC}[\mathcal{F}]$; for studies of \mathbf{BCSK} , see [27, 28]. For positive fixedpoint discriminator logics, the analogue of Theorem 4.1 is

Theorem 6.1. *A deductive system \mathbf{S} over \mathcal{F} is a positive fixedpoint discriminator logic iff it is definitionally equivalent to an axiomatic extension of the canonical expansion of \mathbf{BCSK} by \mathcal{F} -extensional logical connectives.*

Theorem 6.1 solves the logification problem of Font [22, Section 6.5] for the class of fixedpoint $\mathbf{1}$ -discriminator varieties.

The $\{\supset, \square\}$ -fragment of $\mathbf{S5}^G$ constitutes a natural example of a deductive system that is a positive fixedpoint discriminator logic, but not a discriminator logic; numerous other examples may be found in the literature.

Let \mathbf{BCSK}_\neg denote the axiomatic expansion of \mathbf{BCSK} by a unary logical connective \neg (the negation sign) and those axioms of $\Sigma[\mathbf{SBPC}]$ involving \neg . One of the remaining technical results of [37] shows that \mathbf{BCSK}_\neg is definitionally equivalent to its own ‘anti-logic’, wherein the theorems [resp. contradictions] of the former are precisely the contradictions [resp. theorems] of the latter, and conversely. It is shown in [37] that this ‘anti-logic’ has a (comparatively) familiar description: it is a modalised analogue of the hoop logic $\mathbf{S}_{\mathcal{HO}}$ of [6]. The logic $\mathbf{S}_{\mathcal{HO}}$ is itself a natural generalisation of the conjunction-implication fragment of the intuitionistic propositional calculus; for details, see [6, 8].

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